An Introduction to Higher Order Calculations in Perturbative QCD

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Overview of the Lectures

- Lecture I - Higher Order Calculations
  - What are they?
  - Why do we need them?
  - What are the ingredients and where do they come from?
  - Understanding and treating divergences
  - Examples from e^+e^- annihilation
- Lecture II - Examples of Higher Order Calculations
  - Parton Distribution Functions at higher order
  - Lepton Pair Production at higher order
  - Factorization scale dependence

- Lecture III - Hadronic Production of Jets, Hadrons, and Photons
  - Single inclusive cross sections
  - More complex observables and the need for Monte Carlo techniques
  - Overview of phase space slicing methods
- Lecture IV - Beyond Next-to-Leading-Order
  - When is NLO not enough?
  - Large logs and multiscale problems
  - Resummation techniques

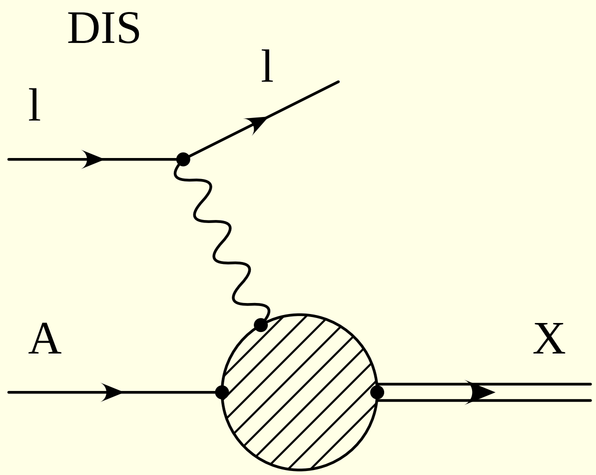
Lecture II - Outline

- Deep Inelastic Scattering
  - Formalism
  - Lowest order contributions to the structure functions
  - Next order corrections
  - PDF definitions and convention dependence
- Lepton Pair Production
  - M^2 and rapidity dependence
  - Next order corrections and the infamous "K factor"
  - Convention dependence
  - Features of the higher order corrections to various distributions

Brief Overview of DIS

- Deep inelastic lepton-nucleon scattering uses the photon as a known probe to investigate the structure of the nucleon
- DIS has played an important role in the determination of PDFs
- Could simply work with the cross section expressed in terms of PDFs
- Historical approach has been based on structure functions
- The basic idea is to remove as much of the known physics of the lepton vertex as possible, constrain the remaining hadronic piece using gauge invariance, current conservation, parity invariance (for the electromagnetic interaction) and time reversal invariance and then express what is left in terms of the hadronic structure functions F_1 and F_2 (plus F_3 for weak interactions
- For PDF determinations it is often preferable to work directly with the cross sections since that avoids any model-dependent assumptions associated the extraction of the structure functions
- I will summarize here the structure function approach

Start with a few definitions for the process $e^-(k) + P(P) \to e^-(k') + X$ in the target rest frame where M denotes the target mass



$$
\begin{aligned}
q^2 &= -Q^2 = (k-k')^2 \qquad x = Q^2/2M\nu \\
E &= k \cdot P/M \qquad E' = k' \cdot P/M \\
\nu &= P \cdot q/M \qquad W^2 = (P+q)^2 = M^2 + Q^2\left(\frac{1}{x} - 1\right) \\
y &= \nu/E = 1 - E'/E
\end{aligned}
$$

The cross section can be written as

$$\sigma = \frac{1}{4ME}\int \frac{d^3k'}{(2\pi)^3 2E'}\frac{1}{4}\sum_{spins}\sum_{X}\prod_{n=1}^{N_X}\int \frac{d^3p_n}{(2\pi)^3 2E_n}|T_{fi}|^2(2\pi)^4\delta(k+P-k'-p_X)$$

- The leptonic and hadronic parts have been written separately
- Can simplify this by being differential in the scattered lepton energy and scattering solid angle.
- Can also express T_{fi} as

$$T_{fi} = e^2\overline{u}(k')\gamma_\mu u(k)\frac{1}{q^2}J^\mu$$

- Here J^μ is the matrix element of the electromagnetic current operator between the initial and final hadronic states

The end result is

$$\begin{aligned}
\frac{d\sigma}{dE'd\Omega'} &= \frac{\alpha^2}{Q^4}\left(\frac{E'}{E}\right) L_{\mu\nu}W^{\mu\nu} \\
L_{\mu\nu} &= 2\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} k\cdot k'\right) \\
W^{\mu\nu} &= \frac{(2\pi)^3}{4M}\sum_{spins}\sum_X \prod_{n=1}^{N_X} \frac{d^3 p_n}{(2\pi)^3 2E_n} J^{\mu\dagger} J^\nu
\end{aligned}$$

- Gauge invariance, current conservation, and parity conservation give

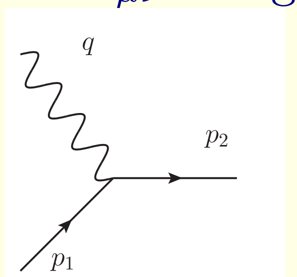
$$W^{\mu\nu} = \frac{F_1}{M}\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right) + \frac{F_2}{M^2\nu}\left(P^\mu + q^\mu \frac{P\cdot q}{Q^2}\right)\left(P^\nu + q^\nu \frac{P\cdot q}{Q^2}\right)$$

- The structure functions F_1 and F_2 contain information on the structure of the hadronic target
- Both depend on the 4-vectors P and q through Lorentz scalars such as Q^2 and x

- In terms of these structure functions, one can write the cross section as

$$\frac{d\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^2}\left[\left(1+(1-y)^2\right)F_1 + \frac{(1-y)}{x}\left(F_2 - 2xF_1\right)\right]$$

- With these definitions, we can now examine the form of the structure functions in the parton model
- Start with the basic definition of $W_{\mu\nu}$ using a parton target of charge e_q



$$W_{\mu\nu} = \frac{(2\pi)^3}{2M}\frac{1}{2}\sum_{spins} N \int \frac{d^3p_2}{(2\pi)^3 2E_2}\delta^4(p_2-q-p_1)e_q^2\left(\overline{u}(p_2)\gamma_\mu u(p_1)\right)^\dagger\left(\overline{u}(p_2)\gamma_\nu u(p_1\right.$$

- N is a normalization factor to be defined below

- Use $\frac{d^3p_2}{2E_2} = d^4p_2\delta(p_2^2)$ to get

$$W_{\mu\nu} = \frac{N}{M} e_q^2 \delta(p_2^2) \left(2p_{1\mu}p_{1\nu} + p_{1\mu}q_\nu + p_{1\nu}q_\mu + \frac{q^2}{2} g_{\mu\nu} \right)$$

- Next, assume that the parton carries a fraction η of the target's 4-momentum and neglect target mass effects. Thus, $p_1 = \eta P$
- With this definition,

$$\begin{aligned} \delta(p_2^2) &= \delta[(p_1+q)^2] = \delta(q^2 + 2p_1 \cdot q) \\ &= \frac{1}{2M\nu}\delta(\eta - \frac{Q^2}{2M\nu}) = \frac{1}{2M\nu}\delta(\eta - x) \end{aligned}$$

- So, to this order, x is a measure of the momentum fraction carried by the struck parton
- The normalization factor N corrects for the flux factor being that of the parton, not the target hadron: $N = 1/\eta$

- The end result is

$$W_{\mu\nu} = \frac{\eta}{2M^2\nu} e_q^2 \delta(\eta - x) \left[2P_\mu P_\nu + \frac{P_\mu q_\nu + P_\nu q_\mu}{\eta} + \frac{q^2}{2\eta^2} g_{\mu\nu} \right]$$

- From this expression one can read off the results

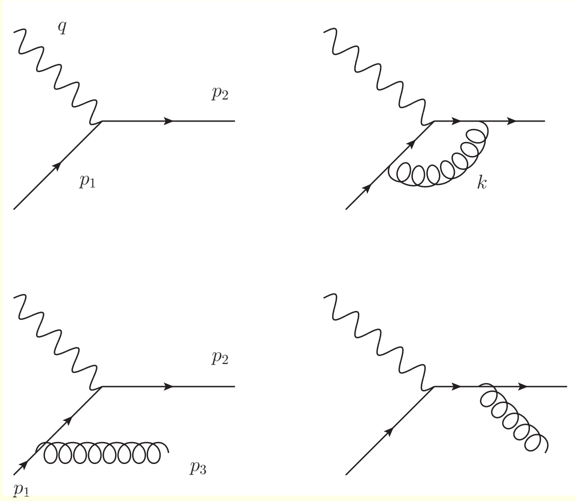
$$\hat{F}_2 = \eta e_q^2 \delta(\eta - x) \quad \hat{F}_2 = 2x\hat{F}_1$$

- I have used the ˆ symbol to denote the contributions to the structure functions at the parton level.
- The hadronic structure functions are given by weighting by the appropriate PDF:

$$\begin{aligned} F_2(x, Q^2) &= 2xF_1(x, Q^2) \\ &= \sum_q e_q^2 \int d\eta q(\eta) \eta \delta(\eta - x) = \sum_q e_q^2 x q(x) \end{aligned}$$

- One can see that to this order the structure functions are independent of Q^2, a phenomenon known as scaling

- We are now prepared to consider the higher order corrections to this result, starting with corrections involving quarks in the initial state
- By now the procedure (if not the details) should be familiar
  - Write the cross section expression in n dimensions to determine the expression for the cross section in terms of the hadronic tensor
  - Write the n-dimensional expression for the hadronic tensor at the parton level for both the one-loop results and the real gluon radiation graphs
  - Add the results, cancelling the ϵ^{-2} contributions and some of the ϵ^{-1} terms, as well
  - Isolate the residual collinear singularities associated with the initial state partons
  - Factorize these collinear singularities and absorb them into the bare quark and gluon PDFs



- I will summarize the results of the steps outlined above. In the following, let $\mathcal{F}_2(x) = F_2(x)/x$. This will simplify the convolution notation. Then, the full structure function can be written in terms of contributions from quarks and gluons as

$$\mathcal{F}_2(x) = \int \sum_q e_q^2 \left[\hat{\mathcal{F}}_2^q(z) q(y) + \hat{\mathcal{F}}_2^g(z) g(y) \right] \delta(x - zy) dz\, dy$$

- With this notation, the lowest order result is $\hat{\mathcal{F}}_2^q(z) = \delta(1 - z)$

- Using this same notation, the one-loop vertex correction to the lowest order quark result is

$$\hat{\mathcal{F}}_2^{q,v}(z) = -\frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \delta(1-z) \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + \frac{\pi^2}{3}\right)$$

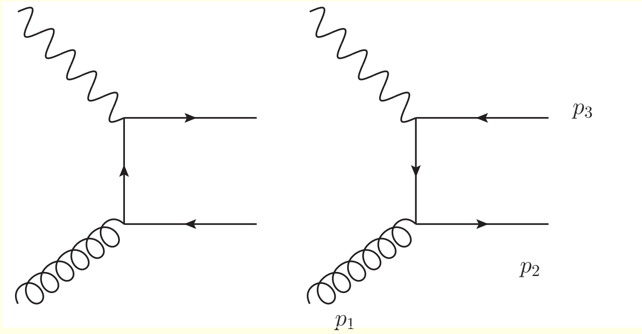
- The contribution from the real emission graphs is

$$\begin{aligned}\hat{\mathcal{F}}_2^{q,r}(z) &= \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &\quad \left[\delta(1-z)\left(\frac{2}{\epsilon^2} + \frac{3}{2\epsilon}\right) - \frac{1}{\epsilon}\frac{1+z^2}{(1-z)_+} + \text{finite terms}\right]\end{aligned}$$

- Adding these two terms together yields the intermediate quark result

$$\hat{\mathcal{F}}_2^q(z) = \delta(1-z) + \frac{\alpha_s}{2\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\left[-\frac{1}{\epsilon}P_{qq}(z) + \tilde{f}_2^q\right]$$

- $\tilde{f}_2^q$ represents a finite correction term which will be detailed shortly



The next contribution to consider is that from the photon gluon fusion process shown above. The sequence of steps is the same as that for the gluon radiation process with the result that

$$\hat{\mathcal{F}}_2^g(z) = \frac{\alpha_s}{2\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\left[-\frac{2}{\epsilon}P_{qg}(z) + \tilde{f}_2^g\right]$$

- It is clear that there are uncancelled poles in ϵ in both the quark and the gluon contributions
- These are collinear divergences which result from configurations where two initial state partons are parallel to each other.
- These divergent terms represent long distance physics reflecting the evolution of the initial quark state before the hard scattering
- As such, they can be absorbed into the bare quark PDF using a procedure analogous to that used for the FFs in Lecture I

Define a scale dependent quark PDF as

$$q(x, M_f^2) = \int dy\, dz \delta(x - yz)$$

$$\left[q(y)\delta(1-z) + \frac{\alpha_s}{2\pi}\left(-\frac{1}{\epsilon}\right)\left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[P_{qq}(z)q(y) + P_{qg}(z)g(y)\right]\right]$$

With this definition, all the remaining collinear divergences have been absorbed into the definition of the scale-dependent PDFs. The finite parton-level structure functions have simple forms

$$\hat{\mathcal{F}}_2^q(z) = \delta(1-z) + \frac{\alpha_s}{2\pi}\left[\ln\left(\frac{Q^2}{M_f^2}\right)P_{qq}(z) + \tilde{f}_2^q\right]$$

$$\hat{\mathcal{F}}_2^g(z) = \frac{\alpha_s}{2\pi}\left[\ln\left(\frac{Q^2}{M_f^2}\right)P_{qg}(z) + \tilde{f}_2^g\right]$$

- The full structure is now given by

$$F_2(x) = \int dy\, dz\, \delta(x - yz) \sum_q e_q^2 \left[\hat{\mathcal{F}}_2^q(z) q(y, M_f^2) + \hat{\mathcal{F}}_2^g(z) g(y, M_f^2) \right]$$

- Now, it must be remembered that the $\hat{\mathcal{F}}$s in the above expression contain dependences on both Q^2 and M_f^2 in the form of $\ln Q^2/M_f^2$
- Suppose that M_f^2 was chosen to be Q^2? Then the log terms vanish
- In this case the result is rather simple:

$$F_2(x, Q^2) = \sum_q e_q^2 x q(x, Q^2)$$
$$+ \frac{\alpha_s}{2\pi} \sum_q e_q^2 x \int \frac{dz}{z} \left[q(\frac{x}{z}, Q^2) \tilde{f}_2^q(z) + g(\frac{x}{z}, Q^2) \tilde{f}_2^g(z) \right]$$

- The last expression shows that the potentially large logs of Q^2 have been absorbed into the quark PDFs leaving an $\mathcal{O}(\alpha_S)$ correction. But wait! It gets even better ...
- When we subtracted the collinear singularities we had the freedom to subtract additional finite terms - that was how we introduced the factorization scale.
- Suppose we also subtracted out the $\tilde{f}$s? Then the last term would be absent and the expression for F_2 would remain the same as for the parton model, but with Q^2 dependent PDFs
- This scheme is referred to as the "DIS scheme." It has seen some use when describing DIS data
- However, the down side is that the PDFs now contain the finite corrections from the $\tilde{f}$s and these must be subtracted out if the PDFs are to be used in any other process
- It is much more common today to use the $\overline{\text{MS}}$ scheme as presented above. That way the finite corrections are calculated on a case-by-case basis for each process

$\overline{\text{MS}}$ DIS Corrections

For completeness, I give here the two finite DIS correction terms for F_2

$$\begin{aligned}\tilde{f}_2^q(z) &= C_F\left[(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{3}{2}\frac{1}{(1-z)_+}\right.\\ &- \left.\frac{1+z^2}{1-z}\ln z + 3 + 2z - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\right]\end{aligned}$$

and

$$\tilde{f}_2^g(z) = \frac{1}{2}\left[\left(z^2+(1-z)^2\right)\ln\frac{1-z}{z} + 8z(1-z) - 1\right]$$

I will refer back to these when we discuss the lepton pair production process shortly

DGLAP Equations

- It is all well and good to have a simple expression for F_2 in terms of scale-dependent PDFs, but where do the PDFs come from and how do you calculate their dependence on the scale?
- Refer back to the definition I introduced for the scale-dependent PDFs
- The scale entered through a term

$$-\frac{1}{\epsilon}\left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} =$$

$$-\frac{1}{\epsilon}+\ln\left(\frac{M_f^2}{\mu^2}\right)-\ln(4\pi)+\gamma_E+\ldots$$

- The partial derivative of this term with respect to $\ln M_f^2$ is just one, so the derivative projects out the coefficient of this term which is just the convolution of the splitting function and the appropriate PDF

- The result is known as the set of DGLAP (Dokshitzer-gribov-Lipatov-Altarelli-Parisi) Equations
- They take the form

$$\begin{aligned}
\frac{\partial q(x,t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi}\int_x^1 \frac{dy}{y}\left[P_{qq}(y)q(\frac{x}{y},t) + P_{qg}(y)g(\frac{x}{y},t)\right] \\
\frac{\partial g(x,t)}{\partial t} &= \frac{\alpha_s(t)}{2\pi}\int_x^1 \frac{dy}{y}\left[P_{gq}(y)q(\frac{x}{y},t) + P_{gg}(y)g(\frac{x}{y},t)\right]
\end{aligned}$$

- here $t = \ln M_f^2/\mu^2$ and I have introduced two additional splitting functions beyond the two we had already encountered.
- These coupled integro-differential equations may be solved iteratively by computer, given a set of initial boundary conditions at some scale
- The boundary conditions on the initial PDFs may be parametrized and then varied to fit a wide variety of data. This is the heart of the global fitting program for determining PDFs, about which more will be said in a later lecture

Extension to Other Processes

We have seen how to define scale-dependent quark PDFs in the case of DIS. The next step is to apply these PDFs in another process. A simple example which builds on what we have done is that of Lepton Pair Production, sometimes referred to as the Drell-Yan process

- At the lowest order there is only one subprocess: $q\overline{q} \rightarrow \gamma^* \rightarrow l^+ l-$
- This process is of historic and practical importance - continue the idea of using a known electromagnetic probe to study hadronic structure
- This is just the time-reversed process from the e^+e^- example
- The cross section at the parton level is just the same as for the e^+e^- example except that we have to average of the numbers of colors in the initial state. The result is

$$\sigma(S) = \frac{4\pi}{9}\frac{\alpha^2}{S}$$

- Denote the parton 4-vectors in the hadron-hadron center-of-mass by

$$p_a = x_a \frac{\sqrt{S}}{2}(1,0,0,1) \qquad p_b = x_b \frac{\sqrt{S}}{2}(1,0,0,-1)$$

- These yield $Q^2 = (p_a + p_b)^2 = x_a x_b S$ where S denotes the overall hadronic center-of-mass energy squared and Q denotes the lepton pair invariant mass.
- The lepton pair mass distribution is then given by

$$\begin{aligned}
\frac{d\sigma}{dQ^2} &= \sigma(Q^2) \int dx_a\, dx_b \sum_q e_q^2 \left[q(x_a, M_f^2)\overline{q}(x_b, M_f^2) + q \leftrightarrow \overline{q}\right] \delta(Q^2 - x_a x_b S) \\
&= \sigma(Q^2) \int_\tau^1 \frac{dx_a}{x_a S} \sum_q e_q^2 \left[q_a(x_a, M_f^2)\overline{q}_b(\tau/x_a, M_f^2) + q \leftrightarrow \overline{q}\right]
\end{aligned}$$

Here I have used $\tau = Q^2/S$

- Can also easily calculate the rapidity distribution at fixed Q^2
  - $y = \frac{1}{2} \ln \left(\frac{E+p_l}{E-p_l} \right)$
  - $E = (x_a + x_b)\sqrt{S}/2 \quad p_l = (x_a - x_b)\sqrt{S}/2$
  - Combining these yields $y = \frac{1}{2} \ln \left(\frac{x_a}{x_b} \right)$
  - Exercise: Combine with $Q^2 = x_a x_b S$ to get $x_{\substack{a \\ b}} = \sqrt{\tau} \exp(\pm y)$
- Insert $\delta \left[y - \frac{1}{2} \ln \left(\frac{x_a}{x_b} \right) \right]$ into the expression for $d\sigma/dQ^2$ and do the x_a integration to get

$$\frac{d\sigma}{dy\, dQ^2} = \frac{\sigma(Q^2)}{S} \sum_q e_q^2 \left[q_a(x_a, M_f^2) \overline{q}_b(x_b, M_f^2) + q \leftrightarrow \overline{q} \right]$$

- Sometimes also use the variable $x_F = \frac{2p_l}{\sqrt{S}}$
- Exercise: Use $dy = \frac{dp_l}{E}$ to show that $\frac{d\sigma}{dQ^2\, dx_F} = \frac{d\sigma}{dQ^2\, dy} \frac{1}{x_a + x_b}$ and that $x_F = x_a - x_b = 2\sqrt{\tau} \sinh(y)$

Time to get on the Soap Box

Some people are fond of saying something like "QCD says that in lowest order the lepton pair is produced with no transverse momentum." This statement is false. Let's see why.

- It is true, that for the $q\overline{q} \to l^+l^-$ subprocess, the lepton pair has the same transverse momentum as the $q\overline{q}$ initial state
- It is also true that we have used kinematics in which we treat the initial partons as being collinear with the beam
- However, the PDFs in the cross section expressions are the scale-dependent PDFs and carry an argument M_f^2.
- This dependence on the factorization scale comes from integrating over the p_T of the additional partons emitted from the initial state (either radiated gluons or quarks and antiquarks created by gluons)
- Thus, QCD radiation causes the incoming partons to have non-zero transverse momenta, but these are integrated out when the scale-dependent PDFs are used
- We make an approximation when we treat the partons given by the integrated PDFs as having zero transverse momenta
- This approximation is valid for the so-called leading logarithm terms

- Thus, QCD predicts that the lepton pair will have a transverse momentum distribution, but we have integrated over it (even if we didn't realize it) when we use the expressions given previously.
- Then, how do we undo the integration? And what value should we use for M_f^2?

Choosing the Factorization Scale

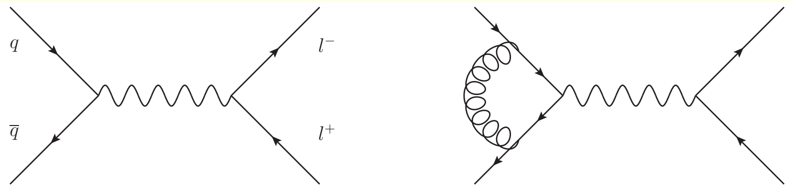
- The factorization scale M_f can be understood as setting the upper limit on the integration over the transverse momenta of the partons emitted in the initial state evolution
- The leading-log contributions from higher order subprocesses have been included in the scale-dependent PDFs
- So, if one wants to calculate the cross section for producing a lepton pair of mass Q then a choice of $M_f \approx Q$ would be appropriate.
- This is not exact, since the true upper limit of the transverse momentum integration would be given by a more complicated expression involving a function of τ and y multiplying Q. But in the leading-log approximation the choice Q is acceptable.
- Of course, any constant times Q is equally acceptable as long as the constant isn't too large (or too small) since then one would generate spurious large logarithms

P_T Distribution

So, given the preceding discussion, how does one calculate the lepton pair p_T distribution? Answer - Go to higher order!

- To calculate the p_T spectrum we will have to consider having the lepton pair recoil against at least one parton. The subprocesses are
  - Compton process: $qg \to l^+ l^- q$
  - Annihilation process $q\overline{q} \to l^+ l^- g$
- Using these subprocesses one can calculate a p_T distribution, but it will be a *leading order* prediction for the p_T dependence
- There is still the issue of the scale choice, but at the leading-log level any choice on the order of Q is acceptable. Another popular choice is something like $\sqrt{Q^2 + p_T^2}$
- Now, suppose that you were to integrate these subprocesses over p_T? This would give an $\mathcal{O}(\alpha_s)$ contribution to the cross section.
- One way of thinking about this is that you would be doing the p_T integration of these two subprocesses exactly, thereby going beyond the leading-log approximation outlined earlier.
- We will return to the issue of the p_T spectrum in Lecture IV

Next Order Correction to $\frac{d\sigma}{dQ^2}$



Let's write the lowest order expression for the cross section as follows:

$$\frac{d\sigma}{dQ^2} = \frac{\sigma(Q^2)}{S} \int \frac{dx_a}{x_a} \frac{dx_b}{x_b} \sum e_q^2 \left[q(x_a)\overline{q}(x_b) + q \leftrightarrow \overline{q}\right] \delta(1-z)$$

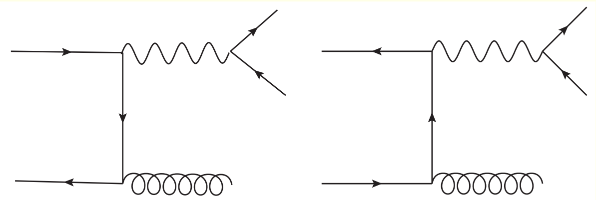
where $z = \frac{Q^2}{x_a x_b S}$

Now, consider the virtual corrections - these are the same as in the e^+e^- example. One simply replaces $\delta(1-z)$ in the above expression by

$$\delta(1-z)\left[1 + \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right]\right]$$

Next, we must consider the contributions from the Compton and annihilation subprocesses

Annihilation Contribution



- By now, the steps should be familiar - square the matrix element in n-dimensions, multiply by 3-body n-dimensional phase space, and divide by the flux factor
- Perform the relevant phase space integrations using the changes of variables and the "+" distributions outlined in the e^+e^- case
- Add to the preceding results for the lowest order and virtual contributions
- The ϵ^{-2} terms will cancel, as will some of the ϵ^{-1} terms, leaving some residual ϵ pole terms.
- Factorize these remaining singular terms and absorb them into the bare PDFs, leaving a residual finite $\mathcal{O}(\alpha_s)$ correction
- As before, the factorization of the initial state collinear singularities will be facilitated by the introduction of a mass factorization scale M_f

The full annihilation contribution, including the lowest order and virtual contributions, at the parton level prior to the mass factorization step is

$$\delta(1-z) + \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

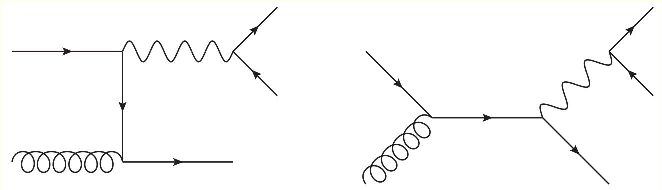
$$\left[\delta(1-z)\left(-\frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right) - \frac{2}{\epsilon}\frac{1+z^2}{(1-z)_+} + 4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\frac{1+z^2}{1-z}\ln\right.$$

One can recognize the familiar splitting function $P_{qq}(z)$ in this expression. The result can be simplified to

$$\delta(1-z) - \frac{2}{\epsilon}\frac{\alpha_s}{2\pi}\left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} P_{qq}(z) + \frac{\alpha_s}{2\pi} 2P_{qq}(z) \ln\frac{Q^2}{M_f^2} + \frac{\alpha_s}{2\pi} f_q(z)$$

where $f_q(z)$ represents a finite $\mathcal{O}(\alpha_S)$ correction as in the DIS example and I have kept the factorization scale dependent term separate from f_q

Compton Subprocess



- The same procedure as outlined on the preceding slides is followed for the Compton subprocess
- This time there is no lower order term and there are no virtual corrections
- The only singularity is the collinear singularity associated with the gluon splitting vertex
- The full Compton result using the same normalization as for the annihilation result is

$$-\frac{1}{\epsilon}\frac{\alpha_s}{2\pi}P_{qg}(z)\left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon}+\frac{\alpha_s}{2\pi}P_{qg}(z)\ln\left(\frac{Q^2}{M_f^2}\right)+\frac{\alpha_s}{2\pi}f_g(z)$$

- At this point the final step is to factorize the remaining collinear terms into the bare PDFs
- This is easily done using the expressions given previously as I have already isolated the appropriate subtraction terms for the $\overline{\mathrm{MS}}$ scheme.
- Restoring the full normalization for the cross section we get

$$
\begin{aligned}
\frac{d\sigma}{dQ^2} &= \frac{\sigma(Q^2)}{S}\int \frac{dx_a}{x_a}\frac{dx_b}{x_b}\left[\sum_q e_q^2\left[q(x_a,M_f^2)\overline{q}(x_b,M_f^2)+a\leftrightarrow b\right]\right. \\
&\quad \cdot\left[\delta(1-z)+\frac{\alpha_s}{2\pi}\left(2P_{qq}(z)\ln\left(\frac{Q^2}{M_f^2}\right)+f_q(z)\right)\right] \\
&+ \sum_q e_q^2\left[\left(q(x_a,M_f^2)+\overline{q}(x_a,M_f^2)\right)g(x_b,M_f^2)+a\leftrightarrow b\right] \\
&\quad \left.\frac{\alpha_s}{2\pi}\left(P_{qg}(z)\ln\left(\frac{Q^2}{M_f^2}\right)+f_g(z)\right)\right]
\end{aligned}
$$

The relatively simple expression on the previous page contains many of the elements that characterize NLO calculations in general

- The chosen form strongly suggests the choice $M_f = Q$ which follows from the fact that the phase space factor $\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}$ is given in terms of Q^2 which sets the natural scale for the process
- There is explicit M_f dependence in the NLO term which partially cancels that contained in the lowest order term
- To see this, take a derivative with respect to $\ln M_f^2$ of the cross section expression
  - The derivative of $q(x_a, M_f^2)$ gives a contribution of

$$\frac{\alpha_s}{2\pi}\left[P_{qq} \otimes q + P_{qg} \otimes g\right]$$

    where $\otimes$ is shorthand for the convolution of the PDF and splitting function. This follows from the DGLAP equations for the scale dependence of the PDFs
  - The derivative of the NLO correction gives a similar term, but with a minus sign coming from the $\ln\left(\frac{Q^2}{M_f^2}\right)$ factors

- The cancellation is not exact, but is correct up to the next order in α_s (Exercise: Show this)
- This is a feature which is typical of NLO calculations and is one of the reasons for why they are important - they generally, but not always, feature a decreased scale dependence relative to the leading-order calculation

For completeness here are the remaining factors in the NLO calculation

$$\begin{aligned} f_q(z) &= C_F \Big[\delta(1-z)(-8+\frac{2\pi^2}{3}) \\ &\quad +4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\frac{1+z^2}{1-z}\ln z\Big] \end{aligned}$$

and

$$f_g(z) = \frac{1}{2}\left[\ln\frac{1+z^2}{z}\left[z^2+(1-z)^2\right] + \frac{1}{2} + 3z - 7z^2\right]$$

Comments

- If you compare the expressions for the fs in the Lepton Pair Production case to those in the DIS case for F_2 you will see some features in common, but also some differences
  - In both cases there are combinations of delta function terms, plus regulators, and other z-dependent terms
  - The dependence on the plus regulators is different
- The differences stem from the fact that in both case we are integrating over an additional parton in the final state, but the phase space is different in the two cases - we have an spacelike photon in the initial state for one and a timelike photon in the final state for the other
- Remember that the plus regulators are related to the limitations placed on parton emission near threshold and these constraints are different in the two cases

- The change from space-like to time-like Q^2 also affects terms involving $\ln Q^2$ since the argument will be negative for one of the processes.
- $Re(-1)^{-\epsilon} = Re\exp(-i\pi\epsilon) = 1 - \epsilon^2\pi^2 + \cdots$
- This multiplies ϵ^{-2} and so generates a contribution proportional to π^2
- Historically, the existence of these π^2 terms played an important role in understanding the QCD description of these two processes

This brings us to the idea of the infamous "K Factors"

Aside: Why infamous? Because they don't have a unique definition and they aren't true factors!!

K factors

- The idea of K factors started innocently enough. In the 1980s the early Lepton Pair Production experimental results were compared with existing predictions based on leading order PDFs and the lowest order hard scattering expressions
- The results were given as the ratio of the data to the predictions and this ratio was called the K factor, *i.e.* the amount one would have to multiply the theoretical predictions by in order to describe the data
- The early comparisons showed that this result was about 2, which seemed like a real problem for QCD
- The explanation came when NLO calculations became available
- Almost all of the NLO correction is associated with a large contribution proportional to $\delta(1-z)$
- To understand this requires several steps...

- First, consider that leading order PDFs fitted to DIS data (that was all we had at first) essentially have all of the higher order corrections absorbed into the PDFs themselves
- This would be equivalent to using the DIS factorization convention where f_2^q and f_2^g are absorbed into the PDFs
- But then, when one calculates the Lepton Pair Production cross section these DIS corrections must be removed from the PDFs. In this DIS scheme we must replace f_q by $f_q - f_2^q$ and similarly for the gluon terms
- The coefficient of the delta function term is then

$$\frac{\alpha_s}{2\pi} C_F \left(1 + \frac{4\pi^2}{3}\right)$$

  relative to which the lowest order term is just 1 (Exercise: Show this)
- For $Q \approx 5$ GeV this correction changes the lowest order term by about 1.8, *i.e.,* a K factor of nearly 2!
- In this case, the bulk of the correction comes from the π^2 terms which appear in the delta function term and so the correction is roughly a constant times the lowest order results
- And, so, the idea of a K factor has been with us ever since

Comments

- One should be worried that the next order correction is so large - is perturbation theory converging?
- Feynman:

$$\frac{1}{1-x} = 1 + x + \ldots \text{ and } e^x = 1 + x + \ldots$$

  For $x \approx 1$ the first diverges while the second gives about 2.7
- As it turns out a significant part of the correction term exponentiates so the second example is closer to what is happening
- In this example the kinematics of the NLO correction delta function piece is the same as for the lowest order, so the correction is essentially a multiplicative constant
- This does not happen very often
- In the more usual case there are many different subprocesses in the NLO calculation and they can have very different dependences on the process kinematics
- Various phase space factors can cause the higher order parton emission contributions to contribute differently in different regions of phase space

- So, in general the ratio of the NLO to LO calculations (the so-called theoretical K factor) will depend on the kinematic variables and *will not be a constant*
- Furthermore, there is the issue of the scale dependence. The NLO and LO terms have different scale dependences. They partially cancel each other, which is a good thing.
- This has the effect that the ratio of the two terms will depend strongly on the chosen factorization scale
- But if the so-called K factor depends on the scale choice, then how can it be a uniquely defined “factor”?

It Can’t!

I’ll have more examples of this in a later lecture

Summary

- We've seen the basics of NLO calculations in DIS and Lepton Pair Production
- We've seen now the collinear singularities associated with initial state radiation can be factorized and absorbed into the PDFs
- We've see the basic structure of the corrections including the plus regulators which are related to an incomplete cancellation between the loop and real emission graphs
- We've seen examples of the convention dependence associated with the process of factorization
- The next step is to investigate how to handle more complicated observables in processes involving more partons in the final state