Properties of Feynman Graph Polynomials

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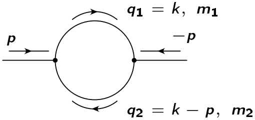
Contents

Scalar Feynman integrals

$$I_G = \int \prod_{r=1}^{L} \frac{d^D k_r}{i\pi^{D/2}} \prod_{j=1}^{N} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}}, \quad D \in \mathbb{C},\ \nu_j \in \mathbb{N},\ \nu = \sum_{j=1}^{N} \nu_j$$

D : regularization parameter, L : loop-number, N : # internal edges, k_r : loop-momenta, q_j : momenta through internal edges, m_j : particle masses

Example:



I_G is a function of D, squared masses m_j^2 and kinematical invariants s_i

i.e. squares of (combinations of) external momenta.

Remark: Tensor-like integrals can be reduced to scalar integrals.
Tarasov '96 (one-loop case: Passarino and Veltman '79):

$$I^{\mu_1 \cdots \mu_n} = \sum_i T^{\mu_1 \cdots \mu_n} I_i^{\text{scalar}}$$

Feynman parameters:

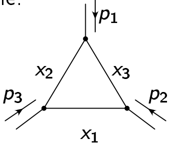
$$I_G = \int \prod_{r=1}^{L} \frac{d^D k_r}{i\pi^{D/2}} \prod_{j=1}^{N} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}}$$

- repeated use of the "Feynman trick": $\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}$,
- evaluation of momentum integrations $\int d^D k_r$

$\Rightarrow$ Feynman parameter representation:

$$I_G = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^1 \left(\prod_{j=1}^{N} dx_j\, x_j^{\nu_j - 1} \right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{\mathcal{F}^{\nu - LD/2}}$$

Example:



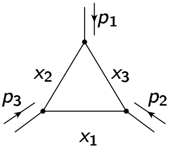
$$\mathcal{U}(G) = x_1 + x_2 + x_3$$

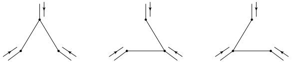
$$\mathcal{F}(G) = x_1 x_2 p_3^2 + x_2 x_3 p_1^2 + x_3 x_1 p_2^2$$

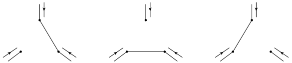
Spanning Forests

- n-forest: graph without loops (cycles) and n connected components,
- tree: 1-forest,
- spanning forest of a graph G: forest, obtained from G by deletion of edges (without deleting vertices)

Example:



spanning trees: 

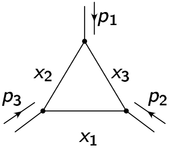
spanning 2-forests: 

Construction/definition of $\mathcal{U}(G)$ and $\mathcal{F}(G)$:

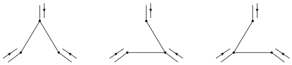
$$\mathcal{U}(G) = \sum_{\text{spanning trees } T \text{ of } G} \; \prod_{\text{edges } \notin T} x_i$$

$$\mathcal{F}_0(G) = -\sum_{\text{spanning 2-forests } (T_1,\, T_2)} \left(\prod_{\text{edges } \notin (T_1,\, T_2)} x_i \right) \left(\sum_{\text{edges } \notin (T_1,\, T_2)} q_i \right)^2,$$

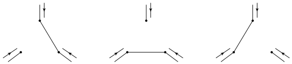
$$\mathcal{F}(G) = \mathcal{F}_0(G) + \mathcal{U}(G) \sum_{i=1}^{N} x_i m_i^2 .$$



spanning trees:



spanning 2-forests:



$$\mathcal{U}(G) = x_1 + x_2 + x_3, \quad \mathcal{F}_0(G) = -x_1 x_2 p_3^2 - x_2 x_3 p_1^2 - x_1 x_3 p_2^2$$

Construction/definition of $\mathcal{U}(G)$ and $\mathcal{F}(G)$:

$$
\begin{aligned}
\mathcal{U}(G) &= \sum_{\text{spanning trees } T \text{ of } G} \prod_{\text{edges } \notin T} x_i \\
\mathcal{F}_0(G) &= - \sum_{\text{spanning 2-forests } (T_1, T_2)} \left(\prod_{\text{edges } \notin (T_1, T_2)} x_i \right) \left(\sum_{\text{edges } \notin (T_1, T_2)} q_i \right)^2,
\end{aligned}
$$

Obvious properties:

- they are linear in each Feynman parameter,
- they are homogeneous in the Feynman parameters, degree: $\deg \mathcal{U}(G) = L$, $\deg \mathcal{F}_0(G) = L + 1$,
- $\mathcal{F}_0(G)$ is also a function of the kinematical invariants (external momenta).

$\mathcal{U}(G)$ and $\mathcal{F}_0(G)$ contain:

$$\prod_{\text{edges } \notin T} x_i, \qquad \prod_{\text{edges } \notin (T_1,\, T_2)} x_i.$$

Alternatively we may consider:

$$U(G) = \sum_{\text{spanning trees } T \text{ of } G} \prod_{\text{edges } \in T} x_i \ \text{(Kirchhoff polynomial)}$$

$$F_0(G) = -\sum_{\text{spanning 2-forests } (T_1,\, T_2)} \left(\prod_{\text{edges } \in (T_1,\, T_2)} x_i \right) \left(\sum_{\text{edges } \notin (T_1,\, T_2)} q_i \right)^2$$

We easily obtain $\mathcal{U}(G)$ and $\mathcal{F}_0(G)$ from $U(G)$ and $F_0(G)$ (and vice versa):

$$\mathcal{U}(G)(x_1, \ldots, x_N) = U(G)\left(\frac{1}{x_1}, \ldots, \frac{1}{x_N}\right) \prod_{i=1}^{N} x_i,$$

$$\mathcal{F}_0(G)(x_1, \ldots, x_N) = F_0(G)\left(\frac{1}{x_1}, \ldots, \frac{1}{x_N}\right) \prod_{i=1}^{N} x_i.$$

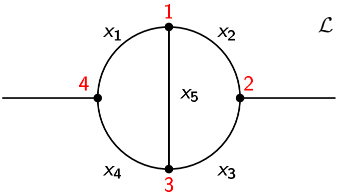
Matrix-Tree Theorems

Laplacian matrices:
Let G be a graph with r edges $E = \{e_1, ..., e_r\}$ and with n vertices.

A Laplacian matrix $\mathcal{L}$ of G is an $n \times n-$matrix with the entries

$$\mathcal{L}_{ij} = \begin{cases} \sum x_k \text{ for } i = j, \ e_k \text{ attached to } v_i \text{ (and not a self-loop)}, \\ -\sum x_k \text{ for } i \neq j, \ e_k \text{ connecting } v_i \text{ and } v_j. \end{cases}$$

Example:



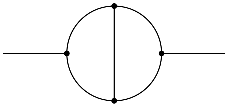
$$\mathcal{L} = \begin{pmatrix} x_1 + x_2 + x_5 & -x_2 & -x_5 & -x_1 \\ -x_2 & x_2 + x_3 & -x_3 & 0 \\ -x_5 & -x_3 & x_3 + x_4 + x_5 & -x_4 \\ -x_1 & 0 & -x_4 & x_1 + x_4 \end{pmatrix}$$

Matrix-tree theorem (Kirchhoff, Tutte):

Let G be a connected graph, $\mathcal{L}$ its Laplacian matrix and $\mathcal{L}[i,i]$ the minor obtained by deleting the ith row and ith column. Then

$$\det(\mathcal{L}[i,i]) = U(G).$$

Example:



$$\mathcal{L} = \begin{pmatrix} x_1+x_2+x_5 & -x_2 & -x_5 & -x_1 \\ -x_2 & x_2+x_3 & -x_3 & 0 \\ -x_5 & -x_3 & x_3+x_4+x_5 & -x_4 \\ -x_1 & 0 & -x_4 & x_1+x_4 \end{pmatrix}$$

$$\det(\mathcal{L}[4,4]) = \begin{vmatrix} x_1+x_2+x_5 & -x_2 & -x_5 \\ -x_2 & x_2+x_3 & -x_3 \\ -x_5 & -x_3 & x_3+x_4+x_5 \end{vmatrix}$$

$$= x_1x_2(x_3+x_4) + (x_1+x_2)x_3x_4 + (x_1x_2 + x_1x_3 + x_2x_4 + x_3x_4) = U(G)$$

Generalization: Remove k rows numbered by $I = (i_1, \ldots, i_k)$ and k columns numbered by $J = (j_1, \ldots, j_k)$.

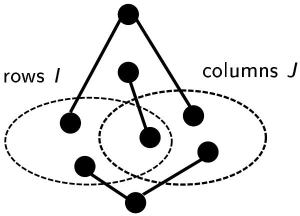
All-minors matrix-tree theorem (Chen, Chaiken, Moon):

$$\det(\mathcal{L}[I, J]) = (-1)^{i_1 + \ldots + i_k + j_1 + \ldots + j_k} \sum_{F \in \mathcal{T}_k^{I, J}} \text{sign}(\pi_F) \prod_{\text{edges } \in F} x_j .$$

For $I = J$:

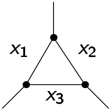
$$\det(\mathcal{L}[I, I]) = \sum_{F \in \mathcal{T}_k^{I, I}} \prod_{\text{edges } \in F} x_j$$

$\mathcal{T}_k^{I, J}$: spanning $k-$forests where each component has exactly one vertex in I and one in J (possibly the same one).



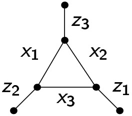
Inroducing variables z_i for the external edges:

G:



$$\mathcal{L}(G) = \begin{pmatrix} x_1 + x_2 & -x_1 & -x_2 \\ -x_1 & x_1 + x_3 & -x_3 \\ -x_2 & -x_3 & x_2 + x_3 \end{pmatrix}$$

$\tilde{G}$:



$$\mathcal{L}(\tilde{G}) = \begin{pmatrix} x_1 + x_2 + z_3 & -x_1 & -x_2 & 0 & 0 & -z_3 \\ -x_1 & x_1 + x_3 + z_2 & -x_3 & 0 & -z_2 & 0 \\ -x_2 & -x_3 & x_2 + x_3 + z_1 & -z_1 & 0 & 0 \\ 0 & 0 & -z_1 & z_1 & 0 & 0 \\ 0 & -z_2 & 0 & 0 & z_2 & 0 \\ -z_3 & 0 & 0 & 0 & 0 & z_3 \end{pmatrix}$$

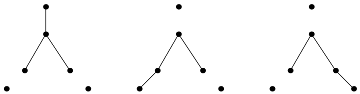
$$\mathcal{L}(\tilde{G})[\text{external vertices}] = \begin{pmatrix} x_1 + x_2 + z_3 & -x_1 & -x_2 \\ -x_1 & x_1 + x_3 + z_2 & -x_3 \\ -x_2 & -x_3 & x_2 + x_3 + z_1 \end{pmatrix}$$

We define: $W(G) := \det\left(\mathcal{L}(\tilde{G})[\text{external vertices}]\right)$

All-minors matrix-tree theorem:

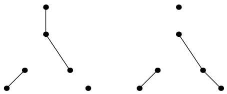
$$\det\left(\mathcal{L}\left[I,\ I\right]\right) = \sum_{F \in \mathcal{T}_k^{I,\ I}} \prod_{\text{edges } \in F} x_j$$

$\Rightarrow$ The polynomial $W(G) = \det\left(\mathcal{L}(\tilde{G})[\text{external vertices}]\right)$ contains:



$$\ldots \Rightarrow x_1 x_2 \left(z_1 + z_2 + z_3\right) + \ldots$$

$$= U(G)(z_1 + z_2 + z_3)$$



$$\ldots \Rightarrow x_2(z_2 z_1 + z_2 z_3) + \ldots$$

$$[z_i z_j = p_i \cdot p_j] \Rightarrow F_0(G)$$

We obtain both Symanzik polynomials from one determinant:

$$W(G) = \det\left(\mathcal{L}(\tilde{G})[\text{external vertices}]\right)$$

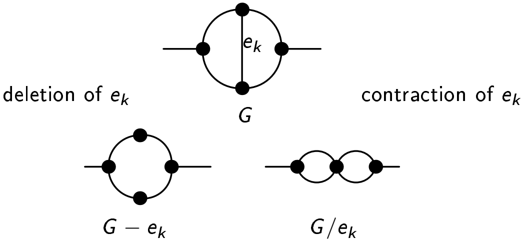
$$W(G) = W^{(1)}(G) + W^{(2)}(G) + \ldots$$

$$W^{(1)}(G) : \text{terms linear in external variables } z_i,$$

$$W^{(2)}(G) : \text{terms quadratic in external variables } z_i$$

$$\begin{aligned} W^{(1)}(G) &= \left(\sum_{\text{external edges}} z_i\right) \cdot U(G) \\ W^{(2)}(G)\left(z_i z_j = p_i \cdot p_j\right) &= F_0(G) \end{aligned}$$

Deletion and Contraction

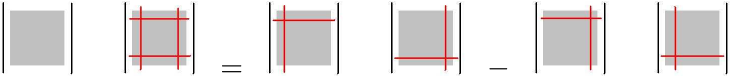


For regular edges (i.e. no bridge or self-loop) we obtain **deletion/contraction identities:**

$$\mathcal{U}(G) = \mathcal{U}(G/e_k) + x_k \mathcal{U}(G - e_k)$$
$$\mathcal{F}_0(G) = \mathcal{F}_0(G/e_k) + x_k \mathcal{F}_0(G - e_k)$$

Dodgson's identity (from 1866) for an arbitrary $n \times n$-matrix A:

$$\det(A)\det(A[\{i,j\};\{i,j\}]) = \det(A[i;\,i])\det(A[j;\,j]) - \det(A[i;\,j])\det(A[j;\,i])$$



Applying Dodgson's identity to an appropriate minor of a Laplacian matrix one obtains (Stembridge '98, Brown '08):

$$\mathcal{U}(G/e_a - e_b)\mathcal{U}(G/e_b - e_a) - \mathcal{U}(G - e_a - e_b)\mathcal{U}(G/e_a/e_b) = \left(\frac{\Delta_1}{x_a x_b}\right)^2$$

$$\Delta_1 = \sum_{F \in \mathcal{T}_2^{(i,k),(j,k)}} \prod_{\text{edges } e_t \notin F} x_t$$

$\mathcal{T}_2^{(i,k),(j,k)}$: spanning 2-forests with a common vertex of e_a and e_b in the first and the other end-points in the second component

W contains both Symanzik polynomials and is a determinant of a minor of a Laplacian.

$\Rightarrow$ From Dodgson's identity we obtain

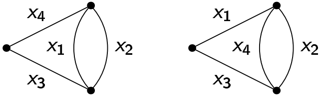
$$\mathcal{U}(G/e_a - e_b)\mathcal{F}_0(G/e_b - e_a) - \mathcal{U}(G - e_a - e_b)\mathcal{F}_0(G/e_a/e_b) +$$
$$\mathcal{F}_0(G/e_a - e_b)\mathcal{U}(G/e_b - e_a) - \mathcal{F}_0(G - e_a - e_b)\mathcal{U}(G/e_a/e_b) = 2\left(\frac{\Delta_1}{x_a x_b}\right)\left(\frac{\Delta_2}{x_a x_b}\right)$$

with Δ_2 being a polynomial in the Feynman parameters, generated by a class of spanning 3-forests.

Again the right-hand side factorizes to polynomials which are linear in each parameter.

Matrois

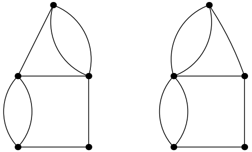
Preliminary remark:



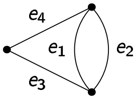
$$x_1 x_2 + (x_1 + x_2)(x_3 + x_4) \text{ is isomorphic to } x_4 x_2 + (x_4 + x_2)(x_3 + x_1)$$

We call two Symanzik polynomials **isomorphic** if they only differ by a bijection on the set of variable names.

Question: When are Kirchhoff polynomials of **two different graphs** isomorphic?



The answer is a corollary of a theorem on cycle matroids.



Example: A graph G with the edges $E = \{e_1, e_2, e_3, e_4\}$
Incidence matrix:

$$\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Linearly independent subsets:

$$\mathcal{I} = \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}\}$$

The ordered pair $(E, \mathcal{I})$ is a cycle matroid of G.

Remark: The bases, i.e. maximal independent sets in $\mathcal{I}$, consist of the edges of the spanning trees of G.

Definition: Matroid

A matroid is an ordered pair $(E, \mathcal{I})$ of a finite set E (ground set) and a collection $\mathcal{I}$ of subsets of E (the independent sets), fulfilling

1. $\emptyset \in \mathcal{I}$.
2. If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.
3. If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then $\exists\ e \in I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

A basis of the matroid is a **maximal independent set**, i.e. a set in $\mathcal{I}$ which is **not a proper subset** of any set in $\mathcal{I}$.

1) The cycle matroid is uniquely determined by the bases and the ground set.
2) The bases $\mathcal{B} = (B_1, B_2, ..., B_k)$ of a cycle matroid are the edge-sets of the spanning trees of the graph.

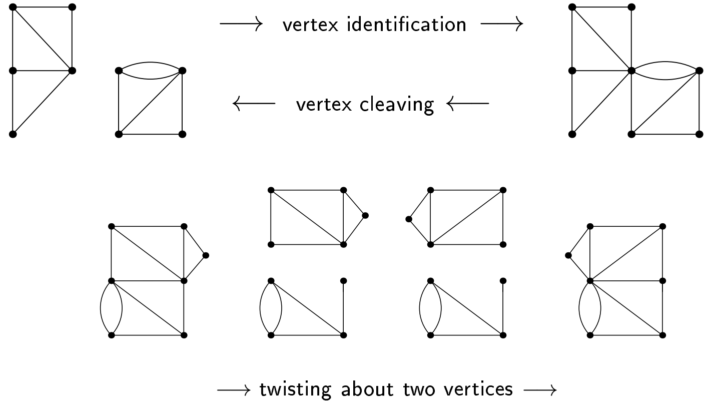
Kirchhoff polynomials and bases of matroids

$$U_G = \sum_{B_i \in \mathcal{B}} \prod_{\mathrm{edges} \in B_i} x_j$$

graphs with isomorphic cycle matroids $\Leftrightarrow$ graphs with isomorphic Kirchhoff polynomials

$\Rightarrow$ When are the **cycle matroids** of two different graphs isomorphic?

Whitney's 2-isomorphism theorem (1933):
Two graphs G and H (without isolated vertices) have isomorphic cycle matroids if and only if G is obtained from H after a sequence of the **transformations**:



⇒ **In these cases, connected graphs G and H have isomorphic Kirchhoff polynomials.**

Summary:

- The Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}_0$ appear in **any** (scalar) **Feynman integral**.
- They are derived/defined by **spanning trees and 2-forests** of the graph.
- **Both** can be derived from the **Laplacian matrix** of a graph.
- **Both** fulfill relations coming from **Dodgson's identity**.
- **Matroid theory** has an answer to the question: "When do different graphs have equal Kirchhoff polynomials?"

Some references:

Review:

C.B. and S. Weinzierl, *Feynman Graph Polynomials*, `arXiv:1002.3458 [hep-ph]`.

Very recent progress:

F. Brown and K. Yeats, *Spanning forest polynomials and the transcendental weight of Feynman graphs*, `arXiv:0910.5429 (math-ph)`.

F. Brown, *On the periods of some Feynman integrals*, `arXiv:0910.0114 (math.AG)`.

E. Patterson, *Singularities of configuration and graph hypersurfaces*, Ph.D. thesis, University of Chicago (2009).

Classical References:

N. Nakanishi, *Graph Theory and Feynman Integrals*, Gordon and Breach, (1971).

W. T. Tutte, *Graph Theory*, Addison-Wesley, (1984).

J. Moon, *Some determinant expansions and the matrix-tree theorem*, Discrete Math. 124 (1994).

J. Oxley, *Matroid Theory*, Oxford Univ. Press, (2006).

(C. Itzykson and J. Zuber, *Quantum Field Theory*, McGraw-Hill, (1980).)

In the language of algebraic topology deletion and contraction are studied in

Eric Patterson's Dissertation (University of Chicago, 2009):

configurations (subspaces of based vectorspaces) $\leftrightarrow$ homology groups of graphs

configuration polynomials $\leftrightarrow$ Symanzik polynomials

(particularly: $\mathcal{F}_0$ is a configuration polynomial)

restrictions on a configuration $\leftrightarrow$ deletion and contraction on a graph