Two Lectures on Cosmology and the CMB

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Topics to be treated

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Based on textbook *The Primordial Density Perturbation (Cosmology, Inflation and the Origin of Structure)*, DHL and A. R. Liddle (CUP 2009).
I'll use units $\hbar = c = k_{\rm B} = 1$ with temperature in eV or MeV.

Part I: *What we know (observable universe after the first second)*
1. The ΛCDM model.
2. The homogeneous universe:
(i) Hubble parameter and Friedmann Equation, (ii) Composition of the cosmic fluid, (iii) brief history, (iii) The particle horizon and the horizon.

3. Departures from homogeneity (perturbations):
(i) Galaxy formation, (ii) Acoustic oscillation of the photon fluid, (iii) Primordial curvature perturbation $\zeta(\mathbf{x})$, (iv) The Cosmic Microwave Background (CMB) anisotropy.

Part II: *What probably happens at very early times*
4. Inflation:
(i) it removes pre-existing inhomogeneity and drives Ω to 1, (ii) it generates scalar field perturbations from their vacuum fluctuation.
5. A scalar field perturbation generates ζ.

and finally ... 6. Possible modifications of the ΛCDM model.

The ΛCDM model: theory

- Observable universe at $t \gtrsim 1$ second described by ΛCDM model
- Theoretical ingredients mostly just
  - General Relativity
  - Electron-photon (Thomson) scattering
  - Atomic physics (H and He)
  - Three species of non-interacting neutrinos
- We also need, at t slightly less than 1 s (to fix the relative p/n and γ/ν abundances and to fix the energy distribution of the photons and the neutrinos):
  - QED for electron-positron annihilation and Fermi theory for neutrino interactions.
- All of this is pre Standard Model physics.

The ΛCDM model: observation

(Simplest) ΛCDM model defined by five parameters, determined by observation:

H_0^{-1}	Ω_B	Ω_c	$\mathcal{P}_\zeta$	$n-1$
13.7×10^9years	0.046	0.23	$(5\times10^{-5})^2$	-0.04

- Most important observations *so far*
  - Anisotropy of the Cosmic Microwave Background (CMB): presently outweighs all others
  - The (uneven) distribution of galaxies
  - Properties of galaxies and galaxy clusters
  - Primordial abundance of H and ^{4}He (also D and ^{7}Li)
  - Galaxy redshifts versus distance, particularly from supernovas.

Hubble parameter

- Observable universe is (almost) homogeneous and isotropically expanding.
- Distance between *any* two points $\propto$ scale factor $a(t)$: set present value $a_0 = 1$.
- Hubble parameter $H(t) \equiv \dot{a}/a$, Hubble time $\equiv 1/H(t)$.
- Roughly, $a \propto t$ (gravity has small effect in one Hubble time)
  - So age of universe is $t \sim H^{-1}(t)$.
    - Present age $t_0 = 13.7 \times 10^9$ yr. (Near exact equality to H_0^{-1} is coincidence.)
- General Relativity gives Friedmann equation:

$$H^2(t) = (1/3M_{\rm P}^2)\rho(t) - K/a^2(t), \qquad M_{\rm P} \equiv (8\pi G)^{-1/2} = 2 \times 10^{18}\ \text{GeV}$$

  - ρ is energy density
  - $K = 0$ if space is flat (Euclidean geometry)
- Density parameter: $\Omega(t) \equiv \rho/3M_{\rm P}^2 H^2 = 1 + K/(aH)^2 = 1 + K/\dot{a}^2$
  - Note: $|\Omega(t) - 1|$ *increases* with time in the early universe because gravity makes $\ddot{a} < 0$.
- ΛCDM model assumes $\Omega = 1$ corresponding to $K = 0$.
  - Observation gives $|\Omega_0 - 1| \lesssim 10^{-2}$.

The measured scale factor $a(t)$

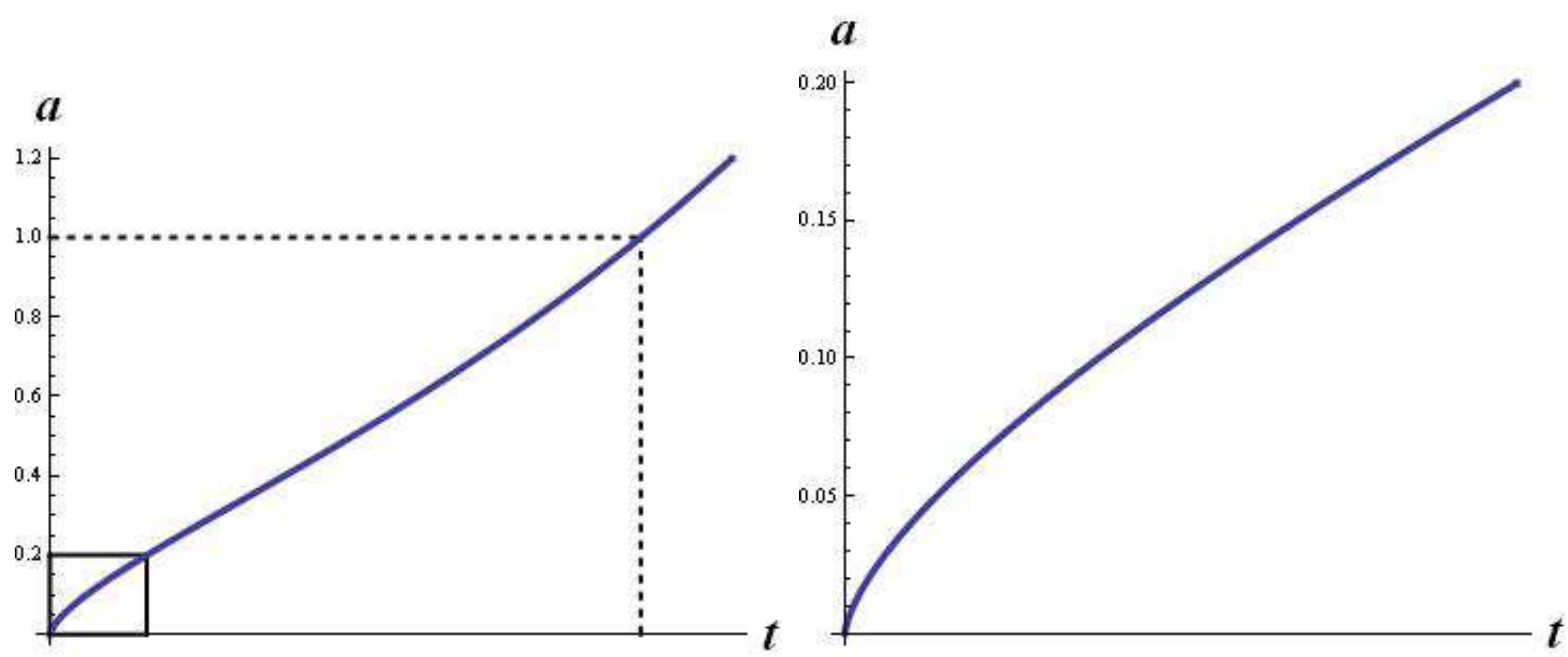

Figure 1: Present value $a_0 = 1$. The lines would be straight in the absence of gravity. Figures courtesy of Mindaugas Karciauskas.

Components of the cosmic fluid

- After $t = 1$ sec, just five components
  - (1) 'Baryonic' matter (protons, neutrons and electrons).
  - (2) Cold Dark Matter (CDM)
    - CDM has negligible interaction and random motion till galaxy formation
      - If CDM is axions it has negligible random motion even then
      - Structure in axion CDM galaxy halos observable?
  - (3) Photons
  - (4) Neutrinos with negligible interaction
  - (5) Cosmological constant ρ_Λ (dark energy)
- $\rho(t) = \sum_i \rho_i(t) = \rho_B(t) + \rho_c(t) + \rho_\gamma(t) + \rho_\nu(t) + \rho_\Lambda$
- PRESENT densities specified by contributions to Ω:
  $\Omega_0 = \Omega_B + \Omega_c + \Omega_\gamma + \Omega_\nu + \Omega_\Lambda$

Values deduced from observation using ΛCDM model with $m_\nu = 0$:

(Ω_0)	Ω_B	Ω_c	Ω_γ	Ω_ν	Ω_Λ
(1)	0.046	0.23	5.0×10^{-5}	3.2×10^{-5}	0.72

Matter and radiation

What happens to the $\rho_i(t)$ going back in time?

- The cosmological constant is time-independent.
- For baryonic matter and CDM, ρ_i is mass density, so $\rho_i \propto a^{-3}$.
- Photons and (massless) ν's form a gas with particle speed $v = 1$
  - Pressure $P_i = \rho_i/3$.
  - Energy conservation $dE = -PdV$ gives $d(a^3\rho_i) = -P_i d(a^3)$ hence $\rho_i \propto a^{-4}$
- Friedmann equation (with $K = 0$) can be written (taking $m_\nu = 0$)

$$
\begin{aligned}
\rho(t) &= \rho_\Lambda + \rho_B(t) + \rho_c(t) + \rho_\gamma(t) + \rho_\nu(t) \\
&= 3M_P^2 H_0^2 \left[\Omega_\Lambda + (\Omega_B + \Omega_c)a^{-3}(t) + (\Omega_\gamma + \Omega_\nu)a^{-4}(t)\right] \\
&= (2.36 \times 10^{-4}\,\text{eV})^4 \times \left[0.72 + 0.28a^{-3}(t) + (8.2 \times 10^{-5})a^{-4}(t)\right]
\end{aligned}
$$

Epoch	Duration	$\rho(t) \simeq$	$a(t) \propto$	$\ddot{a}$
Future	$1 < a(t)$	ρ_Λ	e^{Ht}	> 0
Matter domination	$(3 \times 10^{-4}) \lesssim a(t) \lesssim 1$	$\rho_B + \rho_c$	$t^{2/3}$	< 0
Radiation domination	$a(t) \lesssim (3 \times 10^{-4})$	$\rho_\gamma + \rho_\nu$	$t^{1/2}$	< 0

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History

T	t	What happens
> 1 MeV	$(1\,\text{MeV}/T)^2$ s	*In equilibrium*: e, $\bar{\text{e}}$, γ, ν (their ρ dominates giving T-t relation)
		In equilibrium: p and n giving $\rho_n/\rho_p = \exp[-T/(m_n - m_p)]$
		Not in equilibrium: Cold Dark Matter
1 MeV	1 s	$\bar{\text{e}}$'s annihilate, ν's decouple
10^{-1} MeV	100 s	Big Bang Nucleosynthesis (BBN): n's bind into (mostly) ^{4}He
		There's now a plasma with frequent Thomson scattering.
0.8 eV	10^5 y	Radiation domination gives way to matter domination
0.3 eV	10^5 y	Photon decoupling: e's bind into atoms
10^{-3} eV	10^8 y	First galaxies appear
2×10^{-4} eV	10^{10} y	Present epoch.

- As shown earlier, $\rho_\gamma \propto a^{-4}$ after neutrino decoupling. But blackbody distribution gives $\rho_\gamma \propto T^4$. Hence $T \propto 1/a$ after neutrino decoupling.

Horizon

- Use ‘comoving’ coordinate $\mathbf{x}$, such that any ‘comoving’ point (one moving with the expansion) has constant $\mathbf{x}$.
  - Distance from the origin is $\mathbf{r} = a(t)\mathbf{x}$.
- Since the beginning at $a \simeq 0$, light travels a distance

$$r_{\rm ph}(t) \equiv a(t) x_{\rm ph}(t) \equiv a(t) \int_0^t \frac{dt}{a(t)}$$

- Proof: move origin to position of the photon at some time $t = t_1$.
  At time $t_1 + dt$ photon distance from origin is $a(t)[dt/a(t)] = dt$.
  Hence, instantaneous photon speed measured by observer at photon position is $dt/dt = 1$.
  This is the definition of photon speed according to General Relativity.
- $r_{\rm ph}(t)$ is called the *particle horizon*
  - $r_{\rm ph} = H^{-1}(t)/2$ (matter domination), $r_{\rm ph} = H^{-1}(t)$ (radiation domination).
- Distance $H^{-1}(t)$ is called the *horizon*

Horizon entry

- Region with size $>$ (particle) horizon isn't in causal contact.
- But any region has size $\propto a$, and $d(aH)/dt \equiv \ddot{a} < 0$ before the present.
  - So *any* region is out of causal contact at early times
    - Epoch when size $= H^{-1}(t)$ is called *epoch of horizon entry*
- Particle horizon at present is $r_{\rm obs} \equiv r_{\rm ph}(t_0) = 14,000$ Mpc $= 4 \times 10^{10}$, light years.
  - This is the present size of the observable universe
  - Not exactly equal to ct_0 due to expansion.

Formation of structure

- Present matter density is *very* inhomogeneous (clumped into galaxies).
- To explain this, early matter density must be *slightly* inhomogeneous.
  - Matter falls into over-dense regions.
  - Matter density contrast $\delta_{\rm m} \equiv \delta\rho_{\rm m}/\rho_{\rm m}$ grows. $(\rho_{\rm m} \equiv \rho_B + \rho_{\rm c})$
    - Galaxy-sized over-dense regions collapse to become galaxies.
    - Bigger over-dense regions collapse later to become galaxy clusters
    - Still bigger over-dense regions are still expanding
      - They are 'super-clusters' and under-dense regions are 'voids'.
- Given region must be *inside horizon* before matter can fall into it.
  - Horizon entry is well *after* BBN for galaxy-sized regions.
- CDM has negligible interaction and random motion.
  - Growth of $\delta_{\rm c}$ begins promptly at horizon entry.
- But frequent e-γ collisions prevent growth of δ_B before photon decoupling.
  - And random motion prevents *any* growth of δ_B in region enclosing mass $\lesssim 10^5 M_\odot$.
    - This explains why no galaxies have smaller mass.

Defining cosmological perturbations

- First step is to define a fictitious ‘unperturbed’ universe that is perfectly homogeneous and isotropic. It should closely resemble the actual universe but is otherwise arbitrary.
- Now lay down coordinates $(\mathbf{x}, t)$ in the actual universe. Take them to also describe the unperturbed universe, with $\mathbf{x}$ the comoving coordinate. This defines a mapping, from each point in the unperturbed universe to one in the actual universe.
- A perturbation $\delta\rho$ in (say) energy density ρ is now defined by

$$\delta\rho(\mathbf{x}, t) = \rho(\mathbf{x}, t) - \rho(t),$$

  where $\rho(\mathbf{x}, t)$ refers to the actual universe and $\rho(t)$ to the unperturbed universe.
- Choice of coordinates is called a *gauge*. It should make the perturbations small but is otherwise arbitrary. Physical results gauge-independent.
- Each gauge defines a *slicing* (fixed t) and *threading* (fixed $\mathbf{x}$) of spacetime.

Independent Fourier components

- Instead of $\delta\rho(\mathbf{x},t)$, it's convenient to consider $\delta\rho_{\mathbf{k}}(t) \equiv \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}}\delta\rho(\mathbf{x},t)$
    - Physical wavenumber is $k/a(t)$.
- Similarly for all other perturbations.
- To first order, perturbations with different $\mathbf{k}$ evolve independently — no coupling between them.
- In particular, waves in the universe have fixed $\mathbf{k}$, not fixed $\mathbf{k}/a(t)$: ie. wavelength expands with the universe.
- According to ΛCDM (and much more generally) waves before structure formation are *standing* not *traveling*.
- As we shall see, the CMB anisotropy sees a snapshot of the standing waves in $\delta_{\gamma\mathbf{k}}$ at the epoch of photon decoupling.

Acoustic oscillation

- BEFORE PHOTON DECOUPLING baryonic matter is ionized
  - There's a *baryon-photon fluid* with $P_{B\gamma} = \rho_\gamma/3$ and $\rho_{B\gamma} = \rho_\gamma + \rho_B \simeq \rho_\gamma$.
  - Supports 'sound' waves with speed $c_{\rm s} = \sqrt{\dot{P}_{B\gamma}/\dot{\rho}_{B\gamma}} \sim \sqrt{1/3}$
  - $\delta_\gamma \equiv (\delta\rho_\gamma/\rho_\gamma) = (4/3)(\delta\rho_B/\rho_B)$ oscillates, with damping from photon diffusion

$$\delta_{\gamma\mathbf{k}}(t) = A_\mathbf{k}(t) + e^{-k^2/k_{\rm D}^2(t)} \left[B_\mathbf{k}(t)\cos\left(kx_{\rm s}(t)\right) + C_\mathbf{k}(t)\sin\left(kx_{\rm s}(t)\right)\right]$$

$$x_{\rm s}(t) \equiv \int_0^t c_{\rm s}(t)\frac{dt}{a(t)} \qquad \text{check: } a(t)dx_{\rm s}(t)] = c_{\rm s}dt$$

  - Note: this is the WKB approximation: $B_\mathbf{k}$ and $C_\mathbf{k}$ are proportional to $1/\sqrt{c_{\rm s}}$.
- AT PHOTON DECOUPLING: $x_{\rm s} = 150\,$Mpc, $\qquad k_{\rm D} = (8\,\text{Mpc})^{-1}$
- AFTER PHOTON DECOUPLING:
  - baryons fall into potential wells created by CDM
  - Photons travel freely, seen now as the CMB.
    - $\rho_\gamma(\mathbf{x},t) \propto T^4(\mathbf{x},t)$ so $\delta_\gamma(\mathbf{x},t) = 4\delta T(\mathbf{x},t)/T(t)$

Primordial curvature perturbation $\zeta(\mathbf{x})$

- What determines the density contrasts δ_B, δ_c, δ_γ and δ_ν, present at $T \sim$ MeV?
- According to ΛCDM model, all are determined by *curvature perturbation* $\zeta(\mathbf{x})$
- At $T \sim$ MeV, consider spacetime slicing of uniform total energy density $\rho(t)$.
- Local expansion rate is in general inhomogeneous, $a(\mathbf{x},t) = e^{\zeta(\mathbf{x})} a(t)$.
- This defines $\zeta \equiv \delta[\ln a(\mathbf{x},t)]$.
- $\zeta(\mathbf{x})$ independent of t because all relevant scales are outside horizon:
  - (i) no heat flows
  - (ii) P depends only on ρ: $P(\mathbf{x},t) = \rho(\mathbf{x},t)/3$.
- Proof that ζ is independent of t: use $dE = -PdV$ giving

$$\frac{d\left(a^3(\mathbf{x},t)\rho(t)\right)}{dt} + P(t)\frac{d(a^3(\mathbf{x},t))}{dt} = 0$$

hence $$\dot\rho(t) + 3\left[\rho(t) + P(t)\right]\left[\frac{\dot a(t)}{a(t)} + \frac{d\zeta(\mathbf{x},t)}{dt}\right] = 0$$

Initial density perturbations

- ΛCDM model *assumes* all $\rho_i(t)$ are uniform on slicing of uniform ρ.
- Consider now a different spacetime slicing, of *non-uniform* $\rho(\mathbf{x}, t)$.
- Work to first order in $\delta\rho/\rho$.
- Choose gauge whose threading has locally isotropic expansion, with the slicing orthogonal to the threading (conformal Newtonian gauge).
- It an be shown that $\Delta a \equiv a(\mathbf{x}, t)_{\rm confnewt} - a(\mathbf{x}, t)_{{\rm uniform}\rho} = \zeta(\mathbf{x})/3$.
- At $T \sim$ MeV causal processes ineffective, local evolution is

$$\rho_B \propto \rho_c \propto a^{-3}(\mathbf{x}, t), \qquad \rho_\gamma \propto \rho_\nu \propto a^{-4}(\mathbf{x}, t)$$

- To first order, $\frac{1}{3}\delta_B = \frac{1}{3}\delta_c = \frac{1}{4}\delta_\gamma = \frac{1}{4}\delta_\nu = -\frac{\Delta a}{a} = -\frac{\zeta}{3}$
- So ζ indeed determines initial density perturbations, which are initially time-independent.
- Setting $A_{\mathbf{k}}(t) = 0$ and imposing initial condition, we get at photon decoupling

$$\delta_{\gamma\mathbf{k}} \simeq -\frac{4}{3}\zeta_{\mathbf{k}} e^{-k^2/k_{\rm D}^2} \cos\left(k x_{\rm s}\right), \qquad x_{\rm s} = 150\,\text{Mpc}, \qquad k_{\rm D} = (8\,\text{Mpc})^{-1}$$

We'll see how this contributes to the CMB anisotropy.

Spectrum of ζ

- ΛCDM model assumes $\zeta_{\mathbf{k}}$ uncorrelated ('gaussianity')
- Using Fourier *Series* in box of size aL, define spectrum P_ζ by

$$\langle |\zeta_{\mathbf{k}}|^2 \rangle = L^3 P_\zeta(\mathbf{k})$$

  where $\langle\rangle$ is the average in a cell d^3k (ensemble average).
- Taking again ensemble average we get

$$\langle \zeta^2(\mathbf{x}) \rangle = \frac{1}{(2\pi^3)} \int d^3k P_\zeta(\mathbf{k})$$

  This is independent of $\mathbf{x}$ (statistical homogeneity).
  - Can regard $\langle\rangle$ here as spatial average (example of ergodic theorem).
- Taking $L \to \infty$ we have

$$\langle \zeta_{\mathbf{k}} \zeta^*_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') P_\zeta(\mathbf{k})$$

Observed spectrum and spectral index

- ΛCDM model assumes $P_\zeta(\mathbf{k})$ depends only on magnitude k (statistical isotropy).
- Convenient to define $\mathcal{P}_\zeta(k) \equiv (k^3/2\pi^2) P_\zeta(k)$. Then $\langle \zeta^2(\mathbf{x}) \rangle = \int_0^\infty d(\ln k) \mathcal{P}_\zeta(k)$.
- ΛCDM model assumes power law

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta(k_{\text{pivot}})(k/k_{\text{pivot}})^{n-1}$$

  - Choose $k_{\text{pivot}}^{-1} = 5000\,$Mpc
- Observation gives $n - 1 = -0.04 \pm 0.015$; spectrum is almost scale invariant!
- Value $\mathcal{P}_\zeta(k_{\text{pivot}}) = (5 \times 10^{-5})^2$
- This gives $\langle \zeta^2(\mathbf{x}) \rangle \sim (10^{-4})^2$

The CMB anisotropy

- Photons coming from given direction have blackbody distribution.
  - Average temperature $T = 2.73$K.
- There's a dipole anisotropy due to our velocity $\mathbf{v}$ through CMB
  - Observed in direction $\mathbf{e}$, $\delta T(\mathbf{e})/T = -\mathbf{v}\cdot\mathbf{e}$ with $|v| = 1.2\times 10^{-3}$.
    - According to ΛCDM model, galaxies are on average at rest w.r.t. CMB
      - Confirmed by observation
- We're interested in the INTRINSIC ANISOTROPY
  - $\delta T(\theta,\phi)/T = \sum_{\ell=2}^{\infty} a_{\ell m} Y_{\ell m}(\theta,\phi) \lesssim 10^{-5}$
- According to ΛCDM, $a_{\ell m}$ uncorrelated ('gaussian')
  - $\langle a_{\ell m} a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell$ [note: $C_\ell = \langle |a_{\ell m}|^2\rangle$]
    - Form of RHS corresponds to statistical isotropy
- Expectation value $\langle\rangle$ is average over observer's position.
  - For high ℓ, can approximate this as average over m at our location.
  - Expected error ΔC_ℓ called *cosmic variance*. From gaussianity,

$$(\Delta C_\ell)^2 \equiv \langle \left(|a_{\ell m}|^2 - C_\ell\right)^2 \rangle = 2C_\ell^2/(2\ell+1)$$

- ΛCDM model predicts C_ℓ in terms of the five parameters.

The CMB anisotropy

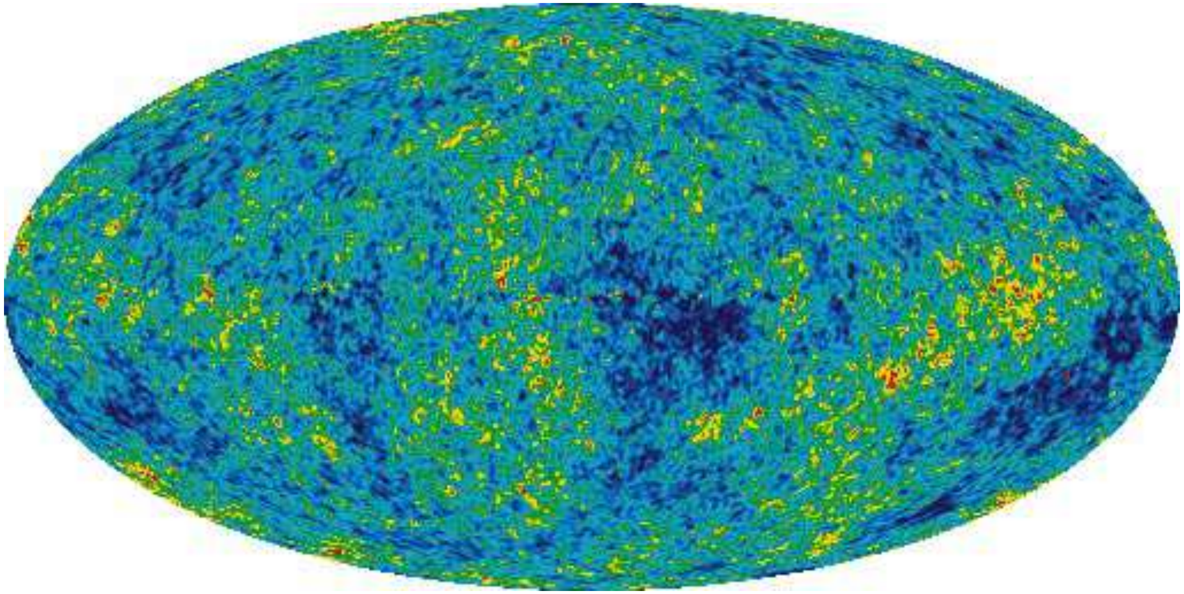

Figure 2: The intrinsic CMB anisotropy.

Best fit to observed CMB anisotropy

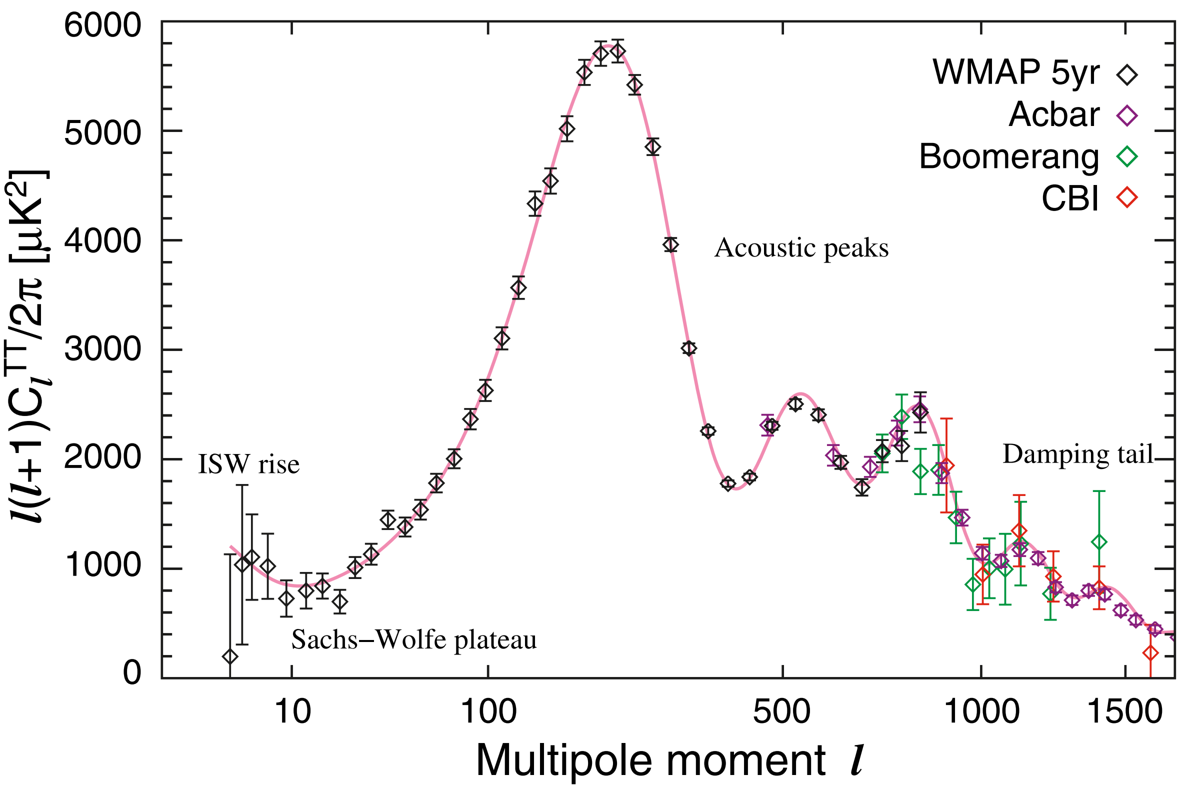

Figure 3: Error bars at $\ell \lesssim 10$ dominated by cosmic variance. PLANCK satellite now flying will give practically zero error bars at $10 \lesssim \ell \lesssim 2000$. Curve is best fit of ΛCDM model to all relevant data. Fit to CMB anisotropy alone gives a similar result.

Peaks of CMB anisotropy

- At $\ell \lesssim 10$, $\delta T/T$ comes mainly from *Sachs-Wolfe effect*. This is the perturbation in the redshift, caused by the inhomogeneity of the matter density along the line of sight.
- Keeping only $\ell \gtrsim 10$, the observed $\delta T(\mathbf{e})/T$ is roughly equal to its value at the epoch of last scattering. Let's see how this gives the peak structure.
- Ignoring the motion of the cosmic fluid, $\delta T/T \simeq \delta_\gamma(\mathbf{x}_{\rm dec}, t_{\rm dec})/4$, where $\mathbf{x}_{\rm dec}$ is position of observed photons at decoupling.
- CMB originates on a sphere with radius close to $r_{\rm obs} = 14,000$ Mpc
- $\ell = kr_{\rm obs}$ where k is typical wavenumber of $\delta_{\gamma\mathbf{k}}$ probed by ℓth multipole
  - (Property of $Y_{\ell m}$, also reason for quantum mechanics $\boldsymbol{\ell} = \mathbf{k}\times\mathbf{r}$)
- Using $\delta_{\gamma\mathbf{k}} \sim -\frac{4}{3}\zeta_{\mathbf{k}} e^{-k^2/k_{\rm D}^2} \cos\left(kx_{\rm s}\right)$ we get

$$
\begin{aligned}
C_\ell &\propto e^{-2(\ell/r_{\rm obs}k_{\rm D})^2}\cos^2(x_{\rm s}\ell/r_{\rm obs}) \simeq e^{-(\ell/1200)^2}\cos^2(\pi\ell/300) \\
r_{\rm obs} &= 14,000\,\text{Mpc}, \qquad x_{\rm s} = 150\,\text{Mpc}, \qquad k_{\rm D} = (8\,\text{Mpc})^{-1}
\end{aligned}
$$

- This would give C_ℓ peaks at $\ell = 0, 300, 600, 900, 1200, 1500$
  - Peak at $\ell = 0$ is actually absent, but other peaks in roughly the right place
- Exact calculation uses Boltzmann code for Thomson scattering, includes neutrinos.

Inflation

What sets the initial condition of the observable universe?

The only good game in town is INFLATION.
Definition: an *early* era with $\ddot{a} > 0$.

Starting with a *Hubble-sized* patch that's *roughly* homogeneous and isotropically expanding, inflation does the following:

(1) Generates an *arbitrarily large* patch that's (practically) *exactly* homogeneous and isotropic at the classical level, with $\Omega = 1$ as required by observation.

(2) The vacuum fluctuations of light scalar fields become classical during inflation, and can generate the observed curvature perturbation $\zeta(\mathbf{x})$.

*

Inflation and the unperturbed universe

FLATNESS PROBLEM: WHY IS PRESENT DENSITY PARAMETER Ω_0 so close to 1?

- As we saw, observation requires $\Omega = 1 \lesssim 10^{-2}$ at the present epoch.
- Also, we saw that $\Omega(t) - 1 \propto 1/(aH)^2$.
- Without inflation, $|\Omega(t) - 1|$ decreases continuosly going back in time.
- At $T \sim$ MeV, $|\Omega(t) - 1| \lesssim 10^{-18}$. Far smaller initial value required if radiation domination persists back to much earlier times.
- Inflation avoids this fine-tuning, if the observable universe starts out well inside the horizon.
  - Indeed, aH then starts out well below its present value and we can have $|\Omega - 1| \sim 1$ initially.

AT CLASSICAL LEVEL, INFLATION CAN REMOVE PERTURBATIONS

- At least within General Relativity, a ‘no-hair’ theorem says that inflation removes all inhomogeneity at the classical level (Wald 1983), within (say) the observable universe.

Light field: classical evolution

Finally, we're going to see how inflation can generate the curvature perturbation $\zeta(\mathbf{x})$.

To do this job, we need *almost exponential inflation* after the observable universe leaves the horizon, ie. $a \propto \exp(Ht)$ with H practically constant. First step is to see how such inflation converts the vacuum fluctuation of any light field ϕ into a classical perturbation $\delta\phi(\mathbf{x})$.

- Light field definition: mass-squared $|m^2| \ll H^2$.
- In the expanding universe it satisfies

$$\ddot{\phi}_{\mathbf{k}}(t) + 3H(t)\dot{\phi}_{\mathbf{k}}(t) + \left[\left(\frac{k}{a(t)}\right)^2 + m^2\right]\phi_{\mathbf{k}}(t) = 0$$

- For now, let's set $m^2 = 0$ and take H exactly constant. Then solutions are

$$\phi_{\mathbf{k}}(t) = \text{ const } e^{\pm ik/a(t)H}\left(\frac{k}{a(t)} \pm iH\right)$$

Quantized zero-mass field

- Promote $\phi_{\mathbf{k}}$ to an operator

$$\begin{aligned}\hat{\phi}_{\mathbf{k}}(t) &= \frac{1}{(2\pi)^3}\left(\phi_k(t)a^\dagger_{\mathbf{k}} + \phi^*_k(t)a_{-\mathbf{k}}\right)\\ [a_{\mathbf{k}}, a_{\mathbf{k}'}] &= (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\end{aligned}$$

- Use Heisenberg picture so $\hat{\phi}_{\mathbf{k}}(t)$ satisfies classical equation.
  - Mode function $\phi_k(t)$ satisfies same equation.
- Choose solution

$$\phi_k = -(2k^3)^{-1/2}e^{ik/aH}\left(\frac{k}{a}+iH\right)k$$

- Well *before* horizon exit (ie. when $aH \ll k$) we have over any interval $\Delta t \ll H$, $\phi_k \propto e^{-i(k/a)t}$.
- Particle interpretation: $a^\dagger_{\mathbf{k}}$ creates particles with momentum $\mathbf{k}/a$.
- No particles exist during inflation, hence initial state is vacuum: $\hat{a}_{\mathbf{k}}|\rangle = 0|\rangle$.
  - Note: state *vector* $|\rangle$ is time-independent.

Quantum to classical transition

- Well *after* horizon exit, $\phi_k(t) = -i(2k^3)^{-1/2} H$.
  - This is purely imaginary giving $\hat{\phi}_{\mathbf{k}}(t) = \frac{1}{(2\pi)^3} \phi_k(t) \left(a_{\mathbf{k}} - a_{-\mathbf{k}} \right)$. Suppose we now measure $\phi_{\mathbf{k}}$, giving an eigenstate: $\hat{\phi}_{\mathbf{k}} |\phi_{\mathbf{k}}\rangle = \phi_{\mathbf{k}} |\phi_{\mathbf{k}}\rangle$. At later times, *same* equation holds. That means, we can regard measured $\phi_{\mathbf{k}}(t)$ as a classical quantity.
- *Before* measurement of $\phi_{\mathbf{k}}$, vacuum expectation value of $\hat{\phi}_{\mathbf{k}} \hat{\phi}^\dagger_{\mathbf{k}}$ is

$$\langle \hat{\phi}_{\mathbf{k}} \hat{\phi}^\dagger_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') |\phi_k|^2 = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') (2\pi^2/k^3)(H/2\pi)^2$$

.

- *After* measurement, spectrum of the classical field defined by

$$\langle \phi_{\mathbf{k}} \phi^*_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') (2\pi^2/k^3) \mathcal{P}_\phi(k)$$

where now $\langle\rangle$ is sum over a cell d^3k.

- Interpretations of $\langle\rangle$ coincide because Fourier components within a cell are uncorrelated.
- Hence $\phi_{\mathbf{k}}$ has *flat* spectrum: $\mathcal{P}_\phi(k) = (k^3/2\pi^2)|\phi_k|^2 = (H/2\pi)^2$.
  - Derived by Bunch and Davies *before* inflation proposed as physical reality.

Spectrum of the curvature perturbation

- Given a field perturbation $\delta\phi(\mathbf{x})$, can generate a curvature perturbation $\zeta(\mathbf{x})$ by several mechanisms.
- Original mechanism assumed the 'slow-roll' inflation model, involving an 'inflaton' field, *and* identified this field with ϕ.
- Later proposals assume that ϕ is *not* the inflaton in a slow-roll model (and don't even assume such a model holds). I'll call such a ϕ the 'curvaton'.
- In any case, absence of causal processes will give a local relation: $\zeta(\mathbf{x}) = f[\phi(\mathbf{x})]$.
- To get a Gaussian ζ we need $\zeta(\mathbf{x}) = A\phi(\mathbf{x})$, giving $\zeta_{\mathbf{k}} = A\phi_{\mathbf{k}}$ and $\mathcal{P}_\zeta(k) = A^2(H/2\pi)^2$, where H is evaluated during inflation and supposed to be time-independent.
- This makes spectral index $n = 1$ in contradiction with observation. But including mass-squared, and slight time-dependence of H, get

$$
\begin{aligned}
n - 1 &= (2m^2/3H^2) + 3\dot{H}/H^2 \qquad \text{inflaton scenario} \\
n - 1 &= (2m^2/3H^2) + \dot{H}/H^2 \qquad \text{curvaton scenario,}
\end{aligned}
$$

with rhs evaluated at horizon exit. Given a suitable model, this can fit observed value $n = 0.96$, and constrains the model.

Possible modifications of ΛCDM model

Small modifications obviously allowed by observation.
Here are some in my order of likelihood, with the magnitude needed for eventually detectability.

- Departure from $\mathcal{P}_\zeta(k) \propto k^{n-1}$, ie. $n(k) - 1 \equiv d\ln\mathcal{P}_\zeta/d\ln k =$ not constant.
  - Running spectral index: $|dn/d\ln k| \gtrsim 10^{-3}$.
- Non-gaussianity of ζ: $|f_{\rm NL}| \gtrsim 1$.
- CDM has significant interaction and/or random motion.
  - Cosmic ray positron excess??
- Cosmic strings: energy per unit length $\gtrsim (10^{12}$ GeV$)^2$.
- Matter isocurvature perturbation: $S_{\rm m} \gtrsim 10^{-2}\delta_{\rm m}$ [$S_{\rm m} \equiv \delta_{\rm m}/3 - \delta_\gamma/4$].
- Tensor perturbation: $r \gtrsim 10^{-3}$ (Planck will detect if $r \sim 0.1$).
- Statistical anisotropy (from vector fields): $\gtrsim 1\%$.
- Neutrino is socurvature perturbation: $S_\nu \gtrsim 10^{-2}$ [$S_\nu \equiv \delta_\nu/4 - \delta_\gamma/4$].
- Time-dependent ρ_Λ (dark energy): $\dot\rho_\Lambda/\rho_\Lambda \sim 10^{-2}$.
- Nonzero spatial curvature: $\Omega_0 \sim 10^{-3}$.
- Statistical inhomogeneity.