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Comments and suggestions to
Fred Olness & Randy Scalise

SMU-HEP-97??

Everything you wanted to know about $\Lambda_{\rm QCD}$, but were afraid to ask

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Abstract

We collect useful formulæ and numerical results for the strong coupling α_S, the beta function β, and the scale parameter Λ.

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1 The Beta Function

The definition of the Callan-Symanzik beta function (with the convention of [RVL] for the coefficients) is

$$\beta(a_S) \equiv \frac{\partial a_S}{\partial \ln \mu^2} \equiv -\sum_{n=0}^{\infty} \beta_n a_S^{n+2} = -\left[\beta_0 a_S^2 + \beta_1 a_S^3 + \beta_2 a_S^4 + \beta_3 a_S^5 + \mathcal{O}(a_S^6)\right] \tag{1}$$

where

$$a_S = \frac{\alpha_S}{4\pi} = \frac{g^2}{16\pi^2} \qquad\qquad \alpha_S = \frac{g^2}{4\pi} \tag{2}$$

and g is the renormalized coupling.

Different conventions exist [RVL, CKS, PDG, Weinberg, Sterman, CTEQ]

$$\begin{array}{lllllllll} \beta_0 & \equiv & \beta_0^{[\text{RVL}]} & = & 2^2\beta_0^{[\text{CKS}]} & = & \beta_0^{[\text{PDG, Weinberg}]} & = & -\beta_1^{[\text{Sterman, CTEQ}]} \\ \beta_1 & \equiv & \beta_1^{[\text{RVL}]} & = & 2^4\beta_1^{[\text{CKS}]} & = & 2\beta_1^{[\text{PDG, Weinberg}]} & = & -\beta_2^{[\text{Sterman, CTEQ}]} \\ \beta_2 & \equiv & \beta_2^{[\text{RVL}]} & = & 2^6\beta_2^{[\text{CKS}]} & = & \frac{1}{2}\beta_2^{[\text{PDG, Weinberg}]} & & \\ \beta_3 & \equiv & \beta_3^{[\text{RVL}]} & = & 2^8\beta_3^{[\text{CKS}]} & & & & \end{array} \tag{3}$$

The coefficients are known to four loop order [RVL]. For QCD they are

$$\begin{array}{lll} \beta_0 & = & 11 - \frac{2}{3}n_f \\ \beta_1 & = & 102 - \frac{38}{3}n_f \\ \beta_2 & = & \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2 \\ \beta_3 & = & \left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3 \end{array} \tag{4}$$

where n_f is the number of active fermion flavors and ζ is the Riemann zeta function $\zeta_p = \sum_{n=1}^{\infty} n^{-p}$ with $\zeta_3 \approx 1.20205690315959428$. For $n \geq 2$, β_n is renormalization scheme dependent. The values shown above are $\overline{\text{MS}}$.

Other relations derivable from Eq. 1 are

$$\mu\frac{\partial \alpha_S}{\partial \mu} = -\frac{\alpha_S^2}{2\pi}\sum_{n=0}^{\infty}\beta_n\left(\frac{\alpha_S}{4\pi}\right)^n = -\left[\beta_0\frac{\alpha_S^2}{2\pi} + \beta_1\frac{\alpha_S^3}{2^3\pi^2} + \beta_2\frac{\alpha_S^4}{2^5\pi^3} + \beta_3\frac{\alpha_S^5}{2^7\pi^4} + \mathcal{O}(\alpha_S^6)\right] \tag{5}$$

and

$$\mu\frac{\partial g}{\partial \mu} = -\frac{g^3}{16\pi^2}\sum_{n=0}^{\infty}\beta_n\left(\frac{g^2}{16\pi^2}\right)^n = -\left[\beta_0\frac{g^3}{2^4\pi^2} + \beta_1\frac{g^5}{2^8\pi^4} + \beta_2\frac{g^7}{2^{12}\pi^6} + \beta_3\frac{g^9}{2^{16}\pi^8} + \mathcal{O}(g^{11})\right] \tag{6}$$

2 The Running Coupling at One Loop Order

We solve the differential equation at one loop order.

$$\mu \frac{\partial \alpha_S(\mu)}{\partial \mu} = -\beta_0 \frac{\alpha_S^2(\mu)}{2\pi} \tag{7}$$

$$\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} = -\frac{2\pi}{\beta_0} \int_{\alpha_S(\mu_0)}^{\alpha_S(\mu)} \frac{d\alpha_S}{\alpha_S^2} \tag{8}$$

$$\ln\left(\frac{\mu}{\mu_0}\right) = \frac{2\pi}{\beta_0}\left[\frac{1}{\alpha_S(\mu)} - \frac{1}{\alpha_S(\mu_0)}\right] \tag{9}$$

$$\alpha_S(\mu) = \frac{1}{\frac{1}{\alpha_S(\mu_0)} + \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right)} \equiv \frac{1}{\frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \tag{10}$$

where Λ is defined to absorb both μ_0 and $\alpha_S(\mu_0)$ into a single parameter, as follows

$$\begin{aligned}
\alpha_S^{-1}(\mu) &= \frac{1}{\alpha_S(\mu_0)} + \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right) \\
&= \ln\left[e^{\frac{1}{\alpha_S(\mu_0)}}\right] + \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right) \\
&= \frac{\beta_0}{4\pi}\ln\left[e^{\frac{1}{\alpha_S(\mu_0)}\frac{4\pi}{\beta_0}}\right] + \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right) \\
&= \frac{\beta_0}{4\pi}\ln\left[\left(\frac{\mu^2}{\mu_0^2}\right) e^{\frac{4\pi}{\beta_0 \alpha_S(\mu_0)}}\right] \\
&= \frac{\beta_0}{4\pi}\ln\left[\frac{\mu^2}{\mu_0^2 e^{-\frac{4\pi}{\beta_0 \alpha_S(\mu_0)}}}\right] \\
&\equiv \frac{\beta_0}{4\pi}\ln\left[\frac{\mu^2}{\Lambda^2}\right]
\end{aligned} \tag{11}$$

so that to one loop order,

$$\Lambda^2 \equiv \mu_0^2 e^{-\frac{4\pi}{\beta_0 \alpha_S(\mu_0)}} \qquad \Rightarrow \qquad \Lambda \equiv \mu_0 e^{-\frac{2\pi}{\beta_0 \alpha_S(\mu_0)}} \tag{12}$$

The mass of the Z^0 boson is

$$m_Z \approx 91.187 \text{ GeV}. \tag{13}$$

At this scale μ, five flavors of quark are active since

$$\begin{aligned}
m_u &\approx 0 \approx m_d \approx m_s \\
m_c &\approx 1.6 \text{ GeV} \\
m_b &\approx 5.0 \text{ GeV} \\
m_t &\approx 175 \text{ GeV}
\end{aligned} \tag{14}$$

and

$$m_b < m_Z < m_t \tag{15}$$

For $n_f = 5$ the first beta function coefficient is

$$\beta_0^{(5)} = \left(11 - \frac{2}{3}n_f\right)\Big|_{n_f=5} = \frac{23}{3} \tag{16}$$

If we take

$$\alpha_S(m_Z) \;\approx\; 0.119 \tag{17}$$

then

$$\Lambda_{(5)} = m_Z \exp\left[\frac{-2\pi}{\beta_0^{(5)}\alpha_S(m_Z)}\right] \approx 93.105 \text{ MeV} \tag{18}$$

Λ must be measured by experiment. Changing $\Lambda_{(5)}$ is tantamount to changing the strength of the QCD coupling. $\Lambda_{(3)}$, $\Lambda_{(4)}$, and $\Lambda_{(6)}$ are determined by matching $\alpha_S(\mu)$ at the quark masses. For example, at $\mu = m_b$

$$\lim_{\epsilon\to 0}\left[\alpha_S(m_b - \epsilon) - \alpha_S(m_b + \epsilon)\right] = 0 \tag{19}$$

determines $\Lambda_{(4)}$

$$\frac{\beta_0^{(4)}}{4\pi}\ln\left[\frac{m_b^2}{\Lambda_{(4)}^2}\right] = \frac{\beta_0^{(5)}}{4\pi}\ln\left[\frac{m_b^2}{\Lambda_{(5)}^2}\right] \tag{20}$$

or

$$\Lambda_{(4)} = \Lambda_{(5)}^{\frac{\beta_0^{(5)}}{\beta_0^{(4)}}}\, m_b^{\left[1-\frac{\beta_0^{(5)}}{\beta_0^{(4)}}\right]} \approx 128.048 \text{ MeV} \tag{21}$$

Similarly,

$$\Lambda_{(3)} = \Lambda_{(4)}^{\frac{\beta_0^{(4)}}{\beta_0^{(3)}}}\, m_c^{\left[1-\frac{\beta_0^{(4)}}{\beta_0^{(3)}}\right]} \approx 154.388 \text{ MeV} \tag{22}$$

$$\Lambda_{(6)} = \Lambda_{(5)}^{\frac{\beta_0^{(5)}}{\beta_0^{(6)}}}\, m_t^{\left[1-\frac{\beta_0^{(5)}}{\beta_0^{(6)}}\right]} \approx \;\; 45.411 \text{ MeV} \tag{23}$$

or, in general,

$$\Lambda_{(n-1)} = \Lambda_{(n)}^{\frac{\beta_0^{(n)}}{\beta_0^{(n-1)}}}\, m_{(n)}^{\left[1-\frac{\beta_0^{(n)}}{\beta_0^{(n-1)}}\right]} \qquad\qquad \Lambda_{(n+1)} = \Lambda_{(n)}^{\frac{\beta_0^{(n)}}{\beta_0^{(n+1)}}}\, m_{(n+1)}^{\left[1-\frac{\beta_0^{(n)}}{\beta_0^{(n+1)}}\right]} \tag{24}$$

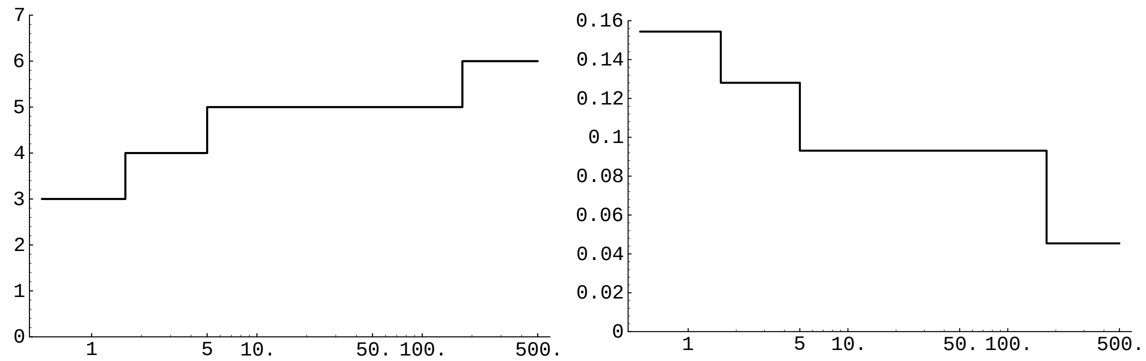

Figure 1: a) n_f *vs.* μ and b) one loop Λ *vs.* μ.

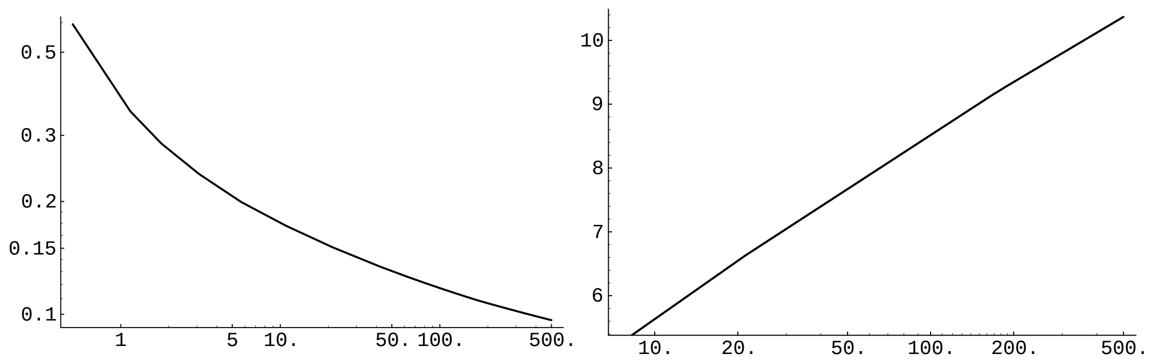

Figure 2: At one loop order, a) α_S *vs.* μ and b) α_S^{-1} *vs.* μ. Notice that the coupling is continuous across the quark mass thresholds. In b), the semi-log plot is piecewise linear with slope changes at the quark masses.

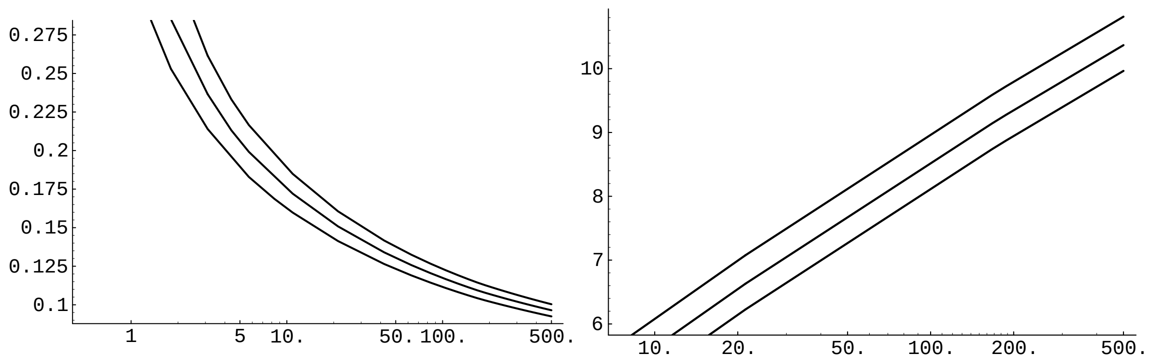

Figure 3: At one loop order, a) α_S *vs.* μ for $\alpha_S(m_Z) = 0.113, 0.119, 0.125$ (lower left to upper right) ($\Lambda_{(5)} = 64.589, 93.105, 129.580$ MeV) and b) α_S^{-1} *vs.* μ for $\alpha_S(m_Z) = 0.113, 0.119, 0.125$ (upper left to lower right).

3 The Running Coupling at Two Loop Order

We solve the differential equation at two loop order.

$$\mu\frac{\partial\alpha_S(\mu)}{\partial\mu} = -\beta_0\frac{\alpha_S^2(\mu)}{2\pi} - \beta_1\frac{\alpha_S^3(\mu)}{8\pi^2} \tag{25}$$

$$\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'} = -2\pi\int_{\alpha_S(\mu_0)}^{\alpha_S(\mu)}\frac{d\alpha_S}{\alpha_S^2\left(\beta_0+\beta_1\frac{\alpha_S}{4\pi}\right)} \tag{26}$$

The integral on the right-hand side is solved easily by partial fractions.

$$\frac{-2\pi}{\alpha_S^2\left(\beta_0+\beta_1\frac{\alpha_S}{4\pi}\right)} = \frac{-2\pi}{\beta_0\alpha_S^2} + \frac{\beta_1}{2\beta_0^2\alpha_S} - \frac{\beta_1^2}{2\beta_0^2(\alpha_S\beta_1+4\pi\beta_0)} \tag{27}$$

$$\ln\left(\frac{\mu}{\mu_0}\right) = \left.\frac{4\pi\beta_0+\alpha_S\beta_1\ln(\alpha_S)-\alpha_S\beta_1\ln(\alpha_S\beta_1+4\pi\beta_0)}{2\alpha_S\beta_0^2}\right|_{\alpha_S(\mu_0)}^{\alpha_S(\mu)} \tag{28}$$

$$\ln\left(\frac{\mu}{\mu_0}\right) = \frac{2\pi}{\beta_0}\left[\frac{1}{\alpha_S(\mu)}-\frac{1}{\alpha_S(\mu_0)}\right] + \frac{\beta_1}{2\beta_0^2}\ln\left[\frac{\alpha_S(\mu)}{\alpha_S(\mu_0)}\right] - \frac{\beta_1}{2\beta_0^2}\ln\left[\frac{\alpha_S(\mu)\beta_1+4\pi\beta_0}{\alpha_S(\mu_0)\beta_1+4\pi\beta_0}\right] \tag{29}$$

$$\alpha_S^{-1}(\mu) = \frac{1}{\alpha_S(\mu_0)} + \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right) + \frac{\beta_1}{4\pi\beta_0}\ln\left[\frac{\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu)}}{\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu_0)}}\right] \tag{30}$$

We can collect all the constant terms into a two loop Λ

$$\begin{aligned}
\alpha_S^{-1}(\mu) &= \frac{\beta_0}{4\pi}\ln\left[e^{\frac{1}{\alpha_S(\mu_0)}\frac{4\pi}{\beta_0}}\right] + \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right) - \frac{\beta_1}{4\pi\beta_0}\ln\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu_0)}\right] + \frac{\beta_1}{4\pi\beta_0}\ln\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu)}\right] \\
&= \frac{\beta_0}{4\pi}\ln\left[\frac{\mu^2}{\mu_0^2 e^{-\frac{4\pi}{\beta_0\alpha_S(\mu_0)}}}\right] + \frac{\beta_0}{4\pi}\ln\left\{\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu_0)}\right]^{-\frac{\beta_1}{\beta_0^2}}\right\} + \frac{\beta_1}{4\pi\beta_0}\ln\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu)}\right] \\
&= \frac{\beta_0}{4\pi}\ln\left\{\frac{\mu^2}{\mu_0^2 e^{-\frac{4\pi}{\beta_0\alpha_S(\mu_0)}}\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu_0)}\right]^{\frac{\beta_1}{\beta_0^2}}}\right\} + \frac{\beta_1}{4\pi\beta_0}\ln\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu)}\right]
\end{aligned} \tag{31}$$

$$\alpha_S^{-1}(\mu) = \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\Lambda^2}\right) + \frac{\beta_1}{4\pi\beta_0}\ln\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu)}\right] \tag{32}$$

where

$$\Lambda \equiv \mu_0 e^{-\frac{2\pi}{\beta_0\alpha_S(\mu_0)}}\left[\beta_1+\frac{4\pi\beta_0}{\alpha_S(\mu_0)}\right]^{\frac{\beta_1}{2\beta_0^2}} \tag{33}$$

However, this is not the usual choice for Λ. The reason is that in the series solution of the implicit function of α_S, Eq. 32, obtained by iteration, in terms of logarithms of Λ and μ

$$\alpha_S(\mu) = \frac{4\pi}{\beta_0\ln\left(\frac{\mu^2}{\Lambda^2}\right)} - \frac{4\pi\beta_1}{\beta_0^3}\frac{\ln\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\ln^2\left(\frac{\mu^2}{\Lambda^2}\right)} - \frac{4\pi\beta_1}{\beta_0^3}\frac{\ln\left(\beta_0^2\right)}{\ln^2\left(\frac{\mu^2}{\Lambda^2}\right)} + \mathcal{O}\left[\ln^{-3}\left(\frac{\mu^2}{\Lambda^2}\right)\right] \tag{34}$$

there are terms proportional to

$$\frac{\text{constant}}{\ln^2\left(\frac{\mu^2}{\Lambda^2}\right)} \tag{35}$$

The following choice for the two loop Λ eliminates these terms [BBDM].

$$\Lambda = \mu_0 e^{-\frac{2\pi}{\beta_0 \alpha_S(\mu_0)}} \left[\beta_1 + \frac{4\pi\beta_0}{\alpha_S(\mu_0)}\right]^{\frac{\beta_1}{2\beta_0^2}} (\beta_0)^{-\frac{\beta_1}{\beta_0^2}} = \mu_0 e^{-\frac{2\pi}{\beta_0 \alpha_S(\mu_0)}} \left[\frac{\beta_1}{\beta_0^2} + \frac{4\pi}{\beta_0\alpha_S(\mu_0)}\right]^{\frac{\beta_1}{2\beta_0^2}} \tag{36}$$

The exact value of the two loop $\Lambda_{(5)}$ is

$$\Lambda_{(5)} = m_Z \exp\left[\frac{-2\pi}{\beta_0^{(5)}\alpha_S(m_Z)}\right] \cdot \left\{\frac{\beta_1^{(5)}}{\left[\beta_0^{(5)}\right]^2} + \frac{4\pi}{\beta_0\alpha_S(m_Z)}\right\}^{\frac{\beta_1^{(5)}}{2\left[\beta_0^{(5)}\right]^2}} \approx 224.026 \text{ MeV} \tag{37}$$

The analog of Eq. 32 is now

$$\begin{aligned} \alpha_S^{-1}(\mu) &= \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\Lambda^2}\right) + \frac{\beta_1}{4\pi\beta_0}\ln\left[\beta_1 + \frac{4\pi\beta_0}{\alpha_S(\mu)}\right] - \frac{\beta_1}{4\pi\beta_0}\ln\left(\beta_0^2\right) \\ &= \frac{\beta_0}{4\pi}\ln\left(\frac{\mu^2}{\Lambda^2}\right) + \frac{\beta_1}{4\pi\beta_0}\ln\left[\frac{\beta_1}{\beta_0^2} + \frac{4\pi}{\beta_0\alpha_S(\mu)}\right] \end{aligned} \tag{38}$$

again an implicit function of α_S and the new expansion is

$$\alpha_S(\mu) = \frac{4\pi}{\beta_0\ln\left(\frac{\mu^2}{\Lambda^2}\right)} - \frac{4\pi\beta_1}{\beta_0^3}\frac{\ln\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\ln^2\left(\frac{\mu^2}{\Lambda^2}\right)} + \mathcal{O}\left[\ln^{-3}\left(\frac{\mu^2}{\Lambda^2}\right)\right] \tag{39}$$

Of course, we can expand further

$$\begin{aligned} \alpha_S(\mu) &= \frac{4\pi}{\beta_0\ln\left(\frac{\mu^2}{\Lambda^2}\right)} - \frac{4\pi\beta_1}{\beta_0^3}\frac{\ln\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\ln^2\left(\frac{\mu^2}{\Lambda^2}\right)} + \frac{4\pi\beta_1^2}{\beta_0^5\ln^3\left(\frac{\mu^2}{\Lambda^2}\right)}\left[\ln^2\ln\left(\frac{\mu^2}{\Lambda^2}\right) - \ln\ln\left(\frac{\mu^2}{\Lambda^2}\right) - 1\right] \\ &\quad + \mathcal{O}\left[\ln^{-4}\left(\frac{\mu^2}{\Lambda^2}\right)\right] \end{aligned} \tag{40}$$

but the β_2 coefficient also enters at $\mathcal{O}\left[\ln^{-3}\left(\frac{\mu^2}{\Lambda^2}\right)\right]$ (see Eq. **??**), so the theoretical error can not be reduced to zero by expanding α_S sufficiently beyond the first two terms.

With the two loop $\alpha_S(m_Z) = 0.119$, the two loop $\Lambda_{(5)}$ is found numerically from the two term expansion (Eq. 39) to be

$$\Lambda_{(5)} \approx 239.178 \text{ MeV} \tag{41}$$

This differs from the exact two loop $\Lambda_{(5)}$ (Eq. 37) because the two term series approximation (Eq. 39) differs from the exact two loop α_S.

Now we want the two loop α_S from Eq. 39 to match at the quark mass thresholds.

$$\lim_{\epsilon\to 0}\left[\alpha_S(m_b - \epsilon) - \alpha_S(m_b + \epsilon)\right] = 0 \tag{42}$$

again determines $\Lambda_{(4)}$

$$\Lambda_{(4)} \approx 346.423 \text{ MeV} \tag{43}$$

and similarly

$$\Lambda_{(3)} \approx 403.729 \text{ MeV} \tag{44}$$

$$\Lambda_{(6)} \approx \ \ 97.228 \text{ MeV} \tag{45}$$

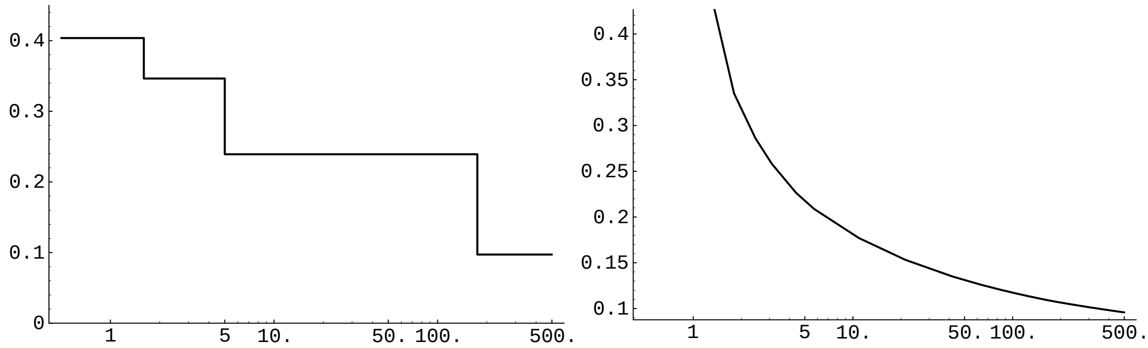

Figure 4: At two loop order, a) Λ *vs.* μ and b) α_S *vs.* μ. Notice that the coupling is continuous across the quark mass thresholds.

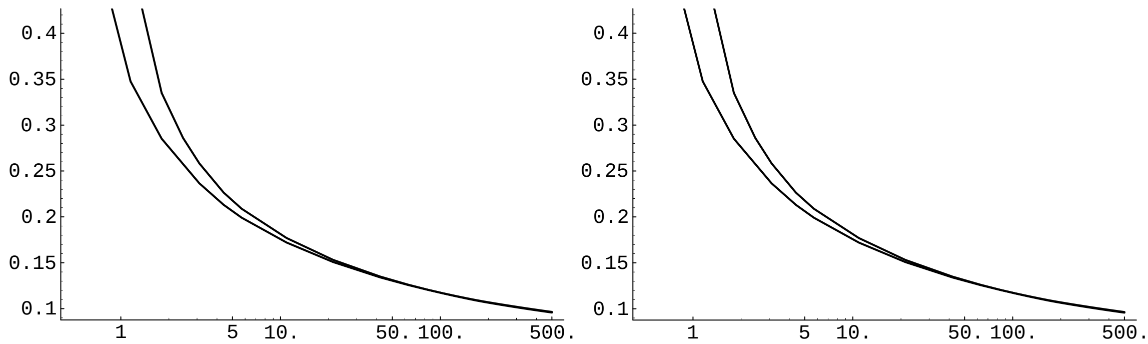

Figure 5: a) A comparison of α_S at one loop (lower) and two loop (upper) orders, and b) detail showing the intersection at $\mu = m_Z$.

We now compare the two-term (Eq. 39) and three-term (Eq. 40) expansions of the two loop α_S. Using the three-term expansion Eq. 40, the matching conditions give:

$$\begin{aligned}\Lambda_{(3)} &\approx 431.271\text{ MeV}\\ \Lambda_{(4)} &\approx 330.936\text{ MeV}\\ \Lambda_{(5)} &\approx 224.929\text{ MeV}\\ \Lambda_{(6)} &\approx 93.585\text{ MeV}\end{aligned} \tag{46}$$

Now $\Lambda_{(5)}$ is closer to the exact two loop value (Eq. 37); the three term series (Eq. 40) is a better approximation to the exact two loop α_S.

4 The Running Coupling at Three Loop Order

We solve the differential equation at three loop order.

$$\mu\frac{\partial\alpha_S(\mu)}{\partial\mu} = -\beta_0\frac{\alpha_S^2(\mu)}{2\pi} - \beta_1\frac{\alpha_S^3(\mu)}{8\pi^2} - \beta_2\frac{\alpha_S^4(\mu)}{32\pi^3} \tag{47}$$

$$\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'} = -2\pi\int_{\alpha_S(\mu_0)}^{\alpha_S(\mu)}\frac{d\alpha_S}{\alpha_S^2\left(\beta_0+\beta_1\frac{\alpha_S}{4\pi}+\beta_2\frac{\alpha_S^2}{16\pi^2}\right)} \tag{48}$$

The integral can be solved exactly by partial fractions, leading to another implicit equation for α_S. The result is unenlightening. We take a different tack – expand the integrand for small α_S.

$$\frac{-2\pi}{\alpha_S^2\left(\beta_0+\beta_1\frac{\alpha_S}{4\pi}+\beta_2\frac{\alpha_S^2}{16\pi^2}\right)} = \frac{-2\pi}{\beta_0\alpha_S^2}+\frac{\beta_1}{2\beta_0^2\alpha_S}+\frac{\beta_0\beta_2-\beta_1^2}{8\pi\beta_0^3}+\frac{(\beta_1^3-2\beta_0\beta_1\beta_2)\alpha_S}{32\pi^2\beta_0^4}+\mathcal{O}(\alpha_S^2) \tag{49}$$

Now integrate

$$\ln\left(\frac{\mu}{\mu_0}\right) = \left[\frac{2\pi}{\beta_0\alpha_S}+\frac{\beta_1\ln(\alpha_S)}{2\beta_0^2}+\frac{(\beta_0\beta_2-\beta_1^2)\alpha_S}{8\pi\beta_0^3}+\frac{(\beta_1^3-2\beta_0\beta_1\beta_2)\alpha_S^2}{64\pi^2\beta_0^4}+\mathcal{O}(\alpha_S^3)\right]\Bigg|_{\alpha_S(\mu_0)}^{\alpha_S(\mu)} \tag{50}$$

The three loop lambda is defined [CKS] to be

$$\Lambda = \mu_0 e^{-\frac{2\pi}{\beta_0\alpha_S(\mu_0)}}\left[\frac{4\pi}{\beta_0\alpha_S(\mu_0)}\right]^{\frac{\beta_1}{2\beta_0^2}}\exp\left[\frac{(\beta_1^2-\beta_0\beta_2)\alpha_S(\mu_0)}{8\pi\beta_0^3}\right] \tag{51}$$

and the exact three loop $\Lambda_{(5)}$ is

$$\begin{aligned}\Lambda_{(5)} &= m_Z\exp\left[\frac{-2\pi}{\beta_0^{(5)}\alpha_S(m_Z)}\right]\cdot\left[\frac{4\pi}{\beta_0^{(5)}\alpha_S(m_Z)}\right]^{\frac{\beta_1^{(5)}}{2\left[\beta_0^{(5)}\right]^2}}\exp\left(\frac{\left\{\left[\beta_1^{(5)}\right]^2-\beta_0^{(5)}\beta_2^{(5)}\right\}\alpha_S(m_Z)}{8\pi\left[\beta_0^{(5)}\right]^3}\right)\\ &\approx 220.865\text{ MeV}\end{aligned} \tag{52}$$

5 The Running Coupling at Four Loop Order

We solve the differential equation at four loop order.

$$\mu \frac{\partial \alpha_S(\mu)}{\partial \mu} = -\beta_0 \frac{\alpha_S^2(\mu)}{2\pi} - \beta_1 \frac{\alpha_S^3(\mu)}{8\pi^2} - \beta_2 \frac{\alpha_S^4(\mu)}{32\pi^3} - \beta_3 \frac{\alpha_S^5(\mu)}{128\pi^4} \tag{53}$$

$$\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} = -2\pi \int_{\alpha_S(\mu_0)}^{\alpha_S(\mu)} \frac{d\alpha_S}{\alpha_S^2 \left(\beta_0 + \beta_1 \frac{\alpha_S}{4\pi} + \beta_2 \frac{\alpha_S^2}{16\pi^2} + \beta_3 \frac{\alpha_S^3}{64\pi^3}\right)} \tag{54}$$

expand the integrand for small α_S.

$$\begin{aligned} \frac{-2\pi}{\alpha_S^2 \left(\beta_0 + \beta_1 \frac{\alpha_S}{4\pi} + \beta_2 \frac{\alpha_S^2}{16\pi^2} + \beta_3 \frac{\alpha_S^3}{64\pi^3}\right)} \quad &= \quad \frac{-2\pi}{\beta_0 \alpha_S^2} + \frac{\beta_1}{2\beta_0^2 \alpha_S} + \frac{\beta_0\beta_2 - \beta_1^2}{8\pi\beta_0^3} \\ &+ \frac{(\beta_0^2\beta_3 - 2\beta_0\beta_1\beta_2 + \beta_1^3)\alpha_S}{32\pi^2\beta_0^4} + \mathcal{O}(\alpha_S^2) \end{aligned} \tag{55}$$

Now integrate

$$\ln\left(\frac{\mu}{\mu_0}\right) = \left[\frac{2\pi}{\beta_0 \alpha_S} + \frac{\beta_1 \ln(\alpha_S)}{2\beta_0^2} + \frac{(\beta_0\beta_2 - \beta_1^2)\alpha_S}{8\pi\beta_0^3} + \frac{(\beta_0^2\beta_3 - 2\beta_0\beta_1\beta_2 + \beta_1^3)\alpha_S^2}{64\pi^2\beta_0^4} + \mathcal{O}(\alpha_S^3)\right]\Bigg|_{\alpha_S(\mu_0)}^{\alpha_S(\mu)} \tag{56}$$

The four loop lambda is defined [CKS] to be

$$\Lambda = \mu_0 e^{-\frac{2\pi}{\beta_0 \alpha_S(\mu_0)}} \left[\frac{4\pi}{\beta_0 \alpha_S(\mu_0)}\right]^{\frac{\beta_1}{2\beta_0^2}} \exp\left[\frac{(\beta_1^2 - \beta_0\beta_2)\alpha_S(\mu_0)}{8\pi\beta_0^3}\right] \exp\left[\frac{(-\beta_0^2\beta_3 + 2\beta_0\beta_1\beta_2 - \beta_1^3)\alpha_S^2(\mu_0)}{64\pi^2\beta_0^4}\right] \tag{57}$$

and the exact four loop $\Lambda_{(5)}$ is

$$\begin{aligned} \Lambda_{(5)} \quad &= \quad m_Z \exp\left[\frac{-2\pi}{\beta_0^{(5)} \alpha_S(m_Z)}\right] \cdot \left[\frac{4\pi}{\beta_0^{(5)} \alpha_S(m_Z)}\right]^{\frac{\beta_1^{(5)}}{2\left[\beta_0^{(5)}\right]^2}} \exp\left(\frac{\left\{\left[\beta_1^{(5)}\right]^2 - \beta_0^{(5)}\beta_2^{(5)}\right\} \alpha_S(m_Z)}{8\pi \left[\beta_0^{(5)}\right]^3}\right) \\ &\cdot \exp\left(\frac{\left\{-\left[\beta_0^{(5)}\right]^2 \beta_3^{(5)} + 2\beta_0^{(5)}\beta_1^{(5)}\beta_2^{(5)} - \left[\beta_1^{(5)}\right]^3\right\} \alpha_S^2(m_Z)}{64\pi^2 \left[\beta_0^{(5)}\right]^4}\right) \approx 220.530 \text{ MeV} \end{aligned} \tag{58}$$

The expansion is

$$
\begin{aligned}
\alpha_S(\mu) \;=\; & \frac{4\pi}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} - \frac{4\pi\beta_1}{\beta_0^3}\frac{\ln\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\ln^2\left(\frac{\mu^2}{\Lambda^2}\right)} + \frac{4\pi\beta_1^2}{\beta_0^5 \ln^3\left(\frac{\mu^2}{\Lambda^2}\right)}\left[\ln^2\ln\left(\frac{\mu^2}{\Lambda^2}\right) - \ln\ln\left(\frac{\mu^2}{\Lambda^2}\right) + \frac{\beta_2\beta_0}{\beta_1^2} - 1\right] \\
& + \frac{4\pi\beta_1^3}{\beta_0^7 \ln^4\left(\frac{\mu^2}{\Lambda^2}\right)}\left[-\ln^3\ln\left(\frac{\mu^2}{\Lambda^2}\right) + \frac{5}{2}\ln^2\ln\left(\frac{\mu^2}{\Lambda^2}\right) + 2\ln\ln\left(\frac{\mu^2}{\Lambda^2}\right) - \frac{1}{2}\right. \\
& \left. - \frac{3\beta_2\beta_0 \ln\ln\left(\frac{\mu^2}{\Lambda^2}\right)}{\beta_1^2} + \frac{\beta_3\beta_0^2}{2\beta_1^3}\right] + \mathcal{O}\left[\ln^{-5}\left(\frac{\mu^2}{\Lambda^2}\right)\right] \qquad (59)
\end{aligned}
$$

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Errata

- [Weinberg]:

    1. Eq. (18.7.11) The last term should be $+\frac{325}{27}n_f^2$

2. Eq. (18.7.12) The middle term in the second line should be $\frac{\beta_2\beta_0}{8\beta_1^2}$

- [Sterman]:

  1. Eq. (12.92) The order of arithmetical operations is not clear. The argument of the logarithm is $\left[\frac{-a\beta_1}{4\left(1+\frac{\beta_2 a}{4\beta_1}\right)}\right]$