The QCD phase diagram at non-zero baryon density

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Introduction

Lattice techniques for finite temperature and density

The phase diagram: the confusion before clarity?

Original work with Ph. de Forcrand (ETH/CERN)
QCD at high temperature/density: change of dynamics

asymptotic freedom \( \alpha_s(p \to \infty) \to 0 \)

\[ T, \mu_B \]

Chiral symmetry: broken (nearly) restored

Chiral condensate, Cooper pairs
QCD at high temperature/density: change of dynamics

Chiral symmetry:
- **broken**
- **(nearly) restored**

Order parameters:
- \( \langle \bar{\psi}\psi \rangle, \langle \psi\psi \rangle \)
  - chiral condensate, Cooper pairs

\( \alpha_s(p \to \infty) \to 0 \) (asymptotic freedom)

Phase transitions?
The QCD phase diagram established by experiment:

Nuclear liquid gas transition, $Z(2)$ end point
QCD phase diagram: theorist’s view

Until 2001: no finite density lattice calculations, sign problem!

Expectation based on models: NJL, NJL+Polyakov loop, linear sigma models, random matrix models, ...
Model predictions for critical end point (CEP)

M. Stephanov, hep-lat/0701002
How to get funding for heavy ion programs:
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Free the Quarks!!!
Thermal QCD in experiment

heavy ion collision experiments at RHIC, LHC, GSI... have finite baryon density!
Phase boundary from hadron freeze-out?

- At fixed collision energy $\sqrt{s}$, abundances well fitted by Boltzmann distribution $(T, \mu_B)$
- $T_{\text{(freeze-out)}} \leq T_c$ but very close? (Braun-Munzinger et al)
Theory: The Monte Carlo method

QCD partition fcn:

\[ Z = \int DU \prod_f \det M(\mu_f, m_f; U) e^{-S_{gauge}(\beta; U)} \]

lattice spacing \(a \ll \) hadron \(\ll L\)!
thermodynamic behaviour, large \(V\)!

typically \(> 10^8 - 10^{10}\) dim. integral

Monte Carlo, importance sampling

\[ T = \frac{1}{aN_t} \]

Continuum limit: \(N_t \to \infty, a \to 0\)

Here: \(N_t = 4, 6\)
\(a \sim 0.3, 0.2\) fm

Light fermions expensive, \(\text{cost}(\det M) \sim \frac{1}{m_n}\), here staggered fermions
How to measure p.t., critical temperature

deconfinement/chiral phase transition $\rightarrow$ quark gluon plasma

"order parameter":
chiral condensate $\langle \bar{\psi} \psi \rangle$

generalized susceptibilities:
$\chi = V(\langle O^2 \rangle - \langle O \rangle^2)$

$\Rightarrow \chi_{max} = \chi(\beta_c) \Rightarrow T_c$

Lee, Yang: only pseudo-critical on finite $V$!
Order of transition:
finite volume scaling
$\chi_{max} \sim V^\sigma$

$\sigma = 1 \quad$ 1st order
$\sigma < 1 \quad$ 2nd order
$\sigma = 0 \quad$ crossover
The order of the p.t., arbitrary quark masses $\mu = 0$

deconfinement p.t.:
breaking of global $Z(3)$

chiral p.t.
restoration of global $SU(2)_L \times SU(2)_R \times U(1)_A$

anomalous
How to identify the order of the phase transition

$$B_4(\bar{\psi}\psi) \equiv \left\langle (\delta \bar{\psi}\psi)^4 \right\rangle \over \left\langle (\delta \bar{\psi}\psi)^2 \right\rangle^2 \quad \nu \to \infty \quad \{1.604 \quad \text{3d Ising}
1 \quad \text{first - order}
3 \quad \text{crossover}\n
\mu = 0:\quad B_4(m, L) = 1.604 + bL^{1/\nu}(m - m_0^c), \quad \nu = 0.63

Crossover

First order

parameter along phase boundary, \( T = T_c(x) \)
Hard part: order of p.t., arbitrary quark masses $\mu = 0$

- physical point: crossover in the continuum
  Aoki et al 06

- chiral critical line on $N_t = 4, a \sim 0.3$ fm
  de Forcrand, O.P. 07

- consistent with tri-critical point at $m_{u,d} = 0, m_s^{\text{tric}} \sim 2.8T$

- But: $N_f = 2$ chiral $O(4)$ vs. 1st still open
  $U_A(1)$ anomaly!
  Di Giacomo et al 05, Kogut, Sinclair 07
  Chandrasekharan, Mehta 07
The ‘sign problem’ is a phase problem

\[ Z = \int DU \left[ \det M(\mu) \right] f e^{-S_g[U]} \]

Dirac operator:
\[ \bar{D}(\mu) = \gamma_5 D(-\mu^*) \gamma_5 \]

\[ \Rightarrow \det(M) \text{ complex for SU}(3), \mu \neq 0 \]
\[ \Rightarrow \text{real positive for SU}(2), \mu = i\mu_i \]
\[ \Rightarrow \text{real positive for } \mu_u = -\mu_d \]

N.B.: all expectation values real, imaginary parts cancel, but importance sampling config. by config. impossible!

Same problem in many condensed matter systems!
1dim. illustration

- Example: $Z(\lambda) = \int dx \exp(-x^2 + i\lambda x)$

- $Z(\lambda)/Z(0) = \exp(-\lambda^2/4)$: exponential cancellations
  - Truncating deep in the tail at $x \sim \lambda$ gives $o(100\%)$ error

“Every $x$ is important” $\leftrightarrow$ How to sample?
Finite density: methods to evade the sign problem

- **Reweighting:**
  \[ Z = \int DU \, \det M(0) \frac{\det M(\mu)}{\det M(0)} \, e^{-S_g} \]

  ~\( \exp(V) \) statistics needed, overlap problem

  use for MC, calculate

  coeffs. one by one, convergence?

- **Taylor expansion:**
  \[ \langle O \rangle(\mu) = \langle O \rangle(0) + \sum_{k=1} \frac{c_k}{\pi T} (\mu T)^{2k} \]

- **Imaginary** \( \mu = i\mu_i \):
  no sign problem, fit by polynomial, then analytically continue

  \[ \langle O \rangle(\mu_i) = \sum_{k=0}^{N} c_k \left( \frac{\mu_i}{\pi T} \right)^{2k}, \quad \mu_i \to -i\mu \]

  requires convergence for anal. continuation

All require \( \mu/T < 1 \)!
The good news: comparing $T_c(\mu)$

de Forcrand, Kratochvila 05

$N_t = 4, N_f = 4$; same actions (unimproved staggered), same mass

Agreement for $\mu/T \lesssim 1$
The (pseudo-) critical temperature

\[
\frac{T_c(\mu)}{T_c(\mu = 0)} = 1 - c(N_f, m_q)\left(\frac{\mu}{\pi T}\right)^2 + \ldots
\]

\[c \approx 0.500(34), 0.602(9), 0.93(10)\]

for light \(N_f = 2, 3, 4\)

cf. Toublan: \(c \propto \frac{N_f}{N_c}\)

- very flat, but not yet physical masses, coarse lattices
- indications that curvature does not grow towards continuum  
  de Forcrand, O.P. 07
- extrapolation to physical masses and continuum is feasible!  
  Budapest-Wuppertal 08
Comparison with freeze-out curve so far

\( T_c(\mu) \) considerably flatter than freeze-out curve (factor \( \sim 3 \) in \( \left. \frac{d^2T_c}{d\mu^2} \right|_{\mu=0} \))
The calculable region of the phase diagram

Upper region: equation of state, screening masses, quark number susceptibilities etc. under control

Here: phase diagram itself, most difficult!
Much harder: is there a QCD critical point?
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Two strategies:
1. follow vertical line: $m = m_{\text{phys}}$, turn on $\mu$
Much harder: is there a QCD critical point?

Two strategies:
1 follow vertical line: $m = m_{\text{phys}}$, turn on $\mu$
2 follow critical surface: $m = m_{\text{crit}}(\mu)$
**Approach 1a: CEP from reweighting**

$N_t = 4, N_f = 2 + 1$ physical quark masses, unimproved staggered fermions

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**Lee-Yang zero:**

\[
(\mu_E^q, T_E) = (120(13), 162(2)) \text{ MeV}
\]

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abrupt change: physics or problem of the method?

(entire curve generated from one point!)  
Splittorf 05, Stephanov 08
Approach 1b: CEP from Taylor expansion

\[ \frac{p}{T^4} = \sum_{n=0}^{\infty} c_{2n}(T) \left( \frac{\mu}{T} \right)^{2n} \]

Nearest singularity = radius of convergence

\[ \frac{\mu_E}{T_E} = \lim_{n \to \infty} \sqrt{\left| \frac{c_{2n}}{c_{2n+2}} \right|}, \quad \lim_{n \to \infty} \left| \frac{c_0}{c_{2n}} \right|^{\frac{1}{2n}} \]

Different definitions agree only for \( n \to \infty \) not \( n=1,2,3 \)

CEP may not be nearest singularity, control of systematics?

Gavai, Gupta

\( N_f = 2 \)
Approach 2: follow chiral critical line

\[ m_c(\mu) = m_c(0) \left( 1 + \sum_{k=1}^{\infty} c_k \left( \frac{\mu}{\pi T} \right)^{2k} \right) \]

1. Tune quark mass(es) to \( m_c(0) \): 2nd order transition at \( \mu = 0, T = T_c \)
   known universality class: 3d Ising

2. Measure derivatives \( \frac{d^k m_c}{d\mu^{2k}} \bigg|_{\mu=0} \):
   Turn on imaginary \( \mu \) and measure \( \frac{m_c(\mu)}{m_c(0)} \)

de Forcrand, O.P. 08,09
Approach 2: imaginary rather than imagined (?) CEP

\[ \frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1}^{\infty} c_k \left( \frac{\mu}{\pi T} \right)^{2k} \]

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\[ \text{de Forcrand, O.P. 08,09} \]
Finite density: chiral critical line $\rightarrow$ critical surface

\[ \frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1}^{\infty} c_k \left( \frac{\mu}{\pi T} \right)^{2k} \]

- **Standard scenario**
  - Pure Gauge
  - 1st order
  - QCD critical point
  - QGP
  - m < m_c(0)

- **Exotic scenario**
  - Pure Gauge
  - 1st order
  - QCD critical point DISAPPEARED
  - QGP
  - m < m_c(0)
Finite density: chiral critical line $\rightarrow$ critical surface

\[ \frac{m_c(\mu)}{m_c(0)} = 1 + \sum_{k=1}^{\infty} c_k \left( \frac{\mu}{\pi T} \right)^{2k} \]

\[ c_1 > 0 \quad \text{Standard scenario} \]

\[ c_1 < 0 \quad \text{Exotic scenario} \]
Curvature of the chiral critical surface

\[ N_f = 3 \]

\[ N_f = 2 + 1, \ m_s = m_s^{\text{phys}} \]

consistent \(8^3 \times 4\) and \(12^3 \times 4\), \(\sim 5 \times 10^6\) traj.

\[
\frac{m_c(\mu)}{m_c(0)} = 1 - 3.3(3) \left( \frac{\mu}{\pi T} \right)^2 - 47(20) \left( \frac{\mu}{\pi T} \right)^4 - \ldots
\]

8th derivative of \(P\)

\[
16^3 \times 4, \ \text{Grid computing,} \ \sim 10^6 \ \text{traj.}
\]

\[
\frac{m_c^{u,d}(\mu)}{m_c^{u,d}(0)} = 1 - 39(8) \left( \frac{\mu}{\pi T} \right)^2 - \ldots
\]

de Forcrand, O.P. 08,09
The chiral critical surface on a coarse lattice:

No chiral crit. pt. at small chem. pot., $\frac{\mu}{T} \lesssim o(1)$, for $N_t=4$ ($a \sim 0.3$ fm)

cf. Ejiri 08 $\rightarrow (\frac{\mu}{T})^{\text{CEP}} \sim 2.4$

- Higher order terms? Convergence?
- Cut-off effects?
- In any case: picture for QCD phase diagram not as clear as anticipated.....
Un-discovering a critical point feels like...
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Scenario unusual? ...the same happens for heavy quark masses!

effective heavy quark theory, same universality class: 3-state Potts model

Real $\mu$: first order region shrinking!
also for finite isospin chemical potential

N.B.: non-chiral critical point still possible!

de Forcrand, Kim, Takaishi
Kogut, Sinclair
Recent model studies with similar results

K. Fukushima 08

NJL-Polyakov loop model with vector-vector interaction

Bowman, Kapusta 08

Linear sigma model with quarks

Qualitative behaviour as in exotic lattice scenario!
Towards the continuum: $N_t = 6, a \sim 0.2 \text{ fm}$

Physical point deeper in crossover region as $a \to 0$

Cut-off effects stronger than finite density effects!

Curvature of crit. surface (so far) consistent with 0; strong quark mass sensitivity!

$\frac{m_c^N (N_t = 4)}{m_c^N (N_t = 6)} \approx 1.77 \quad N_f = 3$

de Forcrand, Kim, O.P. 07
Endrodi et al 07
The interplay of $N_f=2$ and $N_f=2+1$

- $O(4)$ transition for 2 massless flavors
  \[\Rightarrow \text{tricritical points } (m_{u,d} = 0, m_s = \infty, \mu = \mu^*) \text{ and } (m_{u,d} = 0, m_s = m_s^*, \mu = 0)\]

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Pisarski & Wilczek
The interplay of $N_f=2$ and $N_f=2+1$

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- $N_f = 2$ and $N_f = 2 + 1$ analytically connected

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- $N_f = 2$ and $N_f = 2 + 1$ need not be connected
The interplay of $N_f = 2$ and $N_f = 2 + 1$

- $O(4)$ transition for 2 massless flavors

  $\Rightarrow$ tricritical points $(m_{u,d} = 0, m_s = \infty, \mu = \mu^*)$ and $(m_{u,d} = 0, m_s = m_s^*, \mu = 0)$

- $N_f = 2$ and $N_f = 2 + 1$ analytically connected

- $N_f = 2$ and $N_f = 2 + 1$ need not be connected

Pisarski & Wilczek

which critical surface do we need?

$O(4)$

phys.

Real world

Heavy quarks

1st order

crossover

1st order

$\infty$
More (and non-chiral) critical points?

Good time for models: NJL with vector interactions
Ginzburg-Landau approach for quark condensates

Zhang, Kunihiro, Fukushima 09
Baym et al. 06
Conclusions

- Working lattice methods available for $\mu < T$
- $T_c(\mu)$, EoS under control at small density
- On coarse lattices $a \approx 0.3$ fm no chiral critical point for $\mu < T$
- Large cut-off and quark mass effects
- Uncharted territory: do QCD critical points exist?
Conclusions

- Working lattice methods available for \( \mu < T \)
- \( T_c(\mu) \), EoS under control at small density
- On coarse lattices a~0.3 fm no chiral critical point for \( \mu < T \)
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- Uncharted territory: do QCD critical points exist?
The argument: \(N_f = 2, \mu, m = 0\)  
Pisarski, Wilczek 84

spont. chiral symmetry breaking \(SU(2)_V \times SU(2)_A \rightarrow SU(2)_V\);  
restored at finite \(T\)

true order parameter \(\langle \bar{\psi} \psi \rangle \Rightarrow \) separate phases \(\Rightarrow\) non-analytic transition

The assumptions:  
Halasz et al. 98; Rajagopal, Shuryak, Stephanov 98

- \(N_f = 2\) chiral transition is second order \(\Rightarrow O(4)\)
- \(T = 0\) and \(\mu = 0\) transitions are continuously connected

\(N_f = 2 + 1:\) adding \(m_s\) only shifts location of line and (tri-)critical point
The equation of state

Continuum extrapolation with physical quarks feasible in the near future

\[ \frac{\varepsilon_{SB}}{T^4} = \frac{3p_{SB}}{T^4} = \begin{cases} \frac{3 \pi^2}{30}, & T < T_c \\ (16 + \frac{21}{2} N_f) \frac{\pi^2}{30}, & T > T_c \end{cases} \]

\[ T > T_c: \quad \text{more degrees of freedom, but significant interaction!} \]

\[ \text{sQGP or `almost ideal' gas...?} \]
Equation of state at finite baryon density

Taylor expansion, second order  Bielefeld-Swansea  Reweighting  Wuppertal
The nature of the transition for phys. masses

...in the staggered approximation...in the continuum...is a crossover!

Aoki et al. 06