K to $\pi\pi$ Decays on the Lattice

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Introduction

- Quantitative understanding of $K \to \pi\pi$ decays has been an outstanding problem for over 50 years.
- Study is motivated by the CP violating parameter ϵ'/ϵ
- Also of interest is the Δ I$= 1/2$ rule
- Need full non perturbative calculation of decay amplitudes - LQCD

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- Also of interest is the Δ I$= 1/2$ rule
- Need full non perturbative calculation of decay amplitudes - LQCD
- Must overcome problems such as two-particle final state and uncertain chiral extrapolation
- RBC/UKQCD Collaboration - Calculate $K \to \pi\pi$ amplitude directly on the lattice
- To achieve this we use a large lattice, near physical pion mass and Lellouch Luscher factor for the two-particle final state

Effective Hamiltonian for weak decay

- Effective Hamiltonian describes weak interactions and effects of heavy quarks

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{i=1}^{10} V^i_{CKM} C_i(\mu) Q_i(\mu)$$

  C_i are Wilson coefficients, Q_i are 4-quark operators governing decay.

- Weak matrix element is

$$\langle \pi\pi | H_{\text{eff}} | K \rangle = \frac{G_F}{\sqrt{2}} \sum_{i=1}^{10} V^i_{CKM} C_i(\mu) \langle \pi\pi | Q_i(\mu) | K \rangle$$

- Scale dependence of C_i and $\langle \pi\pi | Q_i(\mu) | K \rangle$ must cancel.

Isospin Channels

- Two possible Isospin channels for $K \to \pi\pi$ decays: I=0 & I=2
- I=0 very difficult
- All 10 operators contribute ($\to$ 48 different Wick contractions)
- Operator mixing
- Signal typically very noisy due to contribution from vacuum
- Present calculation on $16^3 \times 32$ lattices with 2+1 flavours of sea quark at unphysical masses ($m_\pi = 420$MeV).

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- I=2 comparatively easy!
- Only three operators to consider (These are linear combinations of the Q_i).
- No mixing with lower dimensional operators
- Avoid problem of vacuum subtraction
- Current calculation on $32^3 \times 64$ lattices, $L_s = 32$ with $2 + 1$ sea quarks at near physical pion mass (valence $m_\pi = 145.6$ MeV)

I=2 Operators

- Classify operators based on how they transform under $SU(3)_L \times SU(3)_R$.
- Three operators to consider:

$$Q_{(27,1)}^{\Delta I=3/2} = (\bar{s}d)_L \left\{(\bar{u}u)_L - (\bar{d}d)_L\right\} + (\bar{s}u)_L(\bar{u}d)_L$$
$$Q_{(8,8)}^{\Delta I=3/2} = (\bar{s}d)_L \left\{(\bar{u}u)_R - (\bar{d}d)_R\right\} + (\bar{s}u)_L(\bar{u}d)_R$$
$$Q_{(8,8)\mathrm{mx}}^{\Delta I=3/2} = (\bar{s}^i d^j)_L \left\{(\bar{u}^j u^i)_R - (\bar{d}^j d^i)_R\right\} + (\bar{s}^i u^j)_L(\bar{u}^j d^i)_R$$

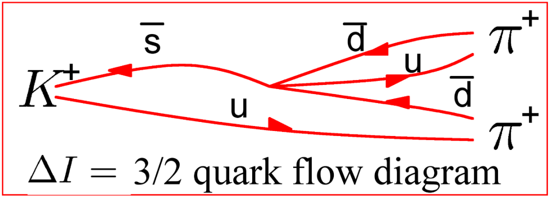
I=2

- We use the Wigner-Eckart theorem to relate $K^+ \to \pi^+\pi^0$ matrix element to the $K^+ \to \pi^+\pi^+$ matrix element
- This simplifies operators

$$Q^{3/2}_{(27,1)} \to Q'^{3/2}_{(27,1)} = (\bar{s}d)_L(\bar{u}d)_L$$

- The relative normalization is

$$\langle \pi^+\pi^0 | \, Q^{3/2} \, | K^+ \rangle = \frac{3}{2} \langle \pi^+\pi^+ | \, Q'^{3/2} \, | K^+ \rangle$$



I=2 Momentum

- For K to $\pi\pi$ decays in the CM frame the pions have non-zero momentum.
- Fitting to excited states is problematic and results are typically very noisy.
- One solution is to use twisted boundary conditions. Instead of using periodic boundary conditions in the spatial directions for quark field, choose boundary conditions which cause the quark field to change by a phase $e^{i\phi}$ when going through the boundary. The spatial direction is twisted by an amount ϕ.
- Instead of momentum $\frac{2\pi n}{L}$, the quark has momentum $\frac{\phi + 2\pi n}{L}$.

I=2 Momentum

- Set $\phi = \pi$ so that the two-pion ground state has zero total momentum.
- Twist d-quark only resulting in kaon at rest and pions with equal and opposite momentum
- For on-shell kinematics we twist in two directions corresponding to $p = \sqrt{2}\pi/L$ for each pion.
- Cosine source reduces number of inversions needed.

$$s_{\mathbf{p}\mathrm{cosine}}(\mathbf{x}) = \cos(p_x x)\cos(p_y y)\cos(p_z z)$$

I=2 Simulation Parameters

- RBC/UKQCD $32^3 \times 64$, $L_s = 32$ lattices with Domain Wall Fermions and 2+1 dynamical quark flavours, generated on BG/P at ANL.
- Large lattice volume to accomodate two-particle final state: $L = 4.51$ fm.
- Simulate with $m_l^{sea} = 0.001$, $m_l^{val} = 0.0001$ $m_s^{sea} = 0.045$ and $m_s^{val} = 0.049$.
- Unitary pion mass ≈ 180 MeV.

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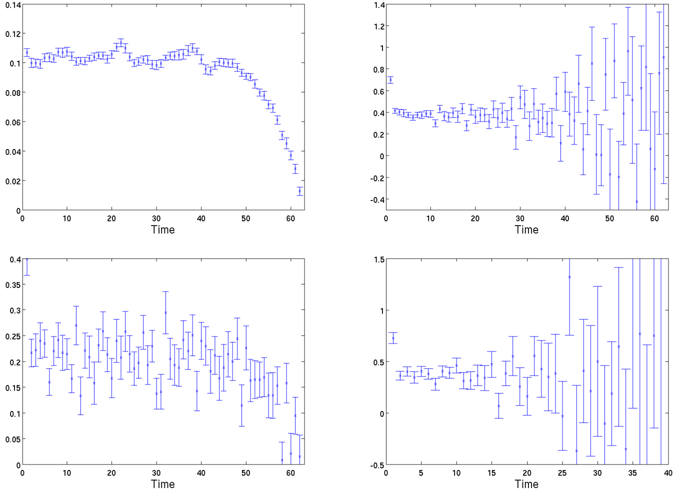
- Unitary pion mass ≈ 180 MeV.
- Inverse lattice spacing $a^{-1} = 1.4$ GeV.

I=2 Details of calculation

- Analyse 47 configurations
- Combine propagators with P+A b.c. and P-A b.c. to double effective time extent of the lattice
- Light quark propagators have a source at $t_\pi = 0$.
- S-quark has source at $t_k = 20$, 24, 28 and 32
- Operator inserted at all times inbetween.

I=2 Masses and Energies

	m_π	m_K	$E_{\pi\pi}(p=0)$	$E_{\pi\pi}(p=\sqrt{2}\pi/L)$
Lattice	0.10400(37)	0.3706(13)	2100(10)	0.3687(61)
MeV	145.6(5)	519(2)	294(1)	516(9)



Extracting the Matrix Element

- Start with 3-point correlation function

$$C_{K\to\pi\pi} = \left\langle O_{\pi\pi}(\mathbf{p}, t_\pi) Q_i(t) O_K^\dagger(t_K) \right\rangle$$
$$\sim \mathcal{M} Z_{\pi\pi} Z_K \exp(-E_{\pi\pi}|t - t_\pi|) \exp(-m_K|t - t_K|)$$

with $Z_{\pi\pi} = |\langle 0| O_{\pi\pi}(0) |\pi\pi\rangle|$ and $Z_K = |\langle 0| O_K(0) |K\rangle|$

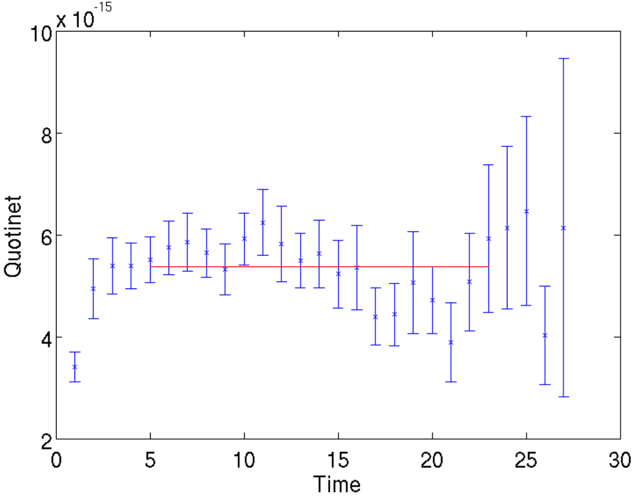
- The quotient

$$\frac{C_{K\pi\pi}(t)}{C_K(t_K - t) \times C_{\pi\pi}(t)} \sim \frac{\mathcal{M}}{Z_K Z_{\pi\pi}}$$

should be constant in time. Fit this to extract matrix element.

I=2 Matrix Element

Quotient of Correlators for $(8,8)$ operator. $t_K = 28$, $p_\pi = \frac{\sqrt{2}\pi}{L}$



Two Pion Phase Shift

- Simple relation exists which relates the two-pion final state energy to the S-wave phase shift (Luscher, M., nucl. Phys. B. 354, p. 531-578)

$$n\pi - \delta(q_\pi) = \phi(q_\pi), \qquad q_\pi = \frac{p_\pi L}{2\pi}$$

where

$$\tan\phi(q) = -\frac{\pi^{3/2} q}{\mathcal{Z}_{00}(1, q^2)}$$

$$\mathcal{Z}_{00}(s, q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n}} (\mathbf{n}^2 - q^2)^{-s}$$

- p_π determined from two-pion energy: $E_{\pi\pi} = 2\sqrt{m_\pi^2 + p_\pi^2}$
- We found

p_π	δ
20(3) MeV	-0.44(17) degrees
213(5) MeV	-19.6(5.6) degrees

Isospin Amplitude A_2

- A_2 is physical quantitiy which can be compared with experiment

$$A(K^0 \to \pi\pi(I=2)) = A_2 e^{i\delta_2}$$

- A_2 related to matrix element via

$$A_2 = \frac{\sqrt{3}}{2\sqrt{2}} \frac{1}{\pi q_\pi} \sqrt{\frac{\partial \phi}{\partial q_\pi} + \frac{\partial \delta}{\partial q_\pi}} L^{3/2} a^{-3} G_F V_{ud} V_{us} \sqrt{m_K} E_{\pi\pi} \times \sum_{i,j} C_i(\mu) Z_{ij}(\mu) \langle \pi\pi | Q_j | K \rangle$$

- $C_i(\mu)$ are Wilson Coefficients
- Z_{ij} operator renormalization constants calculated using NPR. In general operators will mix under renormalization.
- All quantities are real except for Wilson Coefficients which are complex.

Lellouch-Luscher Factor

- The Lellouch Luscher factor is a finite volume correction which takes into account the final state interactions
- Need $\sqrt{\frac{\partial \phi}{\partial q_\pi} + \frac{\partial \delta}{\partial q_\pi}}$
- $\frac{\partial \phi}{\partial q_\pi}$ calculated numerically using Zeta function
- Luscher relation only valid at energies allowed on the lattice, so cannot calculate $\frac{\partial \delta}{\partial q_\pi}$ from $\frac{\partial \phi}{\partial q_\pi}$
- Instead estimate $\frac{\partial \delta}{\partial q_\pi}$ from phenomenological curve:

$$\tan \delta = \sqrt{1 - \frac{4m_\pi^2}{s}} \left(A + B\frac{p^2}{m_\pi^2} + C\frac{p^4}{m_\pi^4} + D\frac{p^6}{m_\pi^6} \right) \left(\frac{4m_\pi^2 - s}{s - s_0} \right)$$

  where s is Mandlestam CM energy^2 and constants are fit from experiment.
- for $p_\pi = 213\text{MeV}$, get

$$\frac{\partial \delta}{\partial q_\pi} = -0.305$$

Re(A_2)

- Main contribution to Re(A_2) comes from (27,1) operator.
- NPR performed on $L_s = 16$ lattices.
- Evaluate $C_{(27,1)}$ and $Z_{(27,1)}$ at 2GeV. A_2 should not depend on scale.
- Our results are:

	Re(A_2)(10^{-8} GeV)
$t_K = 20$	1.52(12)
$t_K = 24$	1.52(10)
$t_K = 28$	1.71(13)
$t_K = 32$	1.35(22)
Error Weighted Average	1.555(73)
Experiment	1.5

Im(A2)

- Main contribution to $\mathrm{Im}(A_2)$ is from $(8,8)$ and $(8,8)\mathrm{mx}$ operators. These mix under renormalization.
- NPR to be done soon. (Nicholas Garron)
- Make crude approximation $Z_{ij} = 0.9 Z_q^2 \delta_{ij}$ for $(8,8)$ and $(8,8)\mathrm{mx}$ operators.

	$\mathrm{Im}(A_2)(10^{-13}$ GeV)
$t_K = 20$	-9.20(50)
$t_K = 24$	-10.03(70)
$t_K = 28$	-9.51(73)
$t_K = 32$	-10.10(84)
Error Weighted Average	-9.58(44)

- Ratio $\mathrm{Im}(A_2)/\mathrm{Re}(A_2)$ needed in calculation of ϵ'. From error weighted averages we find

$$\mathrm{Im}(A_2)/\mathrm{Re}(A_2) = -6.16(29) \times 10^{-5}$$

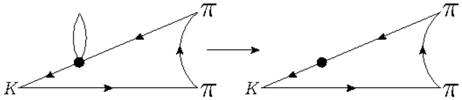
I=2 Summary

- Results presented today are from state of the art lattice simulations.
- Still a work in progress. Need NPR for $(8, 8)$ and $(8, 8)\mathrm{mx}$ operators to reduce systematic error on $\mathrm{Im}(A_2)$
- Details of this calculation will eventually be published in M. Lightman and E. Goode PoS(LATTICE 2010 ***)
- Details of feasibility test preceeding this study can be found M. Lightman and E. Goode, *Proceedings of the XXVII International Symposium on Lattice Field Theory*, PoS(LATTICE 2009)254 [hep-lat/0912.1667].

I=0

Why is $K \to \pi\pi(I = 0)$ so much harder?

1. Final state has same quantum numbers as the vacuum. $\pi\pi$ signal decreases exponentially in time, but contirbution from vacuum is constant (Signal very noisy).
   Improve statistics by averaging over multiple kaon and pion sources (Expensive).
2. Mixing with lower dimensional operators



Details of the calculation

- $16^3 \times 32$ DWF action, $L_s = 16$. $2+1$ dynamical sea quarks. $a^{-1} = 1.73(3)$ GeV.
- Use Coulomb Gauge fixed wall sources for all propagators except quark bubbles. For quark bubbles use stochastic source.
- For 3-point function, Kaons computed at source time $t_K = 1, 2, \cdots 32$. Pions have source at $t_\pi = t_K + 14$. Average over all 32 source positions.
- Sim for two-point functions, average over 32 source positions for kaon and two-pion.
- In contrast to $I = 2$ calculation we do a direct fit to 3-point function data.

$$C_{K\pi\pi} \sim \mathcal{M}|Z_K||Z_{\pi\pi}|e^{-m_K t}e^{-E_{\pi\pi}(14-t)}$$

  is fit to a single parameter $\mathcal{M}$.
- Analysis done for 400 configurations. (Soon to be 800 configurations).

I=0 Results

- Prelimenary results presented by Qi Liu at Lattice 2010 for all 10 operators (zero momentum).
- Unphysical meson masses $m_K = 778\text{MeV}$, $m_\pi = 420\text{MeV}$.
- $\text{Re}(A_0) = 43(12) \times 10^{-8}\text{GeV}$ or $\text{Re}(A_0) = 54.8(3.0) \times 10^{-8}\text{GeV}$ if disconnected diagrams are ignored.
- $\text{Im}(A_0) = -41(31) \times 10^{-12}$ or $\text{Im}(A_0) = -89.2(7.5) \times 10^{-12}$ if disconnected diagrams are ignored.

I=0 Prospects and Conclusions

- This study has shown that it is possible to extract a signal for $K \to \pi\pi(I = 0)$ amplitude.
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- This study has shown that it is possible to extract a signal for $K \to \pi\pi(I = 0)$ amplitude.
- We know we can do better though...
- Improve results by increasing statistics.
- Compute propagators with P+A and P-A BC in order to double effective time extent of lattice and suppress complicated "around the world" effects.
- Eventually wish to recalculate at lighter masses.