Periods, Feynman integrals and Number Theory

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Durham, 18 December 2009

Overview:

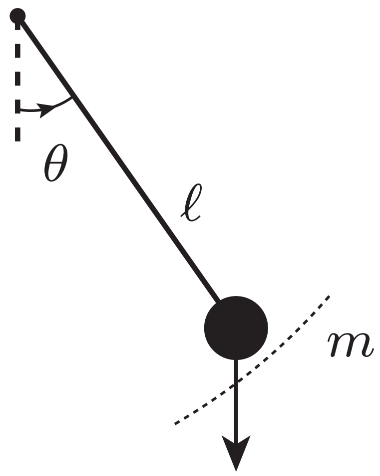
1. Periods, MZVs and motives

2. Parametric Feynman integrals

3. Higher-loop calculations

4. Outlook

Consider a pendulum, of length ℓ.



Its equation of motion is $\ddot{\theta} + \frac{G}{\ell}\sin\theta = 0$.

By a simple substitution, its *period* is given by an elliptic integral of the first kind:

$$T = 4\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\rho^2x^2)}}$$

for some constant $0 < \rho < 1$ which depends on the initial conditions. Rewrite this as:

$$\int_0^\infty \int_0^1 \frac{8\,dxdy}{y^2 + (1-x^2)(1-\rho^2x^2)}$$

so that the denominator is now a polynomial.

Periods

A *period* is defined to be the (absolutely convergent) integral of a rational differential form over a domain given by polynomial inequalities:

$$\int_\Delta \frac{P(x_1,\ldots,x_n)}{Q(x_1,\ldots,x_n)}\, dx_1 \ldots dx_n$$

where Δ is defined by $\{f_i(x_1,\ldots,x_n) \geq 0\}$, and f_i, P, Q are polynomials with rational coefficients. What are the irreducible building blocks of such period integrals?

I will consider two families of periods:

- Massless, single-scale Feynman integrals in ϕ^4 theory.

- Multiple Zeta Values, one of the simplest possible families of periods.

I will try to explain why the two families are the same up to small loop orders.

Multiple Zeta Values

Let $n_1, \ldots, n_r \in \mathbb{N}$, and suppose that $n_r \geq 2$. The multiple zeta value (MZV) $\zeta(n_1, \ldots, n_r)$ is defined by the convergent nested sum:

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \ .$$

Its *weight* is the quantity $w = n_1 + \ldots + n_r$.

To see that MZVs are periods, they can be written as iterated integrals:

$$\zeta(n_1, \ldots, n_r) = \int_{0 \leq t_1 \leq \ldots \leq t_w \leq 1} \frac{dt_1}{\varepsilon_1 - t_1} \ldots \frac{dt_w}{\varepsilon_w - t_w}$$

where w is the weight, and

$$(\varepsilon_1, \ldots, \varepsilon_w) = (1, \underbrace{0, \ldots, 0}_{n_1 - 1}, \ldots, 1, \underbrace{0, \ldots, 0}_{n_r - 1}) \ .$$

MZVs satisfy very many algebraic identities.

- Quasi-shuffle or stuffle relations. Example:

$$\zeta(m)\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n)$$

$$\Big(\sum_{k\geq 1}\frac{1}{k^m}\Big)\Big(\sum_{l\geq 1}\frac{1}{l^n}\Big) = \Big(\sum_{k<\ell} + \sum_{\ell<k} + \sum_{k=\ell}\Big)\frac{1}{k^m l^n}$$

- Shuffle relations, valid for very general iterated integrals. Decompose the product of two simplices as a sum of smaller simplices. This yields formulae such as:

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + 4\zeta(1,3)$$

- Regularization-type identities, e.g., Euler's formula: $\zeta(1,2) = \zeta(3)$.

And many more (e.g., Drinfeld's pentagon equation, *etc*). The product of two MZVs can always be written, in several ways, as a linear combination of multiple zetas.

Weight	1	2	3	4	5
		$\zeta(2)$	$\zeta(3)$	$\zeta(2)^2$	$\zeta(5)$ $\zeta(2)\zeta(3)$
dim	0	1	1	1	2

Weight	6	7	8
	$\zeta(2)^3$ $\zeta(3)^2$	$\zeta(7)$ $\zeta(2)\zeta(5)$ $\zeta(2)^2\zeta(3)$	$\zeta(2)^4$ $\zeta(3)\zeta(5)$ $\zeta(2)\zeta(3)^2$ $\zeta(3,5)$
dim	2	3	4

Let d_k be the dimension in weight k. Zagier conjectured that

$$d_k = d_{k-2} + d_{k-3} \ .$$

But it is not known if

$$\frac{\zeta(5)}{\zeta(2)\zeta(3)} \notin \mathbf{Q}$$

There is a more precise conjecture for the dimension in each weight and depth due to Broadhurst and Kreimer.

Transcendence Results

Euler proved that $\zeta(2) = \frac{\pi^2}{6}$ and more generally that $\zeta(2n)$ is a rational multiple of π^{2n}.

The numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are conjectured to be algebraically independent over $\mathbb{Q}$. In particular, they should be transcendental.

Theorem 1. *(Lindemann) π is transcendental.*

Theorem 2. *(Apéry 1978) $\zeta(3)$ is irrational.*

Theorem 3. *(Rivoal 2000) Infinitely many odd zetas $\zeta(2n+1)$ are irrational.*

It is still not known whether $\zeta(5)$ is irrational, but it is known that one of the four numbers $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ must be irrational. (Zudilin, Rivoal,...)

Lie algebra interpretation

Let $\mathcal{L} = \mathbb{Q}[e_3, e_5, \ldots,]$ denote the free Lie algebra generated by one element in every odd degree. Add a generator e_2 which commutes with all the others to get:

$$\mathcal{F} = \mathbb{Q}[e_2] \oplus \mathcal{L} \ .$$

The underlying graded vector space is generated by, in increasing weight:

$$e_2, e_3, e_5, e_7, [e_3, e_5], e_9, [e_3, e_7], \ldots$$

Let $\mathcal{UF}$ be its universal enveloping algebra.

Conjecture 4. The algebra of multiple zetas is isomorphic to the dual of $\mathcal{UF}$.

In particular, there should be no algebraic relations between MZVs of different weights.

The dimensions in graded weight k of $\mathcal{UF}$ are given by the same d_k, where $d_k = d_{k-2} + d_{k-3}$.

This conjecture comes from the theory of mixed Tate motives over $\mathbb{Z}$, which form a category isomorphic to the category of representations of the previous Lie algebra.

Theorem 5. *(Terasoma, Goncharov, Deligne-Goncharov) The dimension of the space of MZVs in weight k is bounded above by the numbers d_k satisfying Zagier's recurrence.*

It is not known how to reduce any given MZV to some suitable basis of MZVs.

It is not known if the double shuffle (or any other given set of) relations suffice.

But, this has been verified numerically to high weights by Zagier,Broadhurst, . . . , Minh-Petitot, . . . , Broadhurst-Blümlein-Vermaseren.

The motivic theory suggests that there should exist a Galois theory of periods, in the same way that there is a Galois theory of algebraic numbers. There is a large pro-algebraic group which should act on the set of all periods.

In particular, we should get a conjectural *co-product* structure on MZVs. Examples:

$$\Delta\zeta(n) = 1 \otimes \zeta(n) + \zeta(n) \otimes 1$$

$$\Delta\zeta(3,5) = 1\otimes\zeta(3,5) - 5\,\zeta(3)\otimes\zeta(5) + \zeta(3,5)\otimes 1$$

This should have relevance to QFT calculations: Feynman amplitudes should be linear combinations of MVZs which are 'simple' with respect to Δ.

II. Reminder on parametric integrals

We can always reduce Feynman integrals to parametric form using the Schwinger trick.

Let G be a graph with h loops. Let α_e be the Schwinger parameter of each edge e of G. The *graph polynomial*, or first Symanzik polynomial, of G is

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin E_T} \alpha_e$$

where the sum is over all spanning trees T of G. The second Symanzik polynomial of G is

$$\Phi_G = \sum_S \prod_{e \notin S} \alpha_e (q^S)^2$$

where the sum is over cut spanning trees S, and q^S is the moment flow through the cut.

Up to (omitted) Γ-factors, the general shape of parametric Feynman integrals is:

$$I = \int_{[0,\infty]^{E_G}} \frac{\prod_e \alpha_e^{a_e - 1} \Psi_G^{a-(h+1)d/2}}{(\Psi_G \sum_e m_e^2 \alpha_e - \Phi_G)^{a-hd/2}} \delta(\sum_e \alpha_e - 1)$$

where each propagator is raised to the power a_e and $a = \sum_e a_e$.

We only consider the massless case of graphs with trivial momentum dependence (a multiple of a power of q^2). To ensure convergence, we say that G is *primitively divergent* if

- $h_G = 2|E_G|$
- $h_\gamma < 2|E_\gamma|$ for all strict subgraphs $\gamma \subset G$.

In this case the residue can be simply written

$$I_G = \int \frac{\prod_e d\alpha_e}{\Psi_G^2} \delta\left(\sum \alpha_e - 1\right)$$

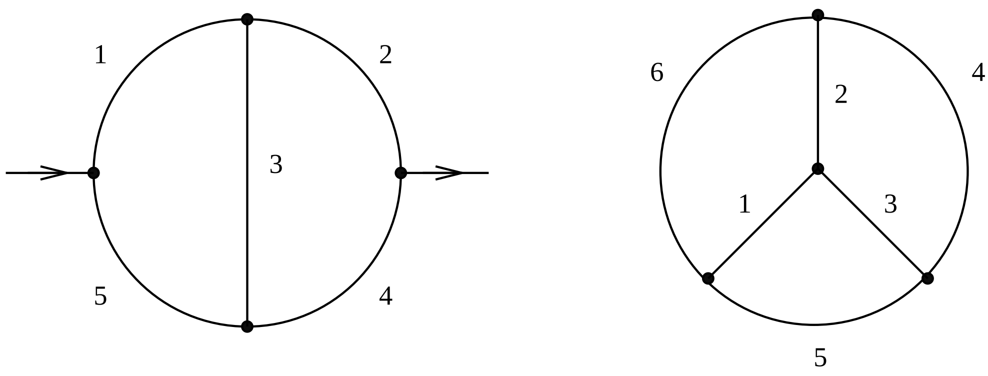
which converges absolutely. By a change of variables, this can be written

$$I_G = \int_{[0,\infty]^{N-1}} \frac{d\alpha_1 \dots d\alpha_{N-1}}{\Psi_G^2\Big|_{\alpha_N=1}}$$

for some choice of edge N.

III. Higher loop calculations

Consider massless single-scale integrals in ϕ^4 theory. For example, the master 2-loop diagram on the left, with external momentum q.



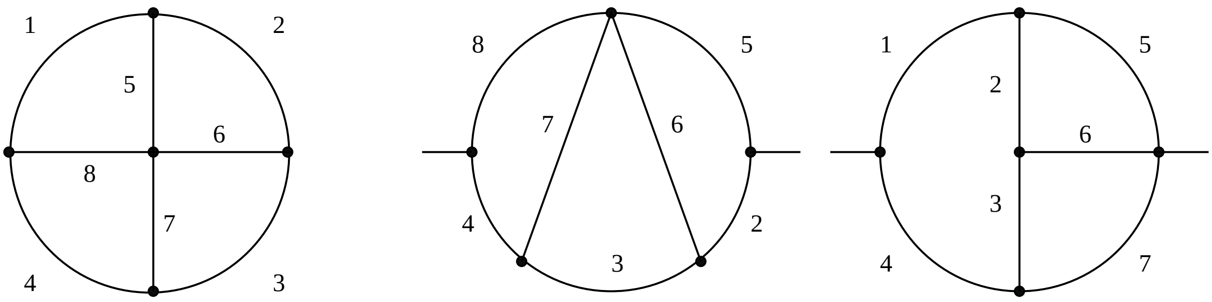
The dressed Feynman integral is:

$$\iint \frac{d^D k_1 d^D k_2}{k_1^{2a_1} k_2^{2a_2} (k_1 - k_2)^{2a_3} (k_2 - q)^{2a_4} (k_1 - q)^{2a_5}} \ ,$$

where $a_i = 1 + n_i\,\varepsilon$, where n_i are positive integers, and $D = 4 - 2\varepsilon$.

By a well-known trick, it suffices to compute the *momentumless* Feynman integral of the graph on the right obtained by closing up the external edges.

Likewise, the wheel with 4 spokes is the unique primitive-divergent graph at 4 loops.

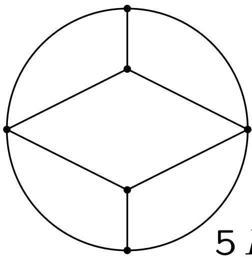


It computes the master integrals for the two 3-loop topologies on the right.

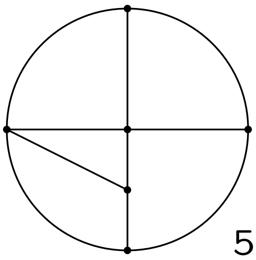
Theorem 6. *The coefficients in the Taylor expansion with respect to* ε *are multiple zetas, for all graphs obtained by breaking open a planar primitive-divergent graph with* ≤ 6 *loops, with any dressing on its edges, in 4 dimensions.*

This theorem also holds for some infinite families of graphs obtained by splitting triangles. The 3-loop case (previous slide) was proved by Bierenbaum and Weinzierl, using Mellin-Barnes methods.

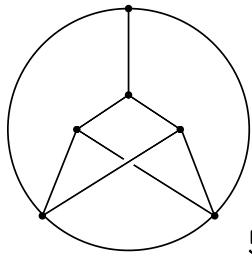
There are 3 primitive divergent at 5 loops:



$5\,R$

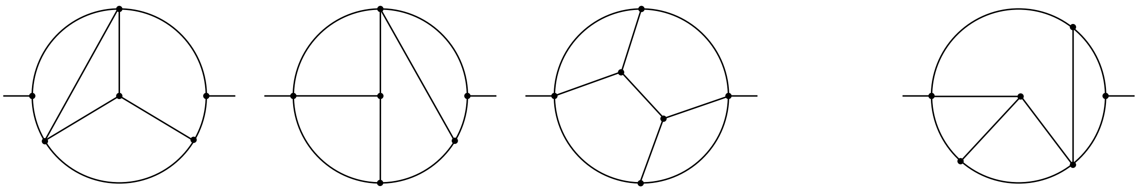


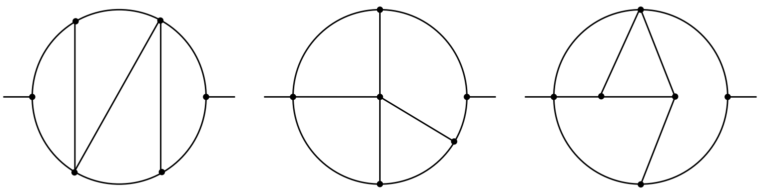
$5\,P$



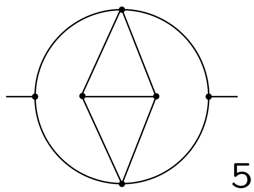
$5\,NP$

They break apart to give the following 4-loop topologies:

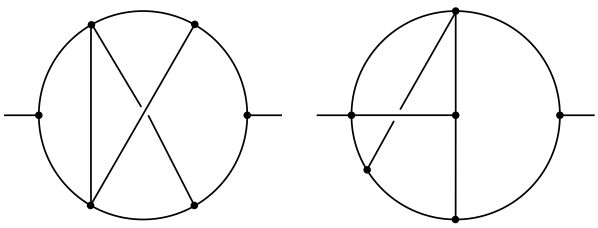


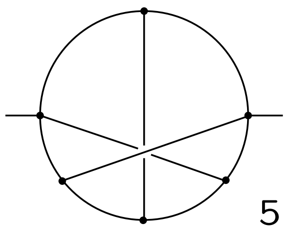


$5\,P$



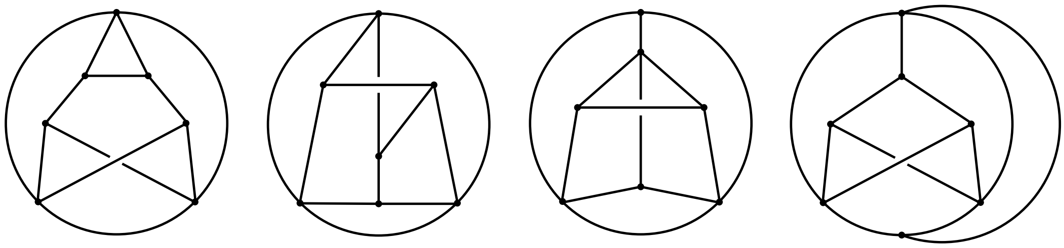
$5\,R$





$5\,NP$

For the non-planar topology at 5 loops, and the following non-planar graphs at 6 loops:



we have the following result:

Theorem 7. *The previous theorem holds, where you replace multiple zeta values with multiple polylogarithms evaluated at 6th roots of unity.*

The method of proof is by integrating directly in parametric space.

It gives an algorithm for the symbolic computation of the Taylor coefficients, and probably works for all graphs up to 7 loops.

The Massless higher loop 2 point function, Communications in Math. Physics 2009

Methods

We illustrate the idea behind the method by showing how to compute the residue:

$$I_G = \int_{[0,\infty]^{N-1}} \frac{d\alpha_1 \dots d\alpha_{N-1}}{\Psi_G^2\Big|_{\alpha_N=1}}$$

In the general, dressed, case, the numerator will be a polynomial in $\alpha_i, \log \alpha_i$ and $\log \Psi_G$. This won't affect the method significantly.

Let $\mathcal{E}_G$ be the reduced incidence matrix of the graph G. Its entries are $0, 1, -1$ and depends on various choices (orientation,...). Let

$$M_G = \left(\begin{array}{ccc|c} \alpha_1 & & & \\ & \ddots & & \mathcal{E}_G \\ & & \alpha_{e_G} & \\ \hline & -{}^T\mathcal{E}_G & & 0 \end{array}\right)$$

It follows from the Matrix-Tree theorem that the graph polynomial $\Psi_G = \det M_G$.

Dodgson polynomials

We need to generalise: define, for any subsets of edges I, J, K of G such that $|I| = |J|$,

$$\Psi_{G,K}^{I,J} = \det M_G(I,J)\Big|_{\alpha_k=0, k\in K}$$

were $M_G(I,J)$ denotes the matrix M_G with rows I and columns J removed. We call $\Psi_{G,K}^{I,J}$ the *Dodgson polynomials* of G.

The key to computing the Feynman integrals is to exploit the many identities between the polynomials $\Psi_{G,K}^{I,J}$. We have:

- The *contraction-deletion formula*:

  $$\Psi_{G,K}^{I,J} = \Psi_{G,K}^{Ie,Je}\alpha_e + \Psi_{G,Ke}^{I,J}$$

  We have $\Psi_{G,K}^{Ie,Je} = \Psi_{G\backslash e,K}^{I,J}$ (deletion of e), and $\Psi_{G,Ke}^{I,J} = \Psi_{G/e,K}^{I,J}$ (contraction of e).

- General *determinantal identities* such as:

$$\Psi^{I,J}_{G,Kabx}\Psi^{Iax,Jbx}_{G,K}-\Psi^{Ix,Jx}_{G,Kab}\Psi^{Ia,Jb}_{G,Kx}=\Psi^{Ia,Jx}_{G,Kb}\Psi^{Ix,Jb}_{G,Ka}$$

  or Plücker-type identities such as:

$$\Psi^{ij,kl}_{G,K}-\Psi^{ik,jl}_{G,K}+\Psi^{il,jk}_{G,K}=0$$

- *Graph-specific identities*. If, for example, K contains a loop, then $\Psi^{I,J}_{G,K}=0$, and many more complicated examples.

Compute the Feynman integral in parametric form by integrating out one variable at a time.

By the contraction-deletion formula, we can write $\Psi=\Psi^{1,1}\alpha_1+\Psi_1$. Therefore

$$I_G=\int_0^\infty \frac{d\alpha_1\ldots d\alpha_{N-1}}{\Psi^2}$$

can be written

$$\int_0^\infty \frac{d\alpha_1\ldots d\alpha_{N-1}}{(\Psi^{1,1}\alpha_1+\Psi_1)^2}=\int_0^\infty \frac{d\alpha_2\ldots d\alpha_{N-1}}{\Psi^{1,1}\Psi_1}$$

By contraction-deletion, the polynomials $\Psi^{1,1}$ and Ψ_1 are linear in the next variable, α_2:

$$\begin{aligned}\Psi^{1,1} &= \Psi^{12,12}\alpha_2 + \Psi_2^{1,1},\\ \Psi_1 &= \Psi_1^{2,2}\alpha_2 + \Psi_{12}\end{aligned}$$

We can write the previous integral $\int \frac{1}{\Psi^{1,1}\Psi_1}$as

$$\int_0^\infty \frac{d\alpha_2 \ldots d\alpha_{N-1}}{(\Psi^{12,12}\alpha_2 + \Psi_2^{1,1})(\Psi_1^{2,2}\alpha_2 + \Psi_{12})}$$

Decompose into partial fractions and integrate out α_2. This leaves an integrand of the form

$$\frac{\log \Psi_2^{1,1} + \log \Psi_1^{2,2} - \log \Psi^{12,12} - \log \Psi_{12}}{\Psi_2^{1,1}\Psi_1^{2,2} - \Psi^{12,12}\Psi_{12}}$$

At this point, we should be stuck since the denominator is quadratic in every variable. Miraculously, there is an identity due to Dodgson:

$$\Psi_2^{1,1}\Psi_1^{2,2} - \Psi^{12,12}\Psi_{12} = (\Psi^{1,2})^2$$

So after two integrations we have $\int \frac{d\alpha_1 d\alpha_2}{\Psi^2}$

$$= \frac{\log \Psi_2^{1,1} + \log \Psi_1^{2,2} - \log \Psi^{12,12} - \log \Psi_{12}}{(\Psi^{1,2})^2}$$

We can then write $\Psi^{1,2} = \Psi^{13,23}\alpha_3 + \Psi_3^{1,2}$ and keep integrating out variables...

As long as we can find a Schwinger coordinate α_i in which all the terms in the integrand are *linear*, then we can always perform the next integration. This requires choosing a good order on the edges of G.

In this case, the integral is expressible as multiple polylogarithms:

$$\mathrm{Li}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = \sum_{1 \le k_1 < \ldots < k_r} \frac{x_1^{k_1} \ldots x_r^{k_r}}{k_1^{n_1} \ldots k_r^{n_r}}$$

typically evaluated at arguments $\Psi_{G,K}^{I,J}$. When this process terminates, the Feynman integral is expressed as values of multiple polylogarithms evaluated at 1 (or roots of unity).

G is *linearly reducible* if this integration process terminates, i.e., we can find a variable with respect to which all the arguments are linear.

Most of the terms which occur are of the form $\Psi^{I,J}_{G,K}$ which are linear in every variable. However, this is not always the case. The first obstruction which can occur is the *five invariant*, defined for any five edges i, j, k, l, m in G:

$$^5\Psi(i,j,k,l,m) = \pm \det \begin{pmatrix} \Psi^{ij,kl}_m & \Psi^{ijm,klm} \\ \Psi^{ik,jl}_m & \Psi^{ikm,jlm} \end{pmatrix}$$

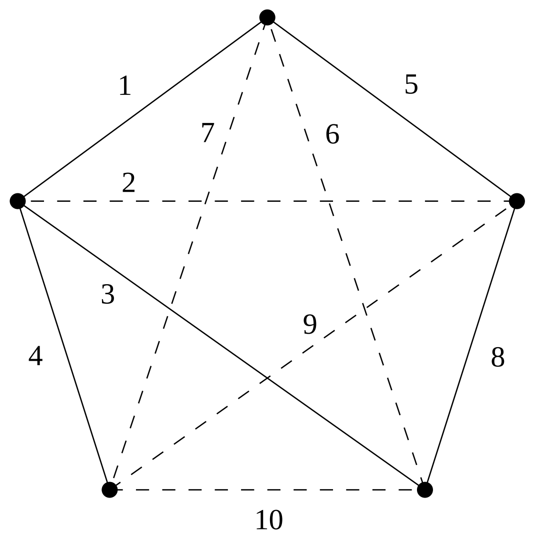
This is quadratic in the general case: i.e. we start to run out of identities!

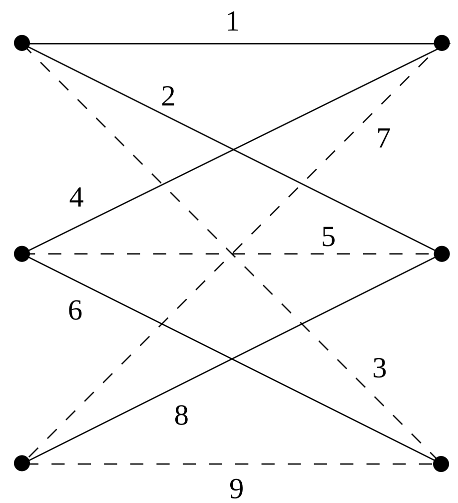
But if i, j, k, l, m contains a triangle, then one of the matrix entries, say $\Psi^{ik,jl}_m$ vanishes, and $^5\Psi(i,j,k,l,m)$ factorizes into a product

$$\Psi^{ijm,klm}\Psi^{ik,jl}_m \ ,$$

and so we can keep on going...

The first non-trivial 5-invariants occur for the non-planar graphs K_5 (fewest vertices, left) and $K_{3,3}$ (fewest edges, right):





For example, the 5-invariant ${}^5\Psi_{K_{3,3}}(1,2,4,6,8)$ for the graph on the right is given by:

$$\alpha_5\alpha_9^2 + \alpha_3\alpha_5\alpha_9 + \alpha_5\alpha_7\alpha_9 + \alpha_3\alpha_5\alpha_7 - \alpha_3\alpha_7\alpha_9$$

It turns out that these graphs are still linearly reducible (just choose a more intelligent order in which to integrate out the edges).

The first *serious* problems occur at 8 loops.

Summary

Consider the partial Feynman integrals

$$I_G^i(\alpha_{i+1}, \ldots, \alpha_n) = \int_{[0,\infty]^i} \frac{d\alpha_1 \ldots d\alpha_i}{\Psi_G^2}$$

This is a function of the Schwinger parameters $\alpha_{i+1}, \ldots, \alpha_n$. Compute its Landau variety. Typically it is given by the zeros of the polynomials $\Psi_{G,K}^{I,J} = 0$. In this case I_G^i can be expressed in terms of multiple polylogarithms, and hence we get multiple zetas. In the general case, we get non-trivial 5 (and higher) invariants, and we do not expect MZVs.

On the periods of some Feynman Integrals, arxiv:0910.0114v1

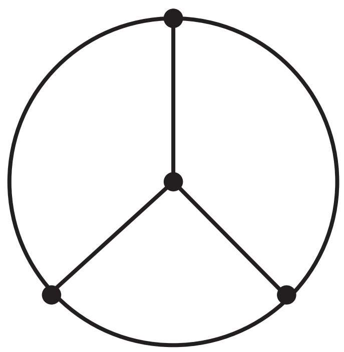
A similar idea should also work for massive Feynman diagrams with more than one external momenta at small loop orders (*in progress*)

Residues

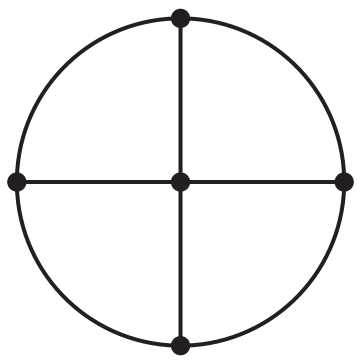
We can say a lot more about the leading term, or residues, of G.

First of all, at small loop orders (≤ 6) they will be *integer* linear combinations of MZVs.

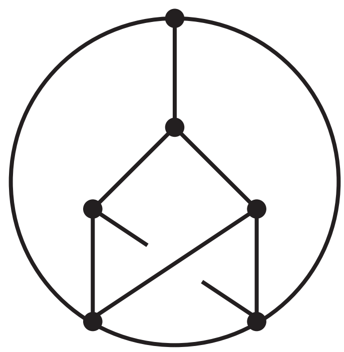
Weights. The typical graph has transcendental weight $2 \times \text{loops} - 3$. This is the case for the left and middle graphs below:



$6\zeta(3)$



$20\zeta(5)$



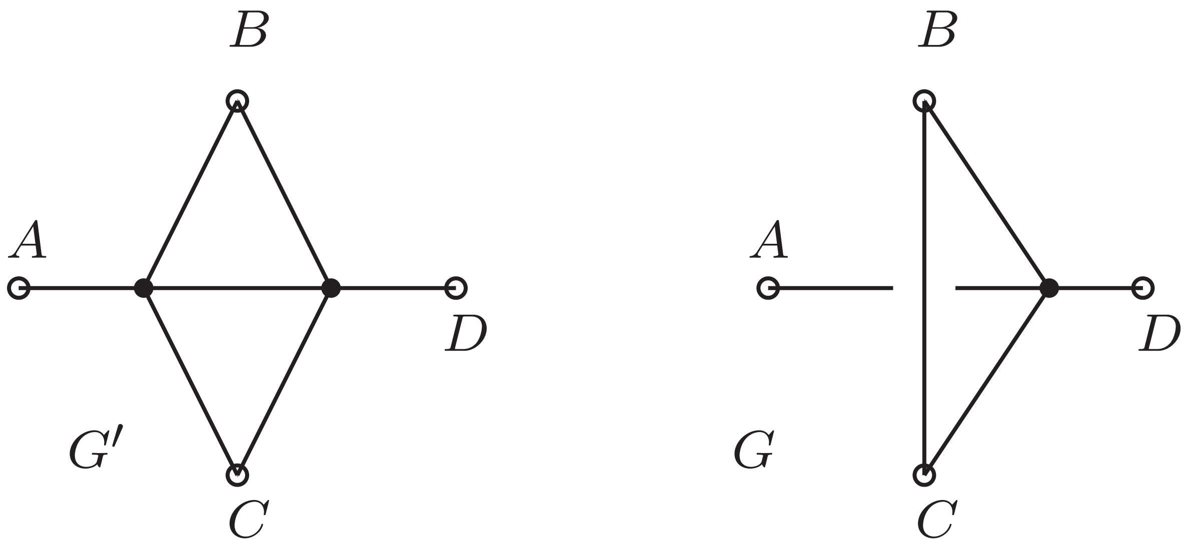
$36\zeta(3)^2$

The graph on the right has 5 loops and should also have weight $2 \times 5 - 3 = 7$. But it has weight 6. There is a *weight drop*.

Theorem 8. *(Joint with Karen Yeats)*

a). A 2-vertex reducible graph has weight drop

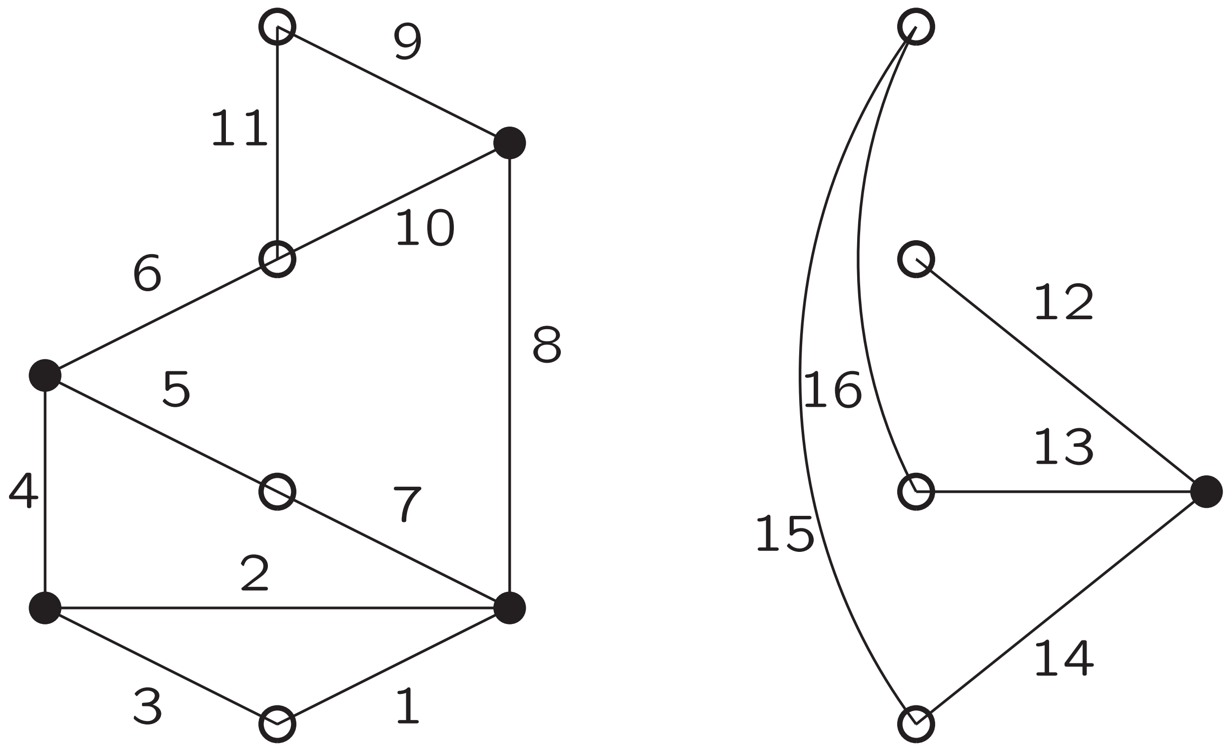
b). If G' is obtained from G by splitting a triangle, then the weight of G is equal to the weight of G'.



This theorem predicts almost all the weight drops up to 7 loops.

What is the physical meaning of the weight drop? Weight drop graphs should play a special role in the perturbative expansion.

Theorem 9. *(with O. Schnetz). Consider the following 8-loop graph obtained by gluing the two sets of white vertices together. It gives rise after 12 reductions to a denominator which defines a singular K_3 surface.*



Therefore we should not expect the residue to be a multiple zeta value.

1. MZVs and periods have a rich structure which is explained by the theory of motives. This gives new ideas such as the coproduct or weight, which should be relevant to physics.

2. There is an algorithm for computing higher-loop massless single-scale Feynman integrals directly in parametric form.

3. The combinatorial reasons for the appearance of MZVs in loop calculations, and more precise information such as the weight, are starting to become apparent.

4. Similar ideas should work for massive and many scale integrals at small loop orders.