An Introduction to Resummation in Perturbative QCD

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Outline

- Review of NLO Calculations
  - Regularization of divergences
  - Cancellation between infrared and soft divergences
  - Factorization of remaining collinear divergences
- Use Lepton Pair Production as an example
  - $d\sigma/dQ^2$ versus $d\sigma/dQ^2\,dp_T$
  - NLO p_T distribution
- Two scale problems - Large logs and the need for resummation
- Leading double log approximation
- k_T resummation – going beyond leading double logs
- Threshold resummation - an overview
- Applications
- Summary

NLO Matrix Element Overview

If the lowest order subprocess has an n-body final state, then at the next order we have

- n-body final state one-loop diagrams. The interference between these and the lowest order diagrams gives a cross section contribution that is one order higher in α_s.
- $n+1$-body final state contributions

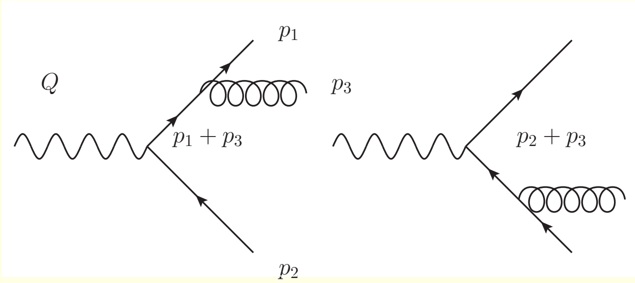
When one tries to calculate these higher order terms one finds:

- Infrared (IR), collinear, and ultraviolet (UV) singularities from virtual diagrams
- Soft singularities from some $n+1$-body processes
- Collinear singularities from some regions of the $n+1$ phase space

Start by considering the relatively simple process of e^+e^- annihilation.

e^+e^- annihilation

First, consider the $2 \to 3$ $e^+e^- \to q\bar{q}g$ subprocess. Actually, it is easier to consider the decay of a virtual photon of 4-momentum Q as shown below:



- Kinematics - use massless quarks and gluons.
- Define $x_i = 2E_i/Q, i = 1, 2, 3$ in the overall center-of-mass system where Q denotes the total energy $\Rightarrow x_1 + x_2 + x_3 = 2$.
- $(p_1 + p_3)^2 = 2p_1 \cdot p_3 = (Q - p_2)^2 = Q^2(1 - x_2)$
- $(p_2 + p_3)^2 = 2p_2 \cdot p_3 = (Q - p_1)^2 = Q^2(1 - x_1)$
- The quark propagators from the above diagrams will give factors of $(1-x_1)$ and $(1-x_2)$ in the denominator. $x_1 \to 1$ corresponds to $\vec{p}_3 \parallel \vec{p}_2$ while $x_2 \to 1$ corresponds to $\vec{p}_3 \parallel \vec{p}_1$. Note that if both $x_1$and $x_2 \to 1$ then $x_3 \to 0$.

3-body Phase Space

Exercise: Show that

$$
\begin{aligned}
dPS_3 &= \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta(Q-p_1-p_2-p_3) \\
&= \frac{Q^2}{16(2\pi)^3}\,dx_1\,dx_2
\end{aligned}
$$

Using this result it is straightforward to show that the differential cross section can be written as

$$\frac{1}{\sigma}\frac{d\sigma}{dx_1\,dx_2} = C_F\frac{\alpha_s}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

For the total cross section, one should integrate over both x_1 and x_2. These integrations diverge when either x_1 or x_2 or both approach unity.

Partial fraction the denominators:

$$\frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left(\frac{1}{(1-x_1)} + \frac{1}{(1-x_2)}\right)$$

- This shows that the double pole when both x_1 and x_2 approach unity is due to a combination of a collinear divergence (x_1 or $x_2 \to 1$) and a soft divergence ($x_3 \to 0$).
- The problem now is how to generate a finite contribution to the total cross section.
- We shall use dimensional regularization
  - Analytically continue in the number of dimensions from $n = 4$ to $n = 4 - 2\epsilon$.
  - For the soft and collinear singularities we will take $\epsilon < 0$
  - Converts logarithmic divergences into poles in ϵ.
  - Note: we will use the substitution $g_s \to g_s\mu^\epsilon$ in order for the strong coupling to remain dimensionless in n dimensions

Phase space becomes

$$dPS_3^n = \frac{Q^2}{16(2\pi)^3}\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}\left(\frac{1-u^2}{4}\right)^{-\epsilon}\frac{1}{\Gamma(2-2\epsilon)}x_1^{-2\epsilon}dx_1\, x_2^{-2\epsilon}dx_2$$

where $u = 1 - \frac{2(1-x_1-x_2)}{x_1 x_2}$

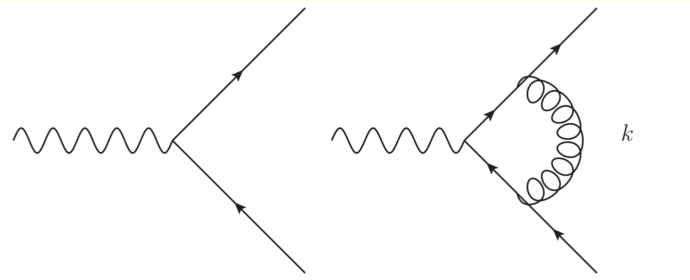
- It is not obvious how this helps until you make a substitution $x_2 = 1 - v x_1$
- The u dependent term introduces factors of $(1-v)^{-\epsilon}$ and $(1-x_1)^{-\epsilon}$
- dx_2 becomes $x_1 dv$
- Then note that

$$\int_0^1 dx(1-x)^{-1-\epsilon} = \frac{1}{-\epsilon}(1-x)^{-\epsilon}|_0^1 = \frac{1}{-\epsilon}$$

  as long as $\epsilon < 0$.

- The logarithmic divergence has, indeed, been converted into a pole in ϵ.

$2 \to 2$ contribution



- The loop graph is $\mathcal{O}(\alpha_s)$ so the interference with the lowest order term gives an $\mathcal{O}(\alpha_s)$ contribution to the cross section
- The loop integral has a denominator of the form: $k^2(p_1+k)^2(p_2-k)^2$
- The denominator vanishes when $k \to 0$ or when k is collinear with either p_1 or p_2
- These singularities correspond to the same types as observed for the $q\bar{q}g$ final state
- Can also use dimensional regularization to evaluate the loop contribution in n-dimensions

Final Results

- After doing both of the integrations for the three-body , one arrives at

$$\sigma_3 = \frac{\alpha_s}{2\pi} C_F \sigma_0 \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right]$$

  where σ_0 is the lowest order result.

- After doing the loop integral for the virtual contribution one gets

$$\sigma_v = \frac{\alpha_s}{2\pi} C_F \sigma_0 \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right]$$

- Adding the two together along with the lowest order result yields

$$\sigma = \sigma_0(1 + \frac{\alpha_s}{\pi})$$

- The poles in ϵ have all cancelled, leaving a finite higher order correction

- In the previous example the integration over the full phase space for the real emission generated poles which cancelled against corresponding poles from the loop graphs
- What if the integration over the real emission phase space was limited?
- What if the integration involved fragmentation functions (FFs) or PDFs whose analytic form was not known?
- We might encounter terms like

$$\int_z^1 dx_1\, x_1^{n-1}\, (1-x_1)^{m-1} D_{h/q}(z/x_1)$$

- Note the non-zero lower limit on the integral which is forced by the argument of the FF
- How are we to do this integral in order to pull out the singular terms when we don't know the analytic form for the FF?
- Enter the "+" distribution!
- This distribution will enable us to extract the poles in ϵ from integrals of the above form

Consider

$$\begin{aligned}
I &= \int_0^1 dw\,(1-w)^{-1-\epsilon}\,f(w) \\
&= \int_0^1 dw\,(1-w)^{-1-\epsilon}\,[f(1)+(f(w)-f(1))] \\
&= -\frac{f(1)}{\epsilon}+\int_0^1 dw\,\frac{f(w)-f(1)}{1-w}\,\left[1-\epsilon\ln(1-w)+\mathcal{O}(\epsilon^2)\right] \\
&= -\frac{f(1)}{\epsilon}+\int_0^1 dw\,\frac{f(w)-f(1)}{1-w}-\epsilon\int_0^1 dw\,\frac{\ln(1-w)}{1-w}\,[f(w)-f(1)]+\mathcal{O}(\epsilon^2) \\
&\equiv -\frac{f(1)}{\epsilon}+\int_0^1 dw\,\frac{f(w)}{(1-w)_+}-\epsilon\int_0^1 dw\,\left(\frac{\ln(1-w)}{1-w}\right)_+ f(w)+\mathcal{O}(\epsilon^2)
\end{aligned}$$

This last expression allows us to make the following identification

$$(1-w)^{-1-\epsilon} = -\frac{\delta(1-w)}{\epsilon}+\frac{1}{(1-w)_+}-\epsilon\left(\frac{\ln(1-w)}{1-w}\right)_+$$

- The astute reader will no doubt have noticed that the previous derivation involved integrals extending from zero to one. What if the lower limit is non-zero?
- The derivation can be repeated and the only difference will be in the δ-function term. There we will get (recall that $\epsilon < 0$)

$$
\begin{aligned}
\frac{1}{\epsilon}(1-w)^{-\epsilon}\Big|_a^1 &= -\frac{1}{\epsilon}(1-a)^{-\epsilon} \\
&= -\frac{1}{\epsilon}\left[1-\epsilon\ln(1-a)+\frac{\epsilon^2}{2}\ln^2(1-a)+\ldots\right] \\
&= -\frac{1}{\epsilon}+\ln(1-a)-\frac{\epsilon}{2}\ln^2(1-a)+\ldots
\end{aligned}
$$

- The regulators under the integral signs behave the same way as when the lower limit was zero.

- Schematically we can write

$$\frac{1}{(1-w)_+} = \frac{1}{(1-w)_a} + \ln(1-a)\delta(1-w)$$

and

$$\left(\frac{\ln(1-w)}{1-w}\right)_+ = \left(\frac{\ln(1-w)}{1-w}\right)_a + \frac{1}{2}\ln^2(1-a)\delta(1-w)$$

There are several important points to notice about these regulators

- We derived these expressions by adding and subtracting $f(1)$ and then rearranging the integrations. When the lower limit is non-zero, the cancellation between these two terms with $f(1)$ is no longer exact and there is a remainder involving logs of $(1-a)$
- As the lower limit, a, approaches 1 these logs can become large.
- This could happen with the fragmentation functions if we were interested in the region of large z.

- These logs are called "threshold" logs and physically what is happening is that the phase space for additional gluon radiation is being limited by the requirement that z be large. These large logs must be resummed via a procedure referred to as "soft gluon" or "threshold" resummation
- Remember the idea of incomplete cancellation between the virtual and real contributions with a finite remainder consisting of potentially large logarithms

When is an NLO Calculation not really NLO?

Consider the example of Lepton Pair Production where a calculation of $d\sigma/dQ^2$ would entail

- LO $q\overline{q} \to l^+l^-$
- NLO $q\overline{q} \to l^+l^-$ (1-loop)
- NLO $q\overline{q} \to l^+l^-g$ and $qg \to l^+l^-q$ (tree graphs)
- Integrate over the additional variables for the radiated gluon or quark in the case of the $2 \to 3$ subprocesses
- Factorize the resulting collinear singularities and absorb them into the definition of scale dependent PDFs
- PDFs with scale $M_f = Q$ interpreted as having the effects of radiated partons with p_Ts up to Q included

But what if we wanted to calculate $d\sigma/dQ^2\, dp_T$?

- The lowest order and the 1-loop virtual contributions are calculated with lowest order kinematics - at this stage the lepton pair has no p_T at the matrix element level
- The $\mathcal{O}(\alpha_S)$ tree graphs give the first non-zero lepton p_T at this stage of the calculation
- *But*, the graphs are convoluted with the bare PDFs

OK - what if we just change the bare PDFs to the scale dependent PDFs?

- Hmmm. We would then be including the effects of radiated parton p_Ts up to the value of the scale chosen for the PDFs
- But we are, at the same time, examining the p_T of the lepton pair which recoils against the radiated partons. Is there an inconsistency here?

No - not if we are content to calculate the p_T distribution at values of p_T which are of the order of Q

Lesson: the tree graphs which, after integrating over the recoiling parton p_T, contributed to the $\mathcal{O}(\alpha_S)$ correction for $d\sigma/dQ^2$ are now giving the LO contribution to the high Q tail of the p_T distribution

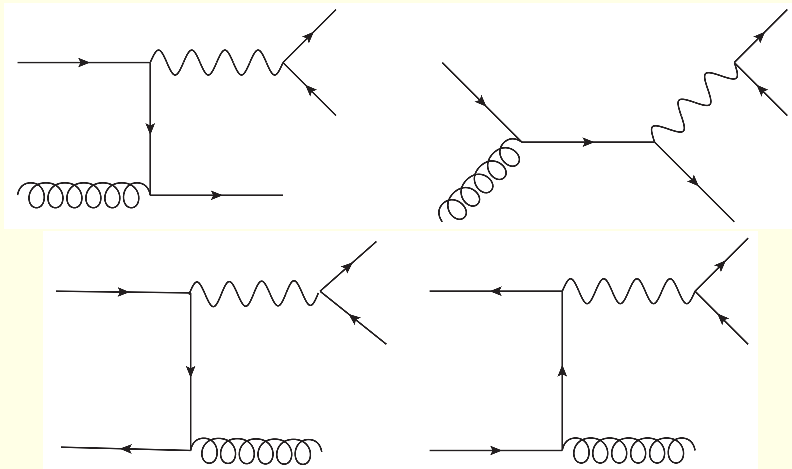
What if we wanted the NLO p_T distribution? We would have to do more!

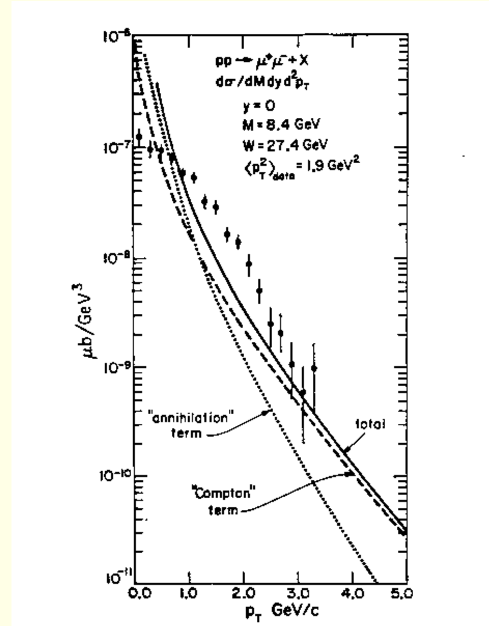
- Include the 1-loop corrections to the $\mathcal{O}(\alpha_s)$ tree graphs
- Also include the $\mathcal{O}(\alpha_s^2)$ tree graphs
- With these ingredients one could generate NLO predictions for the p_T distribution in the region where p_T is of the order of Q

But what if one was interested in the region where $p_T << Q$?

There are several problems

- As noted before, the scale dependent PDFs contain the contributions from integrating the radiated parton p_Ts up to $\mathcal{O}(Q)$ so there is a contradiction if we ask for the p_T of the lepton pair to be much less than Q
- There are now two scales in the problem - p_T and Q and if $p_T << Q$ one can encounter large logs of the ratio Q/p_T which should be resummed





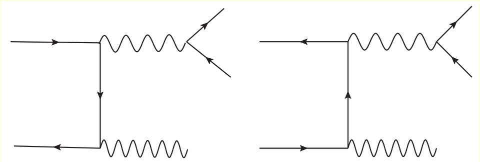
The $\mathcal{O}(\alpha_S)$ subprocesses both give contributions which diverge as p_T^{-2} as p_T goes to zero
These divergent terms are factorized and included in the scale dependent PDFs

We want to do a better calculation in the low p_T region

- Have to figure out what to do with the low p_T radiated partons
- Have to figure out what the scale should be for the PDFs

Simple Example - $e^+e^- \to l^+l^- + \gamma$

- Discussion follows G. Parisi and R. Petronzio, Nucl. Phys. B154, 427 (1979)
- Avoids the complications due to the non-abelian nature of QCD and initial state PDFs, but illustrates the physics



- Cross section for fixed lepton pair mass Q diverges as k_T^{-2}, where k_T is the transverse momentum of the radiated photon
- For $k_T << Q$ we get

$$\frac{d\sigma}{dQ^2\, dk_T^2} = \frac{4\alpha^3}{3k_T^2} \frac{1}{SQ^2} \frac{S^2 + Q^4}{S - Q^2}$$

- Next, integrate over Q: $4m_\mu^2 < Q^2 < S - 2\sqrt{S}k_T$

- Letting $\sigma_0 = \frac{4}{3}\frac{\pi\alpha^2}{S}$ and performing the integration over Q^2 yields (keeping only the most divergent term)

$$\frac{d\sigma}{dk_T^2} = \sigma_0 \frac{\alpha}{\pi} \frac{\ln S/k_T^2}{k_T^2}$$

- Next, consider a partially integrated cross section defined as

$$\Sigma(k_T^2) = \frac{1}{\sigma_0} \int_0^{k_T^2} \frac{d\sigma}{dp_T^2}\, dp_T^2$$

- We know that there is a divergence at $p_T = 0$. But, we also know that the one loop corrections to our tree graphs will also contribute there and that if we were to integrate over all p_T we would get a finite result.
- To logarithmic accuracy we can write

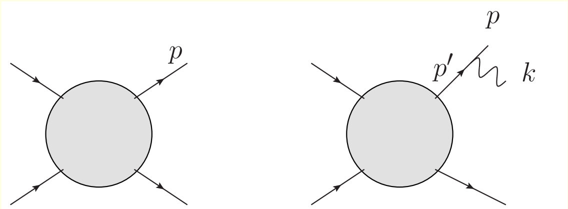
$$\frac{1}{\sigma_0} \int_0^{S} \frac{d\sigma}{dp_T^2}\, dp_T^2 = 1 + \mathcal{O}(\alpha) \times \text{constant} = \frac{1}{\sigma_0} \int_0^{k_T^2} \ldots + \frac{1}{\sigma_0} \int_{k_T^2}^{S} \ldots$$

- Therefore, we can write

$$
\begin{aligned}
\Sigma(k_T^2) &= 1 - \frac{1}{\sigma_0}\int_{k_T^2}^{S} \frac{d\sigma}{dp_T^2}\, dp_T^2 \\
&= 1 - \frac{\alpha}{\pi}\int_{k_T^2}^{S} \frac{dp_T^2}{p_T^2} \ln\frac{S}{p_T^2} \\
&= 1 - \frac{\alpha}{2\pi}\ln^2\frac{S}{k_T^2}
\end{aligned}
$$

- Note that this is result is correct in the leading double log approximation. $\mathcal{O}(\alpha)$ terms which are constants or single logs are not included
- Now, what would happen if there were multiple photons emitted instead of just one?

Consider a process where there is a fermion in the final state and then compare it to one where there is a photon emitted from the fermion



$$\overline{u}(p) \rightarrow \overline{u}(p)\frac{\not{\epsilon}\not{p}'}{p'^2}$$

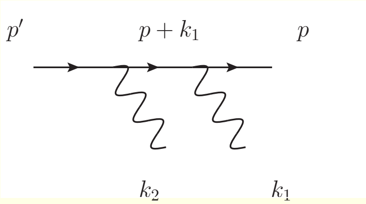
- Use $p' = p + k$ and use the fact that we are interested in soft photons - drop k everywhere except where it would lead to a divergence
- The factor associated with the photon emission is now

$$\overline{u}(p)\frac{\not{\epsilon}\not{p}}{2p \cdot k}$$

- Using the Dirac equation, this may be simplified to

$$\overline{u}(p)\,\frac{p\cdot\epsilon}{p\cdot k}$$

- Now, what about two soft photons?



- Repeat the above analysis and symmetrize the result by interchanging the two photons and dividing by two
- The result is a factor

$$\frac{1}{2}\,\frac{p\cdot\epsilon_1}{p\cdot k_1}\,\frac{p\cdot\epsilon_2}{p\cdot k_2}$$

- Similarly, for n terms one gets

$$\frac{1}{n!}\frac{p\cdot\epsilon_1}{p\cdot k_1}\cdots\frac{p\cdot\epsilon_n}{p\cdot k_n}$$

- Soft photon emission factorizes!
- For n emissions we get a contribution to the cross section

$$\frac{1}{\sigma_0}d\sigma=\frac{\alpha^n}{n!}dk_{T1}^2\ldots dk_{Tn}^2\nu(k_{T1})\ldots\nu(k_{T2})$$

- where $\nu(k_T)=\frac{\ln S/k_T^2}{k_T^2}$ is the result for a single photon
- We want to calculate the contribution of the n photon term to the integrated p_T distribution given by $\Sigma(k_T^2)$
- As a first attempt, ignore the correlations between the transverse momenta of the emitted photons - treat them all as being independent

- Then the n^{th} term is just

$$
\begin{aligned}
\Sigma^{(n)}(k_T^2) &= \frac{1}{n!}\left[\int_0^{k_T^2} dp_T^2\, \frac{\alpha}{\pi}\frac{\ln S/p_T^2}{p_T^2}\right]^n \\
&= \frac{1}{n!}\left[1 - \frac{\alpha}{2\pi}\frac{\ln^2 S/k_T^2}{k_T^2}\right]^n \\
&= \frac{(-1)^n}{n!}\left(\frac{\alpha}{2\pi}\ln^2 S/k_T^2\right)^n + \cdots
\end{aligned}
$$

- Summing over all n yields

$$\Sigma(k_T^2) = \exp(-\frac{\alpha}{2\pi}\ln^2 S/k_T^2)$$

Next, we can recover the differential cross section by taking a derivative of Σ

$$\begin{aligned}\frac{1}{\sigma_0}\frac{d\sigma}{dk_T^2} &= \frac{d}{dk_T^2}\Sigma(k_T^2)\\ &= \frac{\alpha}{\pi}\frac{\ln S/k_T^2}{k_T^2}\exp\left(-\frac{\alpha}{2\pi}\ln^2 S/k_T^2\right)\end{aligned}$$

- Notice that as $k_T \to 0$ the differential cross section now vanishes, rather than diverges
- Summing the leading double logs has tamed the divergence, but at the price of a vanishing cross section. The suppression is too strong, as we will see shortly

  (Exercise: fill in the steps for the derivation of this result)

Interpretation

- Σ is referred to as a Sudakov form factor and it can be interpreted as given the probability for emitting no photons with transverse momenta greater than k_T
- We enforced the independent emission hypothesis and neglected conservation of transverse momentum. The only way to get a lepton pair with zero transverse momentum, was to suppress *all* photon emission.
- The probability of emitting no photons in a collision which creates a massive lepton pair is zero
- The lowest order divergence is actually just the first term in an expansion of the exponential which vanishes at zero transverse momentum

How can we restore transverse momentum conservation?

Insert a δ function which enforces conservation of transverse momentum for the emission of n photons

$$\frac{1}{\sigma_0}\frac{d\sigma^n}{d^2p_T} = \frac{1}{n!}\left(\frac{\alpha}{\pi}\right)^n \int d^2k_{T_1}\dots d^2k_{T_n}\nu(k_{T_1})\dots\nu(k_{T_n})\delta^2(\vec{p}_T - \vec{k}_{T_1} - \dots - k_{T_n})$$

Next, use the Dirac representation of the δ function

$$\delta^2(\vec{p}_T - \vec{k}_{T_1} - \dots - k_{T_n}) = \frac{1}{(2\pi)^2}\int d^2b e^{-i\vec{b}\cdot(\vec{p}_T - \vec{k}_{T_1} - \dots - \vec{k}_{T_n})}$$

- Notice how the integrand still factorizes, even with the δ function included
- Define the Fourier transform of ν by

$$\tilde{\nu}(b) = \frac{1}{\pi}\int d^2k_T e^{i\vec{b}\cdot\vec{k}_T}\nu(k_T)$$

- The n photon emission contribution now looks like

$$\frac{1}{\sigma_0}\frac{d\sigma^n}{d^2p_T} = \frac{\alpha^n}{4\pi^2 n!}\int d^2b e^{-i\vec{b}\cdot\vec{p}_T}\left[\tilde{\nu}(b)\right]^n$$

- We see that the the exponentiation can now take place in impact parameter space

$$\frac{1}{\sigma_0}\frac{d\sigma}{d^2p_T} = \frac{1}{4\pi^2}\int d^2b e^{-i\vec{b}\cdot\vec{p}_T}\,\tilde{\sigma}(b)$$

where $\tilde{\sigma}(b) = \exp[\alpha\tilde{\nu}(b)]$

(Exercise: fill in the steps to derive this result)

Interpretation

- In the first case, the Sudakov form factor entered because we demanded that we approach zero transverse momentum of the lepton pair by limiting the transverse momenta of *all* the emitted photons individually
- By inserting the transverse momentum conserving delta function and exponentiating in impact parameter space, we allowed for the possibility of two or more photons balancing in transverse momentum and giving a zero result
- Formally, these terms are subleading, but the leading terms vanish and so the subleading terms become dominant

Extension to QCD

- This concept was extended to QCD by Collins, Soper, and Sterman (Nucl.Phys.B250,199(1985))

- Need to take into account the transverse momentum of the incoming quarks
  - Normally integrated over, leading to the scale dependence of the PDFs
  - Factorization scale usually chosen to be on the order of the single hard scale
  - Now, the lepton pair p_T will reflect the p_Ts of the incoming quarks
  - PDF scale is chosen to be of the order of $1/b$ where b is the impact parameter seen above
  - b and p_T are conjugate variables - large $p_T \leftrightarrow$ small b
  - A scale of $1/b$ is large for large p_T and small for small p_T
- Classic application is to the lepton pair, W, or Z p_T distributions

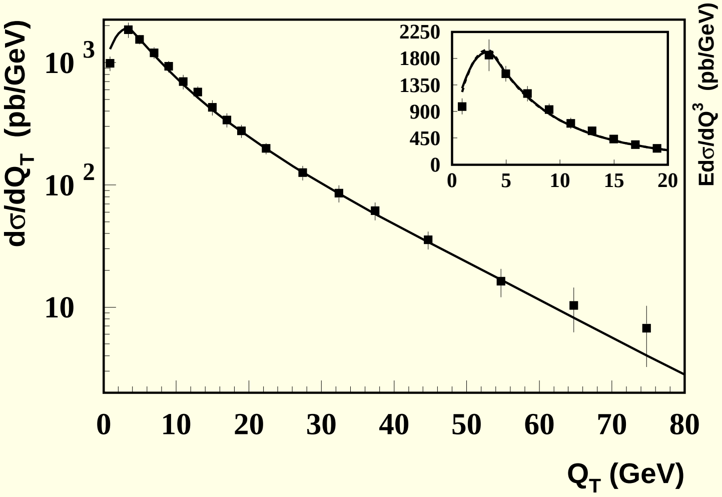
CSS Resummed Result

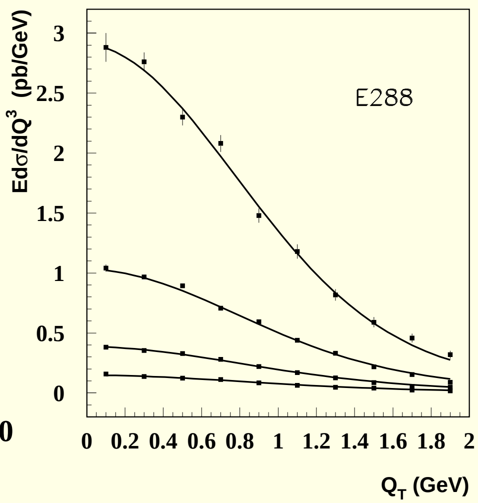
The resummed CSS result takes a relatively simple form with an exponentiation in impact parameter space and a convolution with PDFs evaluated at a scale $1/b$

$$
\begin{aligned}
\frac{d\sigma}{dQ^2\,dy\,dp_T^2} &= \frac{4\pi^2\alpha^2}{9Q^2S}\,(2\pi)^{-2}\int d^2b e^{i\vec{p}_T\cdot\vec{b}}\sum_j e_j^2 \\
&\quad \sum_a\int_{x_a}^1\frac{d\xi_a}{\xi_a}G_{a/A}(\xi_a,1/b)\sum_b\int_{x_b}^1\frac{d\xi_b}{\xi_b}G_{b/B}(\xi_b,1/b) \\
&\quad e^{-S(Q^2,b)}C_{ja}(\frac{x_a}{\xi_a},g(1/b))C_{\bar{j}b}(\frac{x_b}{\xi_b},g(1/b)) \\
&+ \frac{4\pi^2\alpha^2}{9Q^2S}Y(p_T,Q,x_a,x_b)
\end{aligned}
$$

with $S(Q^2,b) = \exp\left[-\int_{1/b^2}^{Q^2}\frac{d\overline{\mu}^2}{\overline{\mu}^2}\left[\ln\left(\frac{Q^2}{\overline{\mu}^2}\right)A(g(\overline{\mu}))+B(g(\overline{\mu}))\right]\right]$

- The Y piece is the residual NLO non-log contribution
- One can see the resemblance to my earlier example modified by the inclusion of the scale dependent PDFs
- In the expression for S, the A term sums the leading logarithms while the B term sums the next-to-leading logs
- Here are some typical resummed results (from J. Qiu and X. Zhang, Phys. Rev. D63:114011,2001) compared to data (D0 and Fermilab E-288)





- Note that by exponentiating in impact parameter space $\frac{d\sigma}{dk_T^2}$ has a non-zero intercept at $k_T = 0$
- The D0 data are shown as $\frac{d\sigma}{dk_T}$ which has a kinematic zero at $k_T = 0$
- For both plots, however, the tree level calculation would diverge as $k_T \to 0$, whereas the b-space exponentiation describes the data nicely
- I will discuss k_T-resummation further in my lecture on vector boson production

Other Resummation Examples

Logarithms of variables other than k_T can also occur - it depends on the type of distribution one is calculating. The logs come from the same basic vertices in the Feynman diagrams - they just appear in different ways and require different types of treatments. Another example is provided by the threshold logs we encountered previously in our discussion of the "plus" distributions

Threshold Resummation – Basic Physics

- For inclusive calculations, singularities from soft real gluon emission cancel against infrared singularities from virtual gluon emission
- Limitations on real gluon emission imposed by phase space constraints can upset this cancellation
- Singular terms still cancel, but there can be large logarithmic remainders
- Classic example is thrust distribution in $e^+e^- \rightarrow jets$

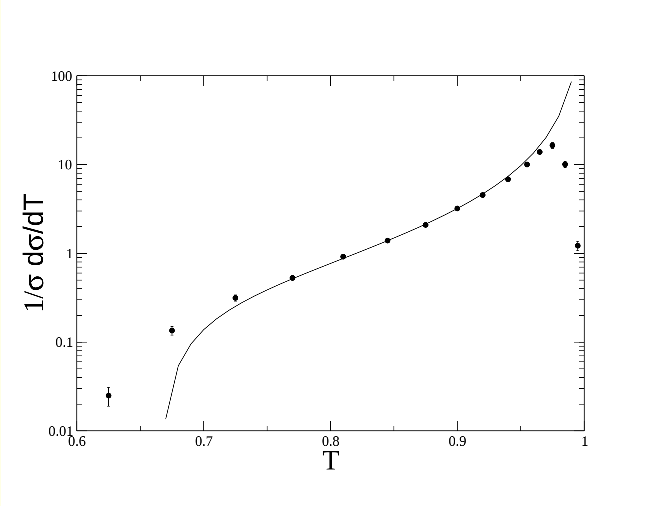
$$T = \max \vec{n} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$

- Vary the choice of the thrust axis $\vec{n}$ in order to maximize T
- 2 parton final state: $\vec{n}$ lies along p_1 and T=1

- If one of the partons emits a collinear parton, then nothing changes and $T = 1$
- If a soft gluon is emitted, then in the limit of zero energy nothing changes and $T = 1$
- The various divergent contributions seen previously all lie at $T = 1$ so that the cancellations still occur
- $T \neq 1$ yields information on the relative angular distributions of the three final state partons
- Note: no jet definition is required in order to study the thrust distribution
- Note: A spherically symmetric multiparton final state: T=1/2

- The thrust distribution is easily calculable: $T = \max[x_i]$
- Integrate $d\sigma/dx_1\,dx_2$ over x_1 and x_2 subject to the above constraint
- Result is

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F\frac{\alpha_s}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)}\right]$$



- Expected divergence as $T \to 1$ is evident
- Perturbative corrections become large in this region - a better treatment is needed

Consider the thrust distribution for small values of thrust, retaining the most divergent term

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{\alpha_s}{2\pi} C_F \frac{4}{1-T} \ln \frac{1}{1-T}$$

- We recognize that as $T \to 1$ the phase space for gluon emission is being restricted
- We can use the same technique we used for the k_T example - split the integral over T into two pieces
- Let $f(T) = \int_T^1 dT\, \frac{1}{\sigma}\frac{d\sigma}{dT}$
- Then we can write

$$\int_{T_{min}}^{1} dT \frac{1}{\sigma}\frac{d\sigma}{dT} = \int_{T_{min}}^{T} dT\, \frac{1}{\sigma}\frac{d\sigma}{dT} + f(T) = 1 + \mathcal{O}(\alpha_S) \times \text{constant}$$

- Hence, $f(T)$ is given by

$$\begin{aligned} f(T) &= 1 - \int_{T_{min}}^{T} dT\,(-4)\,\frac{\alpha_s}{2\pi} C_F\,\frac{\ln(1-T)}{1-T} \\ &= 1 - \frac{\alpha_S}{\pi} C_F \ln^2(1-T) \end{aligned}$$

- In the leading double log approximation this can be exponentiated to give

$$f(T) = e^{-\frac{\alpha_s}{\pi} C_F \ln^2(1-T)}$$

- Taking a derivative yields

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = -2\frac{\alpha_S}{\pi} C_F \frac{\ln(1-T)}{1-T} e^{-\frac{\alpha_s}{\pi} C_F \ln^2(1-T)}$$

(Exercise: fill in the steps for this derivation)

- This gives a turnover as $T \to 1$ which cures the divergence shown for the lowest order expression
- However, the turnover occurs at $\alpha_s \ln(1-T) = \frac{\pi}{2C_F} \approx 1.2$, violating the assumption that $\alpha_s \ln(1-T) << 1$ (Exercise: show this)
- Must go further and include additional log terms, not just the double logs
- However, this simple example serves as an introduction to the idea of threshold resummation

More General Processes

- The general structure of more complex cross sections will be a generalization of what we have already seen
  - Collinear singularities associated with the incoming and outgoing partons - these are collected into *jet functions*
  - Soft singularities associated with the emission of low-energy gluons - these are collected into an overall *soft function*
  - A hard scattering remainder that is finite

- The jet functions that describe the collinear singular region will contain the "plus" distributions that we have already seen that describe the real/virtual cancellations
- For a given observable one must show that
  - The squared amplitude factorizes in the manner described above
  - Phase space factorizes
  - This will generally require some type of transform like the Fourier transform we used in the k_T case
  - For threshold resummation this is usually a Mellin transform involving moments of the cross section with respect to a large scaled energy variable, *e.g.*, $\tau = Q^2/S$ for lepton pair production
  - The physical cross section then obtained by taking the inverse transformation after the exponentiation

Threshold Resummation

- Why use a Mellin transform?
- If Sudakov form factors give a supression, how is it that threshold resummation can give rise to an enhancement of the cross section?
- Useful to run through an example by Catani, Mangano, and Nason in hep-ph/9806484.

Prompt photon production

Consider the E_T distribution integrated over rapidity

$$\frac{d\sigma}{d\,E_T}(x_T, E_T)$$

where $x_T = 2E_T/\sqrt{S}$. One might be more accustomed to thinking of the cross section as a function of S and E_T, but it is convenient to have only one dimensionful variable.

The leading order expression at the parton level will have subprocess cross sections

$$\frac{d\hat{\sigma}}{d\,\hat{E}_T}(\hat{x}_T, \hat{E}_T)$$

where the $\hat{}$ symbol denotes a quantity at the parton level. For this example, we'll consider only subprocesses where the photon is directly produced by the hard scattering, e.g., $q\overline{q} \to \gamma g$ and $qg \to \gamma q$ so that for the photon $\hat{E}_T = E_T$. We can then write the cross section as

$$\begin{aligned} \frac{d\sigma_\gamma(x_T, E_T)}{dE_T} &= \frac{1}{E_T^3}\sum_{ab}\int_0^1 dx_1\, G_a(x_1, \mu_F^2)\int_0^1 dx_2\, G_b(x_2, \mu_F^2) \\ & \int_0^1 dx\,\delta\left(x - \frac{x_T}{x_1 x_2}\right)\hat{\sigma}_{ab\to\gamma}(x, E_T) \end{aligned}$$

where the δ function insures that $\hat{x}_T = x_T$ and $\hat{\sigma} = E_T^3 \frac{d\hat{\sigma}}{d\,E_T}$
The trick is to find a way to disentangle the dependence on $x_1, x_2,$ and x that is enforced by the δ function

Define moments of the cross section and PDFs by

$$\sigma_{\gamma,N}(E_T) = \int_0^1 dx_T^2 \left(x_T^2\right)^{N-1} E_T^3 \frac{d\sigma_\gamma(x_T, E_T)}{dE_T}$$

$$G_{a,N}(\mu_F^2) = \int_0^1 dx x^{N-1} G_a(x, \mu_F^2)$$

It is straightforward to show (this is a good exercise) that

$$\sigma_{\gamma,N}(E_T) = G_{a,N+1} G_{b,N+1} \hat{\sigma}_{\gamma,N}$$

so that the moments of the cross section factorize

One can easily calculate the moments of the PDFs, but we need an expression for the moments of the parton subprocesses which takes into account the threshold logs we want to resum. The result is

$$\begin{aligned}\hat{\sigma}^{(res)}_{ab\to\gamma,N}(E_T,\mu^2,\mu_F^2) &= \alpha\alpha_s(\mu^2)\hat{\sigma}^{(0)}_{ab\to\gamma,N}C_{ab\to\gamma}(\alpha_S(\mu^2),Q^2/\mu^2,Q^2/\mu_F^2)\\ &\quad \Delta^{ab\to\gamma}_{N+1}(\alpha_s(\mu^2),Q^2/\mu^2,Q^2/\mu_F^2)\end{aligned}$$

where $Q^2 = 2E_T^2$. Here the C functions do not contain any dependence on N and are calculable as a power series in α_s.

The N dependent soft gluon factors are given by

$$\begin{aligned}&\Delta^{ab\to\gamma d}_N(\alpha_s(\mu^2),Q^2/\mu^2,Q^2/\mu_F^2) =\\ &\quad \Delta^a_N(\alpha_s(\mu^2),Q^2/\mu^2,Q^2/\mu_F^2)\Delta^b_N(\alpha_s(\mu^2),Q^2/\mu^2,Q^2/\mu_F^2)\\ &\quad J^d_N(\alpha_s(\mu^2),Q^2/\mu^2,Q^2/\mu_F^2)\Delta^{(int)ab\to\gamma d}_N(\alpha_S(\mu^2),Q^2/\mu^2)\end{aligned}$$

In the $\overline{\mathrm{MS}}$ scheme, the factors associated with initial state radiation are given by

$$\Delta_N^a = \exp\left\{\int_0^1 dz \frac{z^{N-1}-1}{1-z}\int_{\mu_F^2}^{(1-z)^2Q^2} \frac{dq^2}{q^2} A_a(\alpha_s(q^2)) + \mathcal{O}(\alpha_s(\alpha_s \ln N)^k)\right\}$$

For the present purpose it is sufficient to evaluate this factor in the the leading double log approximation. Neglecting the running of α_s in the q^2 integral and using $A_a = \frac{\alpha_s}{\pi}C_a$ with $C_a = C_F$ for a quark leg or C_A for a gluon leg, the q^2 integral yields $2\ln(1-z) + \ln(Q^2/\mu^2)$. The leading contribution to the argument of the exponential is then

$$2\frac{\alpha_s}{\pi}C_a \int_0^1 dz \frac{z^{N-1}-1}{1-z}\ln(1-z)$$

The z integral just gives the moments of the plus distribution $\left(\frac{\ln(1-z)}{1-z}\right)_+$
The z integral has a leading term of $\frac{1}{2}\ln^2 N$ so we get

$$\Delta_N^a \simeq \exp\left[2C_a \frac{\alpha_s}{2\pi}\ln^2 N\right]$$

The function J_N^d has a similar form

$$\begin{aligned} J_N^d &= \exp\left\{\int_0^1 dz \frac{z^{N-1}-1}{1-z}\left[\int_{(1-z)^2Q^2}^{(1-z)Q^2}\frac{dq^2}{q^2}A_a(\alpha_s(q^2))\right.\right. \\ &+ \left.\left.\frac{1}{2}B_d(\alpha_s((1-z)Q^2))\right] + \mathcal{O}(\alpha_s(\alpha_s \ln N)^k)\right\} \end{aligned}$$

and contains soft gluon effects related to the final state parton d. The difference in the limits on the q^2 integral gives a result

$$J_N^d \simeq \exp\left\{-C_d \frac{\alpha_s}{2\pi}\ln^2 N\right\}$$

Finally, the factor $\Delta_N^{(int)}$ contains soft gluon interference effects involving gluons from different legs. It is, however, subleading and can be ignored for the purposes of this discussion.

In the leading double log approximation used here we have the following results for the two direct photon subprocesses:

$$\begin{aligned}\hat{\sigma}^{(res)}_{qg\to\gamma q,N} &\simeq \hat{\sigma}^{(0)}_{qg\to\gamma q,N}\exp\left\{[2C_F+2C_A-C_F]\frac{\alpha_s}{2\pi}\ln^2 N\right\}\\ &= \hat{\sigma}^{(0)}_{qg\to\gamma q,N}\exp\left\{(C_F+2C_A)\frac{\alpha_s}{2\pi}\ln^2 N\right\} > \hat{\sigma}^{(0)}_{qg\to\gamma q,N}\end{aligned}$$

and

$$\begin{aligned}\hat{\sigma}^{(res)}_{q\overline{q}\to\gamma g,N} &\simeq \hat{\sigma}^{(0)}_{q\overline{q}\to\gamma g,N}\exp\left\{[2C_F+2C_F-C_A]\frac{\alpha_s}{2\pi}\ln^2 N\right\}\\ &= \hat{\sigma}^{(0)}_{q\overline{q}\to\gamma g,N}\exp\left\{(4C_F-C_A)\frac{\alpha_s}{2\pi}\ln^2 N\right\} > \hat{\sigma}^{(0)}_{q\overline{q}\to\gamma g,N}\end{aligned}$$

We see that both subprocesses are enhanced with the qg getting a larger enhancement than that for $q\overline{q}$

Interpretation

The radiation factor J_N^d associated with the final state parton shows the expected Sudakov suppression. However, the factors associated with the initial state radiation show an enhancement. The explanation lies in realizing that the cross section has been factorized into the product of two $\overline{\rm MS}$ PDFs and the partonic subprocess and that the evolution of the PDFs involves a partial resummation via the DGLAP equations. Thus, what is seen in the Δ_N^a factors is what remains after the factorization.

This leads to a simple pattern

- Expect a Sudakov suppression for legs associated with jets
- Expect an enhancement for legs associated with PDFs or FFs

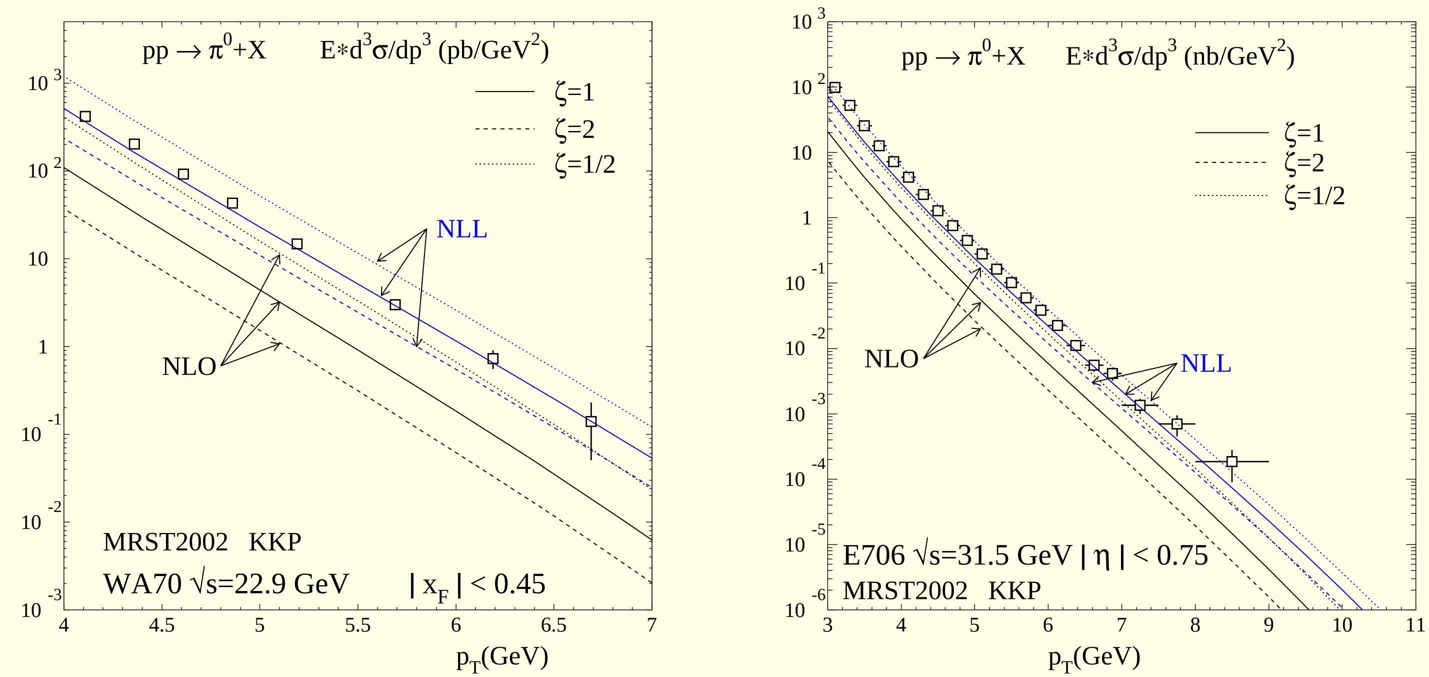
Of course, this is all qualitative since I retained only the leading double log terms

Which processes might have large threshold corrections?

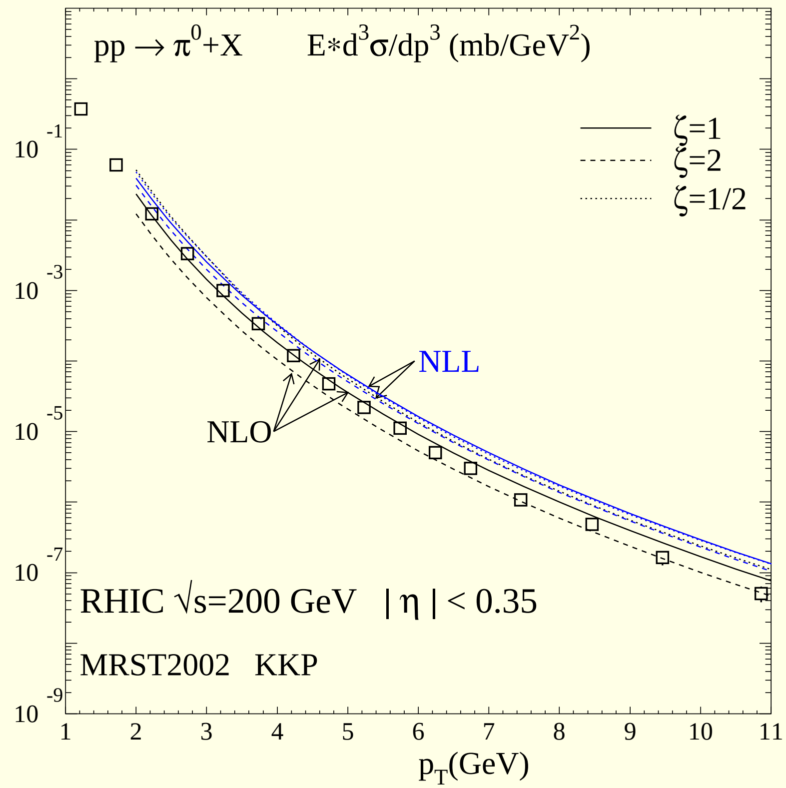
- Dihadron production $AB \to h_1 h_2 + X$: two PDFs and two FFs suggests a large enhancement due to threshold log resummation
- Single hadron production $AB \to h_1 + X$: two PDFs, one FF, and one final state jet suggests a somewhat lesser effect
- Inclusive jet or dijet production: two PDFs and two final state jets suggests even less of an enhancement.

High-$p_T \pi^0$ cross section

- Work by de Florian and Vogelsang (hep-ph/0501258) applies threshold resummation to π^0 production
- For fixed target experiments the center of mass energy is in the 20-40 GeV range while p_T is typically 3-12 GeV $\Rightarrow x_T$ can be large
- The fragmentation fraction z will also be large when one requires a high-p_T hadron - the hadron will take most of the energy of the fragmenting parton (jet) since taking a smaller fraction wastes energy and the parton PDFs fall off rapidly in x (nature doesn't want to waste the available partonic center of mass energy)
- Large values of z relevant for fixed target energies leads to large threshold resummation corrections ($\ln^n(1-z)/(1-z)$)
- Enhancement is strongly energy dependent since the relevant values of z decrease as one goes to higher energies at fixed p_T (more energy is available at fixed p_T and the relevant values of z decrease)



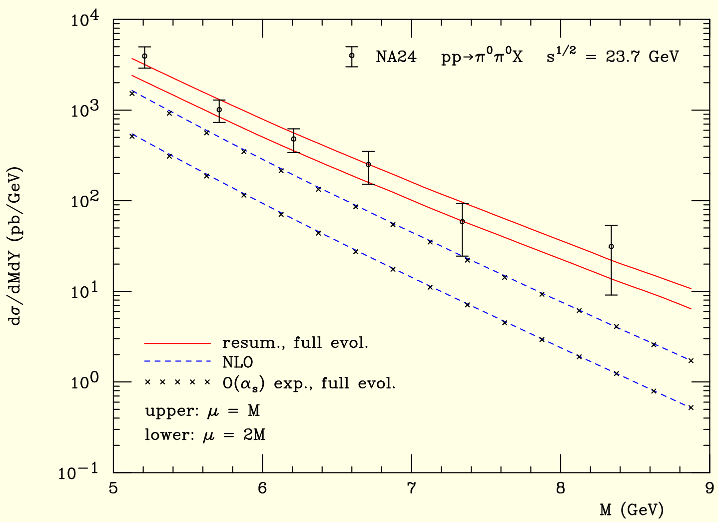
- Blue curves include the resummation corrections properly matched to an existing NLO calculation in order to avoid double counting
- Note the reduced scale dependence of the resummed results

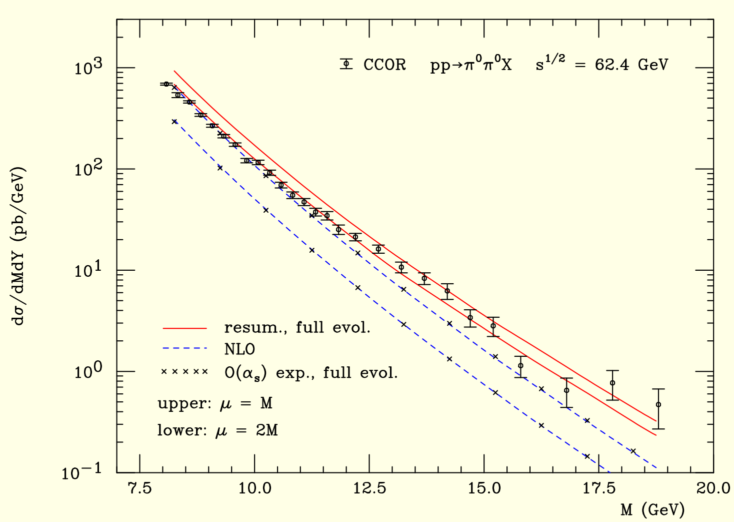


- Note reduced enhancement at RHIC energy compared to the previous fixed target results

Another Example – Dihadron Production

- Two PDFs and two FFs suggests a large enhancement
- In the region of large M^2/S where M is the dihadron mass we expect large threshold logs
- Recent work by Almeidea, Sterman, and Vogelsang (Phys.Rev.D80:074016,2009) confirms that the effects are large for fixed target experiments and that they decrease as S is increased at fixed M
- Results also show decreased scale dependence





Summary

Here are some key points to remember

- NLO calculations are not always adequate for every observable
- NLO calculations are appropriate for processes where there is one large scale
- In some regions of phase space the NLO corrections can actually become LO if the lowest order contribution is suppressed
- Large logs can be generated when the phase space for additional gluon radiation is restricted - look for two or more relevant scales
- Examples include low-p_T lepton pair and vector boson production
- Threshold logs can be resummed and are especially relevant for understanding the energy dependence of fixed target hadron or photon production relative to the high energy collider results
- There is more to perturbation theory than just the next order contribution!