Scattering amplitudes via *AdS*/*CFT*

Luis Fernando Alday

IAS

Annual Theory Meeting - Durham - December 2009

arXiv:0705.0303,..., arXiv:0904.0663, L.F.A & J. Maldacena;
arXiv:0911.4708, L.F.A, D. Gaiotto & J. Maldacena

Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable.

- Weak coupling: Perturbative computations are easier than in QCD. In the last years a huge technology was developed.
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.

Aim of this project

Learn about scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills by means of the AdS/CFT correspondence.

Gauge theory amplitudes (Bern, Dixon and Smirnov, also Anastasiou, Kosower)

- Focus in gluon scattering amplitudes of $\mathcal{N}=4$ SYM, with $SU(N)$ gauge group with N large, in the color decomposed form

$$A_n^{L,Full} \sim \sum_\rho Tr(T^{a_{\rho(1)}} ... T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), ..., \rho(2))$$

- Leading N color ordered $n-$points amplitude at L loops: $A_n^{(L)}$
- The amplitudes are IR divergent.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon) = 1/\epsilon^{2L} + ...$
- Focus on MHV amplitudes and scale out the tree amplitude

$$M_n^{(L)}(\epsilon) = \frac{A_n^{(L)}(\epsilon)}{A_n^{(0)}} \quad \rightarrow \quad \mathcal{M}_n = \sum_L \lambda^L M_n^{(L)}$$

Based on explicit perturbative computations:

BDS proposal for all loops MHV amplitudes

$$\log \mathcal{M}_n = \sum_{i=1}^{n} \left(-\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) - \frac{1}{\epsilon} g^{(-1)} \left(\frac{\lambda \mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) \right) + f(\lambda) Fin_n^{(1)}(k)$$

- $f(\lambda), \ g(\lambda) \rightarrow$ cusp/collinear anomalous dimension.
- Fine for $n = 4, 5$, not fine for $n > 5$.

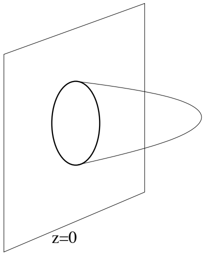
AdS/CFT duality (Maldacena)

Four dimensional maximally SUSY Yang-Mills	$\Leftrightarrow$	Type IIB string theory on $AdS_5 \times S^5$.

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'} \qquad \frac{1}{N} \approx g_s$$

- The AdS/CFT duality allows to compute quantities of $\mathcal{N}=4$ SYM at strong coupling by doing geometrical computations on AdS.

- Remember a similar problem: Expectation value of Wilson loops at strong coupling (Maldacena, Rey)

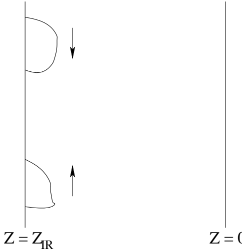


- $ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$
- We need to consider the minimal area ending (at $z = 0$) on the Wilson loop.

$$\langle W \rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- Scattering amplitudes can be computed at strong coupling by considering strings on AdS_5 (L.F.A., Maldacena)
- As in the gauge theory, we need to introduce a regulator.

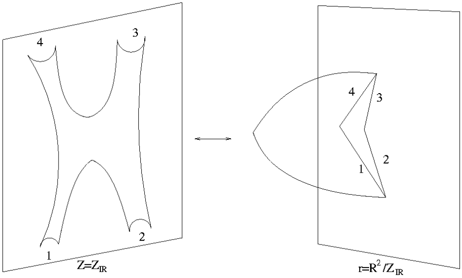
$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$



- Place a D-brane at $z = z_{IR} \gg R$.
- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings (representing the gluons)
- Need to find the world-sheet representing this process...

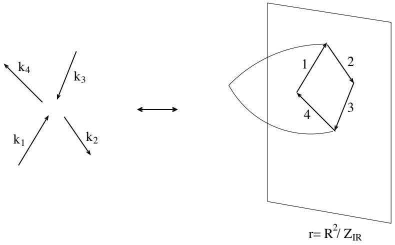
- The world-sheet is easier to find if we go to a dual space: $AdS \to \tilde{AdS}$ (four $T-$dualities plus $z \to r = R^2/z$).

$$ds^2 = \frac{dx_{3+1}^2 + dz^2}{z^2} \to d\tilde{s}^2 = \frac{dy_{3+1}^2 + dr^2}{r^2}$$



- The problem reduces to a minimal area problem!

What is now the boundary of our world-sheet?



- For each particle with momentum k^μ draw a segment $\Delta y^\mu = 2\pi k^\mu$
- Concatenate the segments according to the particular color ordering.
- Polygon of light-like edges.
- Look for the minimal surface ending in such polygon.

- As we have introduced the regulator, the minimal surface ends at $r = R^2/z_{IR} > 0$.
- As $z_{IR} \to \infty$ the boundary of the world-sheet moves to $r = 0$.
- Vev of a Wilson-Loop given by a sequence of light-like segments!

Prescription

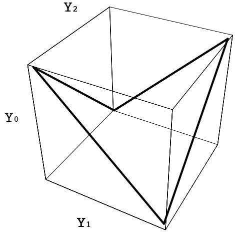
$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
- $A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

Four point amplitude at strong coupling

Consider $k_1 + k_3 \rightarrow k_2 + k_4$

- The simplest case $s = t$.



Need to find the minimal surface ending on such sequence of light-like segments

$$r(y_1, y_2) = \sqrt{(1-y_1^2)(1-y_2^2)}$$
$$y_0 = y_1 y_2$$

In embedding coordinates $(-Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_4^2 = -1)$

$$Y_0 Y_{-1} = Y_1 Y_2, \quad Y_3 = Y_4 = 0$$

- "Dual" $SO(2,4)$ isometries $\rightarrow$ most general solution ($s \neq t$)

Let's compute the area...

- In order for the area to converge we need to introduce a regulator.
- Supergravity version of dimensional regularization: consider the near horizon limit of a $D(3-2\epsilon)-$brane!

Regularized supergravity background

$$ds^2 = \sqrt{\lambda_D c_D}\left(\frac{dy_D^2 + dr^2}{r^{2+\epsilon}}\right) \rightarrow S_\epsilon = \frac{\sqrt{\lambda_D c_D}}{2\pi}\int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- The regularized area can be computed and it agrees precisely with the BDS ansatz!

What about other cases with $n > 4$?

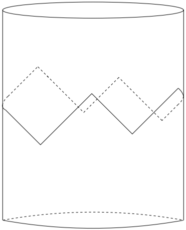
- Dual $SO(2,4)$ symmetry constraints the form of the answer (Drummond et. al.)
- for all n $SO(2,4) \rightarrow A_{strong} = A_{BDS} + R(\frac{x_{ij}x_{kl}}{x_{ik}x_{jl}})$

We can construct such cross-ratios for $n \geq 6$ so for this case the answer will (in principle) differ from BDS.

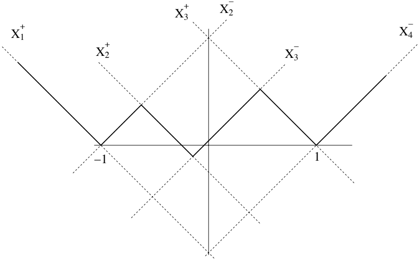
How do we compute the area of minimal surfaces for $n \geq 6$?

- Reduced/baby model: Strings on AdS_3.
- Full problem: Strings on AdS_5

Strings on AdS_3: The external states live in $2D$, *e.g.* the cylinder.



- Consider a zig-zagged Wilson loop of $2n$ sides
- Parametrized by n X_i^+ coordinates and n X_i^- coordinates.
- We can build $2n-6$ invariant cross ratios.



- Consider classical strings on AdS_3.

Strings on AdS_3

Strings on AdS_3 : $\vec{Y}.\vec{Y} = -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$

$$Eoms : \partial\bar{\partial}\vec{Y} - (\partial\vec{Y}.\bar{\partial}\vec{Y})\vec{Y} = 0, \qquad Virasoro : \partial\vec{Y}.\partial\vec{Y} = \bar{\partial}\vec{Y}.\bar{\partial}\vec{Y} = 0$$

Pohlmeyer kind of reduction $\longrightarrow$ generalized Sinh-Gordon

$$\alpha(z,\bar{z}) = \log(\partial\vec{Y}.\bar{\partial}\vec{Y}), \qquad p^2 = \partial^2\vec{Y}.\partial^2\vec{Y}$$

$$\downarrow$$

$$p = p(z), \qquad \partial\bar{\partial}\alpha - e^{2\alpha} + |p(z)|^2 e^{-2\alpha} = 0$$

- $\alpha(z,\bar{z})$ and $p(z)$ invariant under conformal transformations.
- Area of the world sheet: $\mathcal{A} = \int e^{2\alpha} d^2z$

Generalized Sinh-Gordon $\rightarrow$ Strings on AdS_3?

- From α, p construct flat connections $B_{L,R}$ and solve two linear auxiliary problems.

$$(\partial + B^L)\psi^L_a = 0 \qquad B^L_z = \begin{pmatrix} \partial\alpha & e^{\alpha} \\ e^{-\alpha}p(z) & -\partial\alpha \end{pmatrix}$$
$$(\partial + B^R)\psi^R_{\dot{a}} = 0$$

Space-time coordinates

$$Y_{a,\dot{a}} = \begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix} = \psi^L_a M \psi^R_{\dot{a}}$$

One can check that Y constructed that way has all the correct properties.

Relation to Hitchin equations

Consider self-dual YM in 4d reduced to 2d

- $A_{1,2} \rightarrow A_{1,2}$: 2d gauge field, $A_{3,4} \rightarrow \Phi, \Phi^*$: Higgs field.

Hitchin equations

$$F^{(4)} = *F^{(4)} \qquad \rightarrow \qquad \begin{aligned} D_{\bar{z}}\Phi = D_z\Phi^* &= 0 \\ F_{z\bar{z}} + [\Phi, \Phi^*] &= 0 \end{aligned}$$

- We can decompose $B = A + \Phi$.
- $dB + B \wedge B = 0$ implies the Hitchin equations.
- We have a particular solution of the $SU(2)$ Hitchin system.
- Nice relation: $\mathcal{A} = \int Tr\Phi\Phi^*$.

- Classical solutions on $AdS_3 \quad \rightarrow \quad p(z), \alpha(z,\bar{z})$

$$dw = \sqrt{p(z)}dz, \quad \hat{\alpha} = \alpha - \frac{1}{4}\log p\bar{p} \rightarrow \partial_w \bar{\partial}_{\bar{w}} \hat{\alpha} = sinh 2\hat{\alpha}$$

- We need to get some intuition for solutions corresponding to scattering amplitudes...

$n = 2$ "square" solution: $p(z) = 1, \quad \hat{\alpha} = 0$

- For the solutions relevant to scattering amplitudes we require $p(z)$ to be a polynomial and $\hat{\alpha}$ to decay at infinity.

Consider a generic polynomial of degree $n-2$

$$p(z) = z^{n-2} + c_{n-4} z^{n-4} + \ldots + c_1 z + c_0$$

- We have used translations and re-scalings in order to fix the first two coefficients to one and zero.
- For a polynomial of degree $n-2$ we are left with $2n-6$ (real) variables.
- This is exactly the number of invariant cross-ratios in two dimensions for the scattering of $2n$ gluons!

$$\text{Null Wilsons loops of } 2n \text{ sides} \Leftrightarrow p^{(n-2)}(z) \text{ and } \hat{\alpha}(z,\bar{z}) \rightarrow 0$$

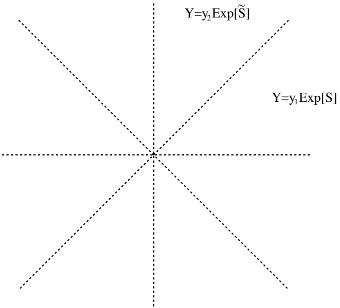
- Degree of the polynomial $\rightarrow$ number of cusps.
- Coefficients of the polynomial $\rightarrow$ shape of the polygon.

- Simplest case: $p(z) = z^{n-2}$
- At infinity the connection becomes very simple (since $\hat{\alpha} \to 0$) and we can solve the inverse problem:

$$\psi^L \approx a_s \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} + b_s \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix}, \quad \psi^R \approx c_s \begin{pmatrix} e^{i(w-\bar{w})} \\ 0 \end{pmatrix} + d_s \begin{pmatrix} 0 \\ e^{-i(w-\bar{w})} \end{pmatrix}$$
$$\Downarrow$$
$$Y = M_s^{++} e^{w+\bar{w}+i(w-\bar{w})} + M_s^{+-} e^{w+\bar{w}-i(w-\bar{w})} + \ldots$$

- As $w = z^{n/2}$, the $z-$plane is naturally divided into n equal angular sectors, called Stokes sectors.
- In each sector only one of the terms in Y will dominate and the space time position (at the boundary, since Y is very large) will be fixed .

z–plane



- Each sector corresponds to a cusp.
- For $p(z) = z^{n-2}$ we obtain a regular polygon of $2n$ sides with Z_{2n}-symmetry.
- The generic situation will be very similar, but the polygon will not be regular.

- Question: For regular polygons, how do we compute the area?

$$A = \int e^{2\hat{\alpha}} d^2w = \int (e^{2\hat{\alpha}} - 1) d^2w + \int 1 d^2w = A_{sinh} + A_{div}$$

- A_{div} gives simply the expected divergent piece.
- A_{sinh} is finite, we don't need to introduce any regulator.

$p = z^{n-2} \rightarrow \hat{\alpha} = \hat{\alpha}(\rho)$: Sinh-Gordon $\rightarrow$ Painleve III

$$\hat{\alpha}''(\rho) + \frac{\hat{\alpha}'(\rho)}{\rho} = \frac{1}{2}\sinh(2\hat{\alpha}(\rho))$$

- Solved in terms of Painleve transcendentals!

$$A_{sinh} = \frac{\pi}{4n}(3n^2 - 8n + 4)$$

Strings on AdS_5:

- We have still a holomorphic quantity $P(z) = \partial^2 \vec{Y}.\partial^2 \vec{Y}$, but not the square of a polynomial anymore!
- Two more physical fields: $\alpha(z,\bar{z}) = \log(\partial \vec{Y}.\bar{\partial} \vec{Y})$ but also $\beta(z,\bar{z})$ and $\gamma(z,\bar{z})$.

How does the counting of cross-ratios work?

- For N gluons: $P(z) = z^{N-4} + c_{N-6} z^{N-6} ... \rightarrow 2N - 10$ real coefficients.
- For AdS_4 we have exactly $2N - 10$ cross ratios, so this is the whole picture (α and β are unique once you have fixed $P(z)$)
- For AdS_5 there are $N - 5$ extra degrees of freedom coming from γ (which are hard to see), giving the expected $3N - 15$ cross-ratios in 4d scattering!

What about the Hitchin equations?

- Not anymore Left+Right factorization but we still have a Hitchin system!
- We obtain a particular case of $SU(4)$ Hitchin system.

General prescription for polynomial $p(z)$ in AdS_3 or $P(z)$ on AdS_5

- Compute the space-time cross-ratios in terms of the coefficients of $P(z)$.
- Compute the area in terms of the coefficients of $P(z)$.
- Write the area in terms of the space-time cross-ratios.

First non trivial cases:

- On AdS_3: $p(z) = z^2 - m$, the "octagon".
- On AdS_5: $P(z) = z^2 - U$, the "hexagon".

but...

- We don't know explicitly the solution for α...
- We cannot perform the inverse map...
- The $w-$plane is complicated...

How do we proceed?

Ask someone else!

- Exactly the same $SU(2)$ Hitchin systems in a completely different context! (Gaiotto, Moore & Neitzke)

Idea: Use integrability to promote the Hitchin system to a family of flat connections (introduce a spectral parameter)

$$B_z^{(\zeta)} = A_z + \frac{\Phi_z}{\zeta}, \quad B_{\bar{z}}^{(\zeta)} = A_{\bar{z}} + \zeta \Phi_{\bar{z}}$$

Why is this any useful?

- Consider the deformed auxiliary linear problems leading to $Y[\zeta]$ (such that $Y[1]$ is the physical solution).
- Conside the cross-ratios as a function of ζ (such that at $\zeta = 1$ we obtain the physical cross-ratios).
- For $\zeta \to 0$ or $\zeta \to \infty$ the connections simplify drastically and we can solve such inverse problems!
- Actually, we observe also Stokes sectors and discontinuities in the ζ plane.
- On the other hand, we expect the cross-ratios as a function of ζ to be analytic away from $\zeta = 0, \infty$.

- We need to find analytic functions with specific jumps when we go from one sector to another.
- This defines a Riemann-Hilbert problem which can be rewritten as an integral equation for the cross-ratios! (as a function of ζ)

For the case of the Hexagon $P(z) = z^2 - U^{3/4}$

$$\begin{aligned}
\epsilon(\theta) = \quad & 2|U|\cosh\theta + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\log(1+e^{-\tilde{\epsilon}}) + \\
& + \frac{1}{2\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\log(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu}) \\
\tilde{\epsilon}(\theta) = \quad & 2\sqrt{2}|U|\cosh\theta + \frac{1}{\pi}\int d\theta' \frac{1}{\cosh(\theta-\theta')}\log(1+e^{-\tilde{\epsilon}}) + \\
& + \frac{\sqrt{2}}{\pi}\int d\theta' \frac{\cosh(\theta-\theta')}{\cosh 2(\theta-\theta')}\log(1+\mu e^{-\epsilon})(1+\frac{e^{-\epsilon}}{\mu})
\end{aligned}$$

- Exactly the form of TBA equations!
- What is the regularized area?

$$A_{reg} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta 2|U|\cosh\theta \log\left(1+e^{-\epsilon}\mu\right)(1+\frac{e^{-\epsilon}}{\mu}) + \\ +\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta 2\sqrt{2}|U|\cosh\theta\log\left(1+e^{-\tilde{\epsilon}}\right)$$

- Exactly the free energy of the TBA system!

Some exact results...

- Large temperature/Conformal limit of the TBA equations
  $U = 0 \rightarrow u_1 = u_2 = u_3$

Hexagonal Wilson loop in AdS_5 in $U \rightarrow 0$ limit

$$R(u,u,u) = \frac{\phi^2}{3\pi} + \frac{3}{8}(\log^2 u + 2Li_2(1-u)), \quad u = \frac{1}{4\cos^2(\phi/3)}$$

Some more exact results...

Eight sided Wilson loop in AdS_3 (the first non trivial)

$$A = \frac{1}{2}\int dt \frac{\bar{m}e^t - me^{-t}}{\tanh 2t} \log\left(1 + e^{-\pi(\bar{m}e^t + me^{-t})}\right)$$

What have we done and what needs to be done

- We have given a further step towards the computation of classical solutions relevant to scattering amplitudes at strong coupling.
- Integrability is the key ingredient of the computation

For the future...

- Could we compute these amplitudes at all values of the coupling?!
- What about other kind of solutions? *e.g.* correlations functions?
- Include fermions and understand non MHV amplitudes?