An Introduction to Higher Order Calculations in Perturbative QCD

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2010 JET Summer School
June 14-17, 2010
Lawrence Berkeley National Laboratory

Overview of the Lectures

- Lecture I - Higher Order Calculations
  - What are they?
  - Why do we need them?
  - What are the ingredients and where do they come from?
  - Understanding and treating divergences
  - Examples from e^+e^- annihilation
- Lecture II - Examples of Higher Order Calculations
  - Parton Distribution Functions at higher order
  - Lepton Pair Production at higher order
  - Factorization scale dependence

- Lecture III - Hadronic Production of Jets, Hadrons, and Photons
  - Single inclusive cross sections
  - More complex observables and the need for Monte Carlo techniques
  - Overview of phase space slicing methods
- Lecture IV - Beyond Next-to-Leading-Order
  - When is NLO not enough?
  - Large logs and multiscale problems
  - Resummation techniques

Lecture I - Outline

- Next-to-Leading-Order Calculations - what are they?
- Overview and ingredients
- Dealing with divergences
- Examples from e^+e^- annihilation
  - Total cross section
  - Infrared safe observables
  - Thrust
  - Jet cross sections
  - Particle distributions

NLO Calculations - What are they?

Consider a generic hard scattering process for $A + B \rightarrow C + X$ in the leading-logarithm approximation:

$$
\begin{aligned}
d\sigma(AB \rightarrow C + X) \quad = \quad & \frac{1}{2\hat{s}} \sum_{a,b,c} G_{a/A}(x_a, M_f^2)\, dx_a\, G_{b/B}(x_b, M_f^2)\, dx_b \\
& \alpha_s^n(\mu_r^2) |M_{ab\rightarrow cd}|^2\, D_{C/c}(z_c, M_f^2)\, dz_c\, dPS^{(n)}
\end{aligned}
$$

- $G_{a/A}(x_a, M_f^2)dx_a$ is the probability of finding a parton a in a hadron A with a momentum fraction x between x_a and $x_a + dx_a$.
- $D_{C/c}(z_c, M_f^2)dz_c$ is the probability of finding a hadron C in a parton c with a momentum fraction z between z_c and $z_c + dz_c$.
- M_f denotes the factorization scale which serves to separate the long- and short-distance parts of the scattering process

- The lowest order partonic subprocesses give an $\mathcal{O}(\alpha_s^n)$ contribution
- α_s is the one-loop running coupling
- μ_r is the renormalization scale chosen for α_s
- $\hat{s}$ is the partonic center of mass energy squared
- The parton distribution functions (PDFs) and the fragmentation functions (FFs) are solutions of the appropriate DGLAP equations in the leading-log approximation

To include higher order contributions we must examine how each of these ingredients is modified.

Modifications for Next-to-Leading-Order Calculations

Basically, there are three steps to follow, some being easier than others

1. Choose an appropriate observable and calculate the contributing subprocesses to the next order in the strong coupling
   - These matrix elements must be integrated over the appropriate phase space variables
   - Some methods must be applied to deal with the divergences which will appear
   - Most of the material in these lectures will deal with this step
2. Use the two-loop approximation for the running coupling
3. Use NLO PDFS and FFs

Running Coupling

The running coupling α_s is the solution to the following equation

$$Q^2 \frac{d\alpha_s}{dQ^2} = \beta(\alpha_s)$$

where the β function has a perturbative expansion

$$\beta(\alpha_s) = -b\alpha_s^2(1 + b'\alpha_s + ...)$$

with $b = \frac{(33-2n_f)}{12\pi}$ and $b' = \frac{153-19n_f}{2\pi(33-2n_f)}$
where n_f is the number of active parton flavors.

One can solve the above equation to get

$$t \equiv \ln \frac{Q^2}{\mu^2} = \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}$$

- Neglecting b', this equation has the solution

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2) bt}$$

- Suppose one is calculating an observable that is characterized by a single large scale Q and that it has a perturbative expansion

$$\sigma = \sigma_1 \alpha_s(\mu^2) + \sigma_2 \alpha_s^2(\mu^2) + \ldots$$

- It would seem that our answer depends on both the scale Q and the choice of μ
- But, physically, the answer shouldn't depend on our choice of the scale used to define α_s, at least to the order in perturbation theory that we are using. This is formally encoded in the Renormalization Group Equation.
- The solution of the RGE results in the introduction of the running coupling and allows us to calculate the dependence on the choice of μ in any given order

- For example, in the above case in lowest order we would have

$$
\begin{aligned}
\sigma &\approx \sigma_1 \alpha_s(Q^2) = \sigma_1 \alpha_s(\mu^2) \sum_j \left(-\alpha_s(\mu^2) bt\right)^j \\
&= \sigma_1 \alpha_s(\mu^2) \left[1 - \alpha_s(\mu^2) bt + \alpha_S^2(\mu^2)(bt)^2 + \ldots\right]
\end{aligned}
$$

- In this case the *leading logarithms* depending on μ have been summed into the running coupling $\alpha_s(Q^2)$
- Note how in this example the terms are of the form

$$\alpha_s(\mu^2)^{j+1} t^j$$

- For the leading-log approximation, only the one-loop term (b) is retained.

What about an NLO Calculation?

- In the simple example I have outlined, an NLO calculation would include the term $\sigma_2 \alpha_s^2$
- Expressing this in terms of the running coupling we have

$$\sigma_2 \alpha_s^2(\mu^2)[1 - 2\alpha_s(\mu^2) bt + \ldots]$$

- Notice that there is one less logarithm per power of α_S than for the lowest order term. We have included some subleading logarithms.
- We must examine other sources of subleading logs. One such place is the expression used for the running coupling - using the two-loop expression for β (keeping the b' term) generates contributions which are down by one logarithm
- So, for NLO calculations the two-loop term (b') must be retained.

NLO PDFs

The PDFs satisfy a set of coupled equations (the DGLAP equations)

$$
\begin{aligned}
Q^2 \frac{dG_q(x,Q^2)}{dQ^2} &= \frac{\alpha_s(Q^2)}{2\pi} \int \frac{dy}{y} \left[P_{qq}(y) G_q(\frac{x}{y}, Q^2) + P_{qg}(y) G_g(\frac{x}{y}, Q^2) \right] \\
Q^2 \frac{dG_g(x,Q^2)}{dQ^2} &= \frac{\alpha_s(Q^2)}{2\pi} \int \frac{dy}{y} \left[\sum_q P_{gq}(y) G_q(\frac{x}{y}, Q^2) + P_{gg}(y) G_g(\frac{x}{y}, Q^2) \right]
\end{aligned}
$$

The splitting functions have perturbative expansions

$$
P_{ij}(y) = P_{ij}^{(0)} + \frac{\alpha_s}{2\pi} P_{ij}^{(1)}
$$

For the leading-log expansion only $P_{ij}^{(0)}$ is retained, while for NLO PDFs the $P_{ij}^{(1)}$ contribution is retained, as well.

Comments

- The issue of how the PDFs are obtained from data will be discussed in a later lecture, once we understand how NLO calculations are performed
- A similar set of DGLAP equations holds for the FFs (it involves the transpose of the splitting function matrix) and the same perturbative expansion is used.

The remaining ingredient for an NLO calculation is to include the next order contribution from the squared matrix elements along with the appropriate phase space factor.

- When one tries to perform the phase space integrals, divergences are encountered
- The challenge is two-fold:
  - Regulate the divergences (render them finite)
  - Interpret and remove them, leaving a finite correction for some observable

Why should we care about NLO calculations?

- LO calculations have a monotonic decrease as μ_r increases
- LO calculations have a monotonic decrease (increase) with increasing M_f for $x \gtrsim .1$ ($x \lesssim .1$)
- NLO calculations cancel some, but not all, of the scale dependences
- NLO calculations can improve the accuracy of the theoretical predictions as the hard scattering is now calculated to one order higher in α_s.
- Since there is one additional parton in the final state, one can gain new information on jet substructure, angular distributions, etc.

NLO Matrix Element Overview

If the lowest order subprocess has an n-body final state, then at the next order we have

- n-body final state one-loop diagrams. The interference between these and the lowest order diagrams gives a cross section contribution that is one order higher in α_s.
- $n+1$-body final state contributions

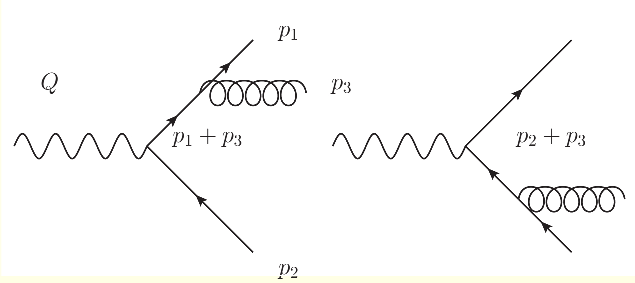
When one tries to calculate these higher order terms one finds:

- Infrared (IR), collinear, and ultraviolet (UV) singularities from virtual diagrams
- Soft singularities from some $n+1$-body processes
- Collinear singularities from some regions of the $n+1$ phase space

Start by considering the relatively simple process of e^+e^- annihilation.

e^+e^- annihilation

First, consider the $2 \rightarrow 3$ $e^+e^- \rightarrow q\bar{q}g$ subprocess. Actually, it is easier to consider the decay of a virtual photon of 4-momentum Q as shown below:



- Kinematics - use massless quarks and gluons.
- Define $x_i = 2E_i/Q, i = 1, 2, 3$ in the overall center-of-mass system where Q denotes the total energy $\Rightarrow x_1 + x_2 + x_3 = 2$.
- $(p_1 + p_3)^2 = 2p_1 \cdot p_3 = (Q - p_2)^2 = Q^2(1 - x_2)$
- $(p_2 + p_3)^2 = 2p_2 \cdot p_3 = (Q - p_1)^2 = Q^2(1 - x_1)$
- The quark propagators from the above diagrams will give factors of $(1-x_1)$ and $(1-x_2)$ in the denominator. $x_1 \rightarrow 1$ corresponds to $\vec{p}_3 \parallel \vec{p}_2$ while $x_2 \rightarrow 1$ corresponds to $\vec{p}_3 \parallel \vec{p}_1$. Note that if both $x_1$and $x_2 \rightarrow 1$ then $x_3 \rightarrow 0$.

3-body Phase Space

Exercise: Show that

$$\begin{aligned} dPS_3 &= \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta(Q-p_1-p_2-p_3) \\ &= \frac{Q^2}{16(2\pi)^3}\,dx_1\,dx_2 \end{aligned}$$

Using this result it is straightforward to show that the differential cross section can be written as

$$\frac{1}{\sigma}\frac{d\sigma}{dx_1\,dx_2} = C_F\frac{\alpha_s}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

For the total cross section, one should integrate over both x_1 and x_2. These integrations diverge when either x_1 or x_2 or both approach unity.

Partial fraction the denominators:

$$\frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left(\frac{1}{(1-x_1)} + \frac{1}{(1-x_2)}\right)$$

- This shows that the double pole when both x_1 and x_2 approach unity is due to a combination of a collinear divergence (x_1 or $x_2 \longrightarrow 1$) and a soft divergence ($x_3 \to 0$).
- The problem now is how to generate a finite contribution to the total cross section.
- We shall use dimensional regularization
  - Analytically continue in the number of dimensions from $n = 4$ to $n = 4 - 2\epsilon$.
  - For the soft and collinear singularities we will take $\epsilon < 0$
  - Converts logarithmic divergences into poles in ϵ.
  - Note: we will use the substitution $g_s \to g_s\mu^\epsilon$ in order for the strong coupling to remain dimensionless in n dimensions

Phase space becomes

$$dPS_3^n = \frac{Q^2}{16(2\pi)^3}\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}\left(\frac{1-u^2}{4}\right)^{-\epsilon}\frac{1}{\Gamma(2-2\epsilon)}x_1^{-2\epsilon}dx_1\, x_2^{-2\epsilon}dx_2$$

where $u = 1 - \frac{2(1-x_1-x_2)}{x_1 x_2}$

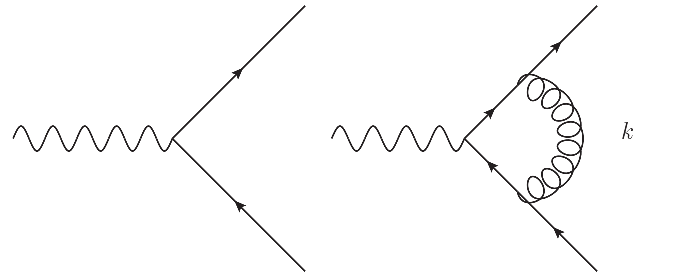
- It is not obvious how this helps until you make a substitution $x_2 = 1 - v x_1$
- The u dependent term introduces factors of $(1-v)^{-\epsilon}$ and $(1-x_1)^{-\epsilon}$
- dx_2 becomes $x_1 dv$
- Then note that

$$\int_0^1 dx(1-x)^{-1-\epsilon} = \frac{1}{-\epsilon}(1-x)^{-\epsilon}|_0^1 = \frac{1}{-\epsilon}$$

  as long as $\epsilon < 0$.

- The logarithmic divergence has, indeed, been converted into a pole in ϵ.

$2 \to 2$ contribution



- The loop graph is $\mathcal{O}(\alpha_s)$ so the interference with the lowest order term gives an $\mathcal{O}(\alpha_s)$ contribution to the cross section
- The loop integral has a denominator of the form: $k^2(p_1+k)^2(p_2-k)^2$
- The denominator vanishes when $k \to 0$ or when k is collinear with either p_1 or p_2
- These singularities correspond to the same types as observed for the $q\bar{q}g$ final state
- Can also use dimensional regularization to evaluate the loop contribution in n-dimensions

Final Results

- After doing both of the integrations for the three-body , one arrives at

$$\sigma_3 = \frac{\alpha_s}{2\pi} C_F \sigma_0 \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right]$$

  where σ_0 is the lowest order result.

- After doing the loop integral for the virtual contribution one gets

$$\sigma_v = \frac{\alpha_s}{2\pi} C_F \sigma_0 \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right]$$

- Adding the two together along with the lowest order result yields

$$\sigma = \sigma_0 (1 + \frac{\alpha_s}{\pi})$$

- The poles in ϵ have all cancelled, leaving a finite higher order correction

Infrared Safety

- This is an example which will play out over and over in the following – for a suitable defined inclusive observable there is a cancellation between the soft and collinear singularities occurring in the real contributions and those which occur in the loop contributions.
- It is imperative that this cancellation be allowed to occur when calculating any observable!
- Care must be taken when designing new observables to insure that they do not distinguish between a configuration of partons and the same one where a soft or collinear parton is added.
- Observables that respect this constraint are called infrared safe observables
- The requirement of infrared safety is a necessary condition for an observable to be calculable in perturbation theory.

Differential Observables

- In order to further test the theory one would like to have more information than that provided buy the total cross section
- The phase space integrations obscure a lot of information which should be tested by comparison with data
- An example of an infrared safe observable - Thrust

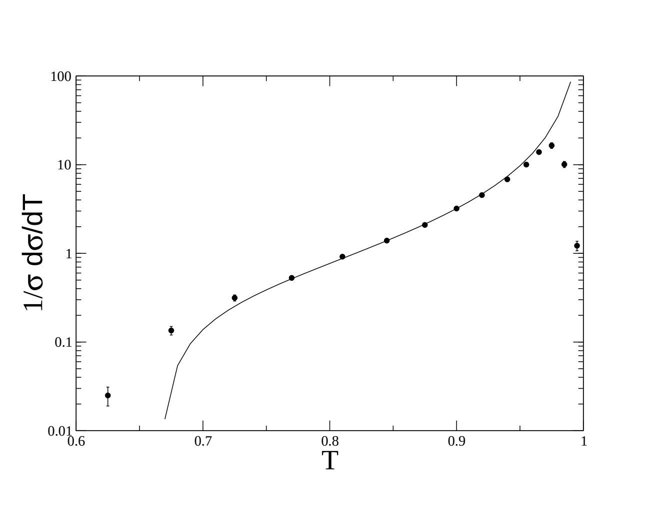
$$T = \max \vec{n} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$

- Vary the choice of the thrust axis $\vec{n}$ in order to maximize T
- 2 parton final state: $\vec{n}$ lies along p_1 and T=1

- If one of the partons emits a collinear parton, then nothing changes and $T = 1$
- If a soft gluon is emitted, then in the limit of zero energy nothing changes and $T = 1$
- The various divergent contributions seen previously all lie at $T = 1$ so that the cancellations still occur
- $T \neq 1$ yields information on the relative angular distributions of the three final state partons
- Note: no jet definition is required in order to study the thrust distribution
- Note: A spherically symmetric multiparton final state: T=1/2

- The thrust distribution is easily calculable: $T = \max[x_i]$
- Integrate $d\sigma/dx_1\,dx_2$ over x_1 and x_2 subject to the above constraint
- Result is

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_s}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)}\right]$$



- Expected divergence as $T \to 1$ is evident
- Perturbative corrections become large in this region - a better treatment is needed

Comments on Jet Algorithms

- At lowest order, one associates final state partons with jets. One might therefore expect that the $\mathcal{O}(\alpha_s)$ calculation would shed information on both the 2- and 3-jet cross sections
- In order to define a jet cross section, one needs an infrared safe jet definition
- Such a definition must not distinguish between a parton and two collinear partons or between a parton and a parton plus a soft parton
- Examples include Sterman-Weinberg jets, cone jets, k_T algorithms, and many more
- The key point I wish to make is that whatever algorithm is used, it must allow for the cancellation between the soft and collinear singularities from the real emission graphs for an $n-$body process and those from the $(n-1)$-body virtual graphs
- This point will become of great importance when we discuss NLO programs based on phase space slicing techniques

Fragmentation Functions

- What if one wants to study the hadronic composition of the final state?
- What if you don't want to use a jet observable which depends on choosing a specific jet algorithm?
- What if your detector is optimized for particle detection, but not for reconstructing jets?
- One solution is to introduce Fragmentation Functions (FFs)

  $D_{C/c}(z)\,dz$ is the probability of getting a hadron C from a parton c with a fraction of the parton's momentum fraction between z and $z + dz$
- Lowest order form for $e^+e^- \rightarrow hadrons$

$$\frac{1}{\sigma}\frac{d\sigma^h}{dz} = \sum_q e_q^2 \left[D_{h/q}(z) + D_{h/\overline{q}}(z)\right]$$

- How can we extend this concept to include the $\mathcal{O}(\alpha_s)$ corrections that we have been studying?

Virtual Contributions

- These are easy, as we have already done the work! The virtual contribution, σ_v, calculated previously has the same final state structure as the lowest order term.
- It can be included along with the lowest order term by just multiplying the previous expression by $1 + \frac{\sigma_v}{\sigma_0}$

Three-body Contribution

- This one is more complicated - there are now three partons in the final state and each can give rise to hadrons
- Not only do we have quark and antiquark FFs, but now we also have to include a possible gluon FF.
- The basic structure should be familiar:

$$\frac{d\sigma}{dz} = \frac{1}{2Q}\,(\text{PhaseSpace})\,(\text{Squared matrix elements})\,(\text{FFs})$$

- Of course, this is all done in n dimensions in order to regularize the soft and collinear divergences

- Bear with me - this is going to get complicated, but there is a reason for all of this
- Plugging the appropriate terms into the above expression yields

$$
\begin{aligned}
\frac{d\sigma}{dz} \quad &= \frac{1}{2Q}\frac{Q^2}{16(2\pi)^3}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-2\epsilon}\int\int\left(\frac{1-u^2}{4}\right)^{-\epsilon}\frac{1}{\Gamma(2-2\epsilon)}x_1^{-2\epsilon}dx_1\, x_2^{-2\epsilon}dx_2 \\
&\frac{8(e\,g\,\mu^{2\epsilon})^2}{(1-x_1)(1-x_2)}\left[(n-2)(x_1^2+x_2^2)+2(n-4)(n-2)(2(1-x_1-x_2)+x_1x_2)\right] \\
&e_q^2\left[D_{h/q}(y)\delta(z-yx_1)+D_{h/\overline{q}}(y)\delta(z-yx_2)+D_{h/g}(y)\delta(z-yx_3)\right]dy
\end{aligned}
$$

- We recognize some familiar structures from the total cross section calculation, but the structure is more complex
- How can we do the integrations with the unknown FFs in the integrands?
- How can we ensure that the proper soft and collinear cancellations take place?
- Proceed as in the total cross section case. Consider the first term and make the substitution $x_2 = 1 - vx_1$.

- This introduces factors of $(1-x_1)^{-\epsilon}$ and $(1-v)^{-\epsilon}$
- The v integrations can be done using

$$\int_0^1 dv\, v^{n-1}\,(1-v)^{m-1} = B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

- This generates explicit poles in ϵ through terms like

$$B(-\epsilon,1-\epsilon) = \frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}$$

- Can do the y integration using the δ function

- We are left with terms like

$$\int_z^1 dx_1\, x_1^{n-1}\, (1 - x_1)^{m-1} D_{h/q}(z/x_1)$$

- Note the non-zero lower limit on the integral which is forced by the argument of the FF
- How are we to do this integral in order to pull out the singular terms when we don't know the analytic form for the FF?
- Enter the "+" distribution!
- This distribution will enable us to extract the poles in ϵ from integrals of the above form

Consider

$$
\begin{aligned}
I &= \int_0^1 dw\,(1-w)^{-1-\epsilon}\,f(w) \\
&= \int_0^1 dw\,(1-w)^{-1-\epsilon}\,[f(1) + (f(w) - f(1))] \\
&= -\frac{f(1)}{\epsilon} + \int_0^1 dw\,\frac{f(w) - f(1)}{1-w}\,\left[1 - \epsilon\ln(1-w) + \mathcal{O}(\epsilon^2)\right] \\
&= -\frac{f(1)}{\epsilon} + \int_0^1 dw\,\frac{f(w) - f(1)}{1-w} - \epsilon\int_0^1 dw\,\frac{\ln(1-w)}{1-w}\,[f(w) - f(1)] + \mathcal{O}(\epsilon^2) \\
&\equiv -\frac{f(1)}{\epsilon} + \int_0^1 dw\,\frac{f(w)}{(1-w)_+} - \epsilon\int_0^1 dw\,\left(\frac{\ln(1-w)}{1-w}\right)_+ f(w) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

This last expression allows us to make the following identification

$$
(1-w)^{-1-\epsilon} = -\frac{\delta(1-w)}{\epsilon} + \frac{1}{(1-w)_+} - \epsilon\left(\frac{\ln(1-w)}{1-w}\right)_+
$$

- The astute reader will no doubt have noticed that the previous derivation involved integrals extending from zero to one. What if the lower limit is non-zero?
- The derivation can be repeated and the only difference will be in the δ-function term. There we will get (recall that $\epsilon < 0$)

$$\begin{aligned}
\frac{1}{\epsilon}(1-w)^{-\epsilon}\Big|_a^1 &= -\frac{1}{\epsilon}(1-a)^{-\epsilon} \\
&= -\frac{1}{\epsilon}\left[1 - \epsilon \ln(1-a) + \frac{\epsilon^2}{2}\ln^2(1-a) + \ldots\right] \\
&= -\frac{1}{\epsilon} + \ln(1-a) - \frac{\epsilon}{2}\ln^2(1-a) + \ldots
\end{aligned}$$

- The regulators under the integral signs behave the same way as when the lower limit was zero.

- Schematically we can write

$$\frac{1}{(1-w)_+} = \frac{1}{(1-w)_a} + \ln(1-a)\delta(1-w)$$

and

$$\left(\frac{\ln(1-w)}{1-w}\right)_+ = \left(\frac{\ln(1-w)}{1-w}\right)_a + \frac{1}{2}\ln^2(1-a)\delta(1-w)$$

There are several important points to notice about these regulators

- We derived these expressions by adding and subtracting $f(1)$ and then rearranging the integrations. When the lower limit is non-zero, the cancellation between these two terms with $f(1)$ is no longer exact and there is a remainder involving logs of $(1-a)$
- As the lower limit, a, approaches 1 these logs can become large.
- This could happen with the fragmentation functions if we were interested in the region of large z.

- These logs are called "threshold" logs and physically what is happening is that the phase space for additional gluon radiation is being limited by the requirement that z be large. These large logs must be resummed via a procedure referred to as "soft gluon" or "threshold" resummation
- Remember the idea of incomplete cancellation between the virtual and real contributions with a finite remainder consisting of potentially large logarithms
- After this interlude, we can go back to the fragmentation calculation

- We have done the y and v integrations, leaving integrals of the form

$$\int_z^1 dx_1 \, x_1^{n-1} \, (1-x_1)^{m-1} D_{h/q}(z/x_1)$$

- Terms with $m = -\epsilon$ will give poles proportion to $\delta(1-x_1)$
- Doing the x_1 integration will give pole terms proportional to $D_{h/q}(z)$ which can now be combined with the lowest order terms
- In this way we can extract the divergent pieces needed for the cancellation with the virtual contributions

- The intermediate answer for the next order contribution after adding the virtual contribution to the three real fragmentation pieces is as follows

$$\frac{d\sigma}{dz} = \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$\sum_q e_q^2 \int dx\, dy\, \delta(z-xy) \left(D_{h/q}(y) + D_{h/\overline{q}}(y)\right)$$

$$\left[\delta(1-x)\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \frac{2}{\epsilon^2} + \frac{3}{2\epsilon}\right) - \frac{1}{\epsilon}\frac{1+x^2}{(1-x)_+} + \tilde{f}_q(x)\right]$$

$$+ \quad \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \sum_q e_q^2 \int dx\, dy\, \delta(z-xy)\, 2\, D_{h/g}(y)$$

$$\left(-\frac{1}{\epsilon}\frac{1+(1-x)^2}{x} + \tilde{f}_g(x)\right)$$

- See that the $\frac{1}{\epsilon^2}$ terms cancel, but that there are some remaining $\frac{1}{\epsilon}$ pieces
- How should these be interpreted and what can we do about them?

- These are residual collinear singularities associated with the quark propagators going on shell in the collinear quark+gluon configuration
- On-shell propagators are associated with long range physics and should not be associated with the hard scattering correction that we are calculating.
- Factorize the remaining collinear singularities and absorb them into the bare FFs.
- We need to define a scheme to tell us how much of the finite contributions to subtract along with the ϵ pole terms. Use

$$\frac{1}{\epsilon}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{\epsilon} + \ln(4\pi) - \gamma_E - \ln\frac{M_f^2}{\mu^2} - \ln\frac{Q^2}{M_f^2} + \ldots$$

- The MS scheme (Minimal Subtraction) says to subtract only the pole term
- The $\overline{\mathrm{MS}}$ scheme (Modified Minimal Subtraction) says to also subtract the $\ln(4\pi) - \gamma_E$ terms

- In addition, I have introduced a factorization scale M_f and I will subtract the $\ln \frac{M_f^2}{\mu^2}$ term, as well. Technically, each choice of M_f defines a new scheme, but we usually refer to all of them as being the $\overline{\mathrm{MS}}$ scheme
- In order to absorb the collinear singularities in the bare FFs, introduce a scale-dependent FF

$$D_{h/q}(z, M_f^2) = D_{h/q}(z) + \frac{\alpha_s}{2\pi} \int dx\, dy\, \delta(z - xy) \left(-\frac{1}{\epsilon}\right) \left(\frac{M_f^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[C_F \left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right) D_{h/q}(y) + C_F \frac{1+(1-x)^2}{x} D_{h/g}(y)\right]$$

- Replacing the bare FFs by the above scale-dependent FFs will cancel the remaining collinear singularities

- The final result takes the form

$$\frac{1}{\sigma}\frac{d\sigma}{dz} =$$

$$\sum_q e_q^2 \left[D_{h/q}(z, M_f^2) + D_{h/\overline{q}}(z, M_f^2)\right]$$

$$+ \quad \frac{\alpha_s}{2\pi}\sum_q e_q^2 \int_z^1 \frac{dx}{x} \left[D_{h/q}(\frac{z}{x}, M_f^2) + D_{h/\overline{q}}(\frac{z}{x}, M_f^2)\right] \left[\ln \frac{Q^2}{M_f^2} P_{qq}(x) + \tilde{f}_q\right]$$

$$+ \quad \frac{\alpha_s}{2\pi}\sum_q e_q^2 \int_z^1 \frac{dx}{x}\, 2D_{h/g}(\frac{z}{x}, M_f^2) \left[\ln \frac{Q^2}{M_f^2} P_{gq}(x) + \tilde{f}_g\right]$$

- Here I have used two splitting functions defined as
$P_{qq}(x) = C_F \left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right)$ and $P_{gq}(x) = C_F \left(\frac{1+(1-x)^2}{x}\right)$

- Note that I have substituted the scale-dependent FFs on the right hand side - this is allowed to this order
- Note that the results simplify considerably if we choose $M_f = Q$
- In this case the $\ln \frac{Q^2}{M_f^2}$ terms disappear and all of the logs have been absorbed in the scale-dependent FFs

Summary

In this lecture we have seen the following

- The typical ingredients for the hard scattering subprocesses include
  - The lowest order expressions for the relevant subprocesses
  - The 1-loop virtual corrections to these subprocesses
  - The expressions for the relevant next order subprocesses
- The real processes generally have both soft and collinear singularities
- After renormalization, the loop graphs also contribute soft and collinear singularities
- For suitable observables these singularities cancel, leaving finite higher order corrections
- Observables for which this occurs are said to be infrared safe
- If one wants to study specific details of the hadronic final state, then Fragmentation Functions can be introduced.
- In the next order there will be uncancelled collinear singularities which can be absorbed into the bare FFs by defining scale-dependent FFs