Non-perturbative B physics on the Lattice

Nicolas Garron

Lattice meets phenomenology

ALPHA
Collaboration

In collaboration with B. Blossier, M. Della Morte, P. Fritzsch, J. Heitger, G. von Hippel, F. Knechtli, B. Leder, T. Mendes, S. Schaefer, F. Virotta, H. Simma, R. Sommer, N. Tantalo

Outline

- Motivations
- Lattice implementation of the b-quark
- HQET on the lattice
- The b-quark mass
- The heavy-light decay constant
- Results and summary

motivations

- $\Delta m_{\rm d} \propto f^2_{\rm B_d} B_{\rm B_d} |V_{\rm td} V^*_{\rm tb}|^2$ and $\Delta m_{\rm s} \propto f^2_{\rm B_s} B_{\rm B_d} |V_{\rm ts} V^*_{\rm tb}|^2$
- $\mathcal{B}_R({\rm B_s} \to \mu^+\mu^-) = F^2_{B_s}(C_{\rm SM} + \tan^6 \beta_{\rm MSSM})$
- Theoretical uncertainty on the inclusive determination on $|V_{ub}|$ dominated by the one of the b-quark mass $\delta V_{ub}/V_{ub} \sim 4\, \delta m_{\rm b}/m_{\rm b}$

  Now $\delta m_{\rm b} = 40\ {\rm MeV} \Rightarrow \delta V_{ub}/V_{ub} = 3.5\%$ [Hitlin et al. 09]

Need to decrease the errors in the b-quark sector, in particular the ones coming from the non-perturbative effects.
$\Rightarrow$ A task for lattice QCD

Unfortunately, with today's computer resources, it is not possible
(Realistic simulations of a b-quark from lattice QCD)*
$\Rightarrow$ Instead one uses lattice effective theory

motivations

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Unfortunately, with today's computer resources, it is not possible
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* With one recent and noticeable exception, HPQCD with HISQ [HPQCD PRD'10]
not covered in this talk
but was discussed yesterday by C. Davies and by E. Follana

Heavy quark on the lattice

b-quark on the lattice

B meson contains both light and heavy degrees of freedom
$m_s \sim 100\ \mathrm{MeV}$ and $m_b \sim 4\ \mathrm{GeV}$
$\Rightarrow$ need a large volume and a small lattice spacing

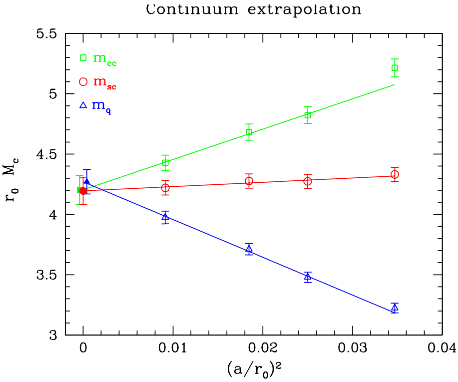
- Discretization errors $\propto (am_\mathrm{q})^\alpha$
  Choose bare heavy quark mass $am_\mathrm{b} \ll 1$, eg $am_\mathrm{b} = 0.1$
  $\Rightarrow$ For a O(a)-improved action, leading discr error $\mathcal{O}(am_\mathrm{b})^2 \sim 1\%$

- Spatial extent $L = aN$. For instance impose $L > 2\ \mathrm{fm}$
  $\Rightarrow$ Requires a large number of points (per space time direction)

$$N > \frac{2\ \mathrm{fm}}{a} = (2\ \mathrm{fm}) \times (10 m_\mathrm{b}) = 80\ \mathrm{GeV\ fm} \sim 400$$

Not doable with nowadays computers $\Rightarrow$ Effective theory

The charm quark



The simulation of the charm mass is just doable ...

... and $M_b \simeq 4M_c$.

Effective theories for heavy quark (I)

Momentum of a heavy quark (inside a hadron) $p = m_Q v + k$
Interaction with light dof $k \sim \Lambda_{\rm QCD} \ll m_Q$
Separate the higher and lower components of the heavy quark, and find an effective Lagrangian (see eg [Grozin '02])

$$\mathcal{L}_{\rm eff}^{\rm heavy} = \bar{\psi}_{\rm h}(x) \left[iv.D + \frac{(iD_\perp)^2}{2m_Q} + \frac{g\sigma.G}{4m_Q} + ... \right] \psi_{\rm h}(x)$$

Different choices of lattice implementation

- Expansion in $\Lambda_{\rm QCD}/m_Q$: HQET
- Expansion in v and $1/am_Q$: NRQCD
- Fermilab Method [El-Khadra et al '96]
- Relativistic heavy quarks [Aoki et al '01, Christ et al, Lin et al '06]
- Recent proposal by ETMC [ETMC '10]

Effective theories for heavy quark (II)

Main advantages and disadvantages of the different methods

method	pros	cons
NRQCD	heavy-heavy possible many results available	non-renormalizable (no continuum limit)
HQET	np renormalization	only heavy-light not many unquenched results
Fermilab	from light to heavy many results available	perturbative matching
RHQ	from light to heavy, np matching	numerically hard

Should you like (lattice) HQET ?

- pros
  - Theoretically well defined, (continuum limit, renormalization)
  - Can be implemented non-perturbatively
  - The static propagator is numerically cheap
  - In many cases the $1/m$ terms are doable
  - Convergence expected to be fast

- cons
  - Effective theory, not QCD
  - Linear divergence in the static energy [Eichten & Hill '90]

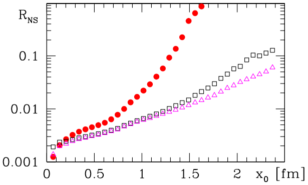
$$E^{\mathrm{stat}} \simeq \frac{19.95}{12\pi^2} \times \frac{g_0^2}{a} + \dots$$

  - Ratio $\mathrm{Noise/signal} \longrightarrow exp\left(E^{\mathrm{stat}} x_o\right)$
    $\Rightarrow$ Can one get a signal ?

"Recent" improvements in HQET

- Conceptual improvement:
  Non Perturbative matching with HQET [Heitger & Sommer 03]
  ⇒ Subtractions of the divergences

- Technical improvement:
  1. Reduction of the Ratio Noise/Signal
  [D Morte, Dürr, Heitger, Molke, Rolf, Shindler, Sommer '03]



  2. Application on variational techniques and all to all propagators [Blossier, D Morte, V Hippel, Mendes, Sommer '09]

Non-perturbative HQET on the lattice

HQET at zero velocity on the lattice

The static part is given by the Eichten-Hill action [Eichten & Hill 90]

$$S_{\rm stat} = a^4 \sum_x \overline{\psi}_{\rm h}(x) D_0 \psi_{\rm h}(x)$$

with $P_+\psi_{\rm h} = \psi_{\rm h}\,, \quad \overline{\psi}_{\rm h} P_+ = \overline{\psi}_{\rm h}\,, \quad P_+ = \frac{1}{2}(1+\gamma_0)$

The static energy contains a linear divergence ($\propto 1/a$) which is absorbed by $m_{\rm bare}$

$$m_{\rm B} = E^{\rm stat} + m_{\rm bare}$$

The $1/m$ corrections are the kinetic and chromomagnetic terms

$$\mathcal{O}_{\rm kin} = -\overline{\psi}_{\rm h}(\mathbf{D}^2)\psi_{\rm h} \qquad \mathcal{O}_{\rm spin} = -\overline{\psi}_{\rm h}(\sigma\cdot\mathbf{B})\psi_{\rm h}$$

with coefficient $\omega_{\rm kin}, \omega_{\rm spin}$ $\Rightarrow$ Classically $\omega_{\rm kin} = \omega_{\rm spin} = 1/(2m)$

HQET coefficients $m_{\rm bare}, \omega_{\rm kin}, \omega_{\rm spin}$ are determined non-perturbatively $\Rightarrow$ renormalizability

HQET computation on the lattice

We want to compute hadronic quantities at the $1/m$ order of hqet, for example

$$
\begin{aligned}
m_{\mathrm{B}} &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{spin}} + \omega_{\mathrm{spin}} E^{\mathrm{spin}} \\
\langle 0|A_0^{\mathrm{HQET}}|\mathrm{B}\rangle &= Z_{\mathrm{A}}^{\mathrm{HQET}} \Big(\langle 0|A_0^{\mathrm{stat}}|\mathrm{B}\rangle + \omega_{\mathrm{kin}} \langle 0|\mathrm{A}_0^{\mathrm{kin}}|\mathrm{B}\rangle + \omega_{\mathrm{spin}} \langle 0|\mathrm{A}_0^{\mathrm{spin}}|\mathrm{B}\rangle \Big)
\end{aligned}
$$

⇒ To achieve such a computation, one needs:

- large volume matrix element and energies $E^{\mathrm{stat}}, E^{\mathrm{kin}}, \langle 0|A_0^{\mathrm{stat}}|\mathrm{B}\rangle, \ldots$
- HQET parameters $m_{\mathrm{bare}}, \omega_{\mathrm{kin}}, Z_{\mathrm{A}}^{\mathrm{HQET}}, \ldots$

⇒ In this talk, I will focus on the second point.

Strategy [Heitger & Sommer 03]

QCD ($am_b \ll 1$)

HQET

$L_1 \sim 0.4$ fm — $\Phi^{\mathrm{QCD}}(L_1, am_{\mathrm{b}})$ ⟷ Matching ⟷ $\Phi^{\mathrm{HQET}}(L_1)$

$L_2 = 2L_1$ — $\left[\Phi^{\mathrm{HQET}}(L_2) - \Phi^{\mathrm{HQET}}(L_1)\right]$

$L_{\mathrm{inf}} = 2L_2$ — $\Phi(L_{\mathrm{inf}}, am_{\mathrm{b}})$ ⟷ $\left[\Phi^{\mathrm{HQET}}(L_{\mathrm{inf}}) - \Phi^{\mathrm{HQET}}(L_2)\right]$

Experiment

First application

The b-quark mass

Observable: the B meson mass [Heitger & Sommer 03]

In infinite volume $m_{\rm B} = m_{\rm bare} + E^{\rm stat}$, where $m_{\rm bare}$ cancels the $1/a$ divergence

- Simulate QCD in small volume $L_1 \sim 0.5\ \mathrm{fm}$ with $am_{\rm b} \ll 1$.

  Compute a "finite volume mass" $\Gamma(L_1, m_{\rm q}) = \lim_{a\to 0} \Gamma(L_1, m_{\rm q}, a)$

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- Compute the corresponding quantities ($E^{\rm stat}$) in the effective theory for various a's.

  Impose the matching $\Gamma^{\rm QCD}(L_1, m_{\rm q}) = m_{\rm bare}(m_{\rm q}, a) + \Gamma^{\rm stat}(L_1, a)$

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- Perform another simulation of HQET, with the same a's but in a larger volume, for example $L_2 = 2L_1$, and compute $\Gamma^{\rm stat}(L_2, a)$

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- To obtain the meson mass in the volume L_2, compute:

$$\begin{aligned} \Gamma(L_2, m_{\rm q}) &= \lim_{a\to 0} \left(\Gamma^{\rm stat}(L_2, a) + m_{\rm bare}(m_{\rm q}, a)\right) \\ &= \lim_{a\to 0} \left(\Gamma^{\rm stat}(L_2, a) - \Gamma^{\rm stat}(L_1, a)\right) + \Gamma^{\rm QCD}(L_1, m_{\rm q}) \end{aligned}$$

(note that the divergence cancels out in the difference)

Observable: the B meson mass [Heitger & Sommer 03]

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  (note that the divergence cancels out in the difference)

- Re-iterate until the volume is large enough

Strategy [Heitger & Sommer 03]

For an observable Φ with an additive renormalization (in our case $\Phi = L\Gamma = L(\Gamma + m_{\rm bare})$)

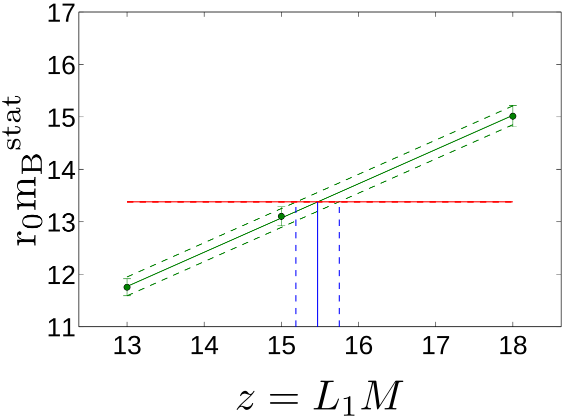
$$
\begin{aligned}
\Phi(L_\infty, m_{\rm q}) &= \lim_{a\to 0}\left[\Phi^{\rm HQET}(L_\infty, a) - \Phi^{\rm HQET}(L_2, a)\right] && a \sim 0.1\ \mathrm{fm} \\
&+ \lim_{a\to 0}\left[\Phi^{\rm HQET}(L_2, a) - \Phi^{\rm HQET}(L_1, a)\right] && a \sim 0.05\ \mathrm{fm} \\
&+ \lim_{a\to 0}\Phi^{\rm QCD}(L_1, m_{\rm q}, a) && a \sim 0.025\ \mathrm{fm}
\end{aligned}
$$

To extract the b-quark mass, we match the B mesons mass to its experimental value, and solve

$$\Phi(m_{\rm b}, L_\infty) = \Phi^{\rm exp} = L_\infty m_{\rm B}^{\rm exp}$$

by an interpolation in the quark mass

Static b-quark mass



b quark mass, the $1/m$ correction

At the NLO, in infinite volume

$$m_{\mathrm{B}} = \underbrace{E^{\mathrm{stat}} + m_{\mathrm{bare}}}_{\mathrm{LO}} + \underbrace{\omega_{\mathrm{kin}} E^{\mathrm{kin}} + \omega_{\mathrm{spin}} E^{\mathrm{spin}}}_{\mathrm{NLO}}$$

Note that in the literature

$$m_{\mathrm{B}} = \bar{\Lambda} + m_{\mathrm{bare}} + \frac{\lambda_1}{2m} + \frac{3\lambda_2}{2m}$$

$\Rightarrow$ Need 3 observables Φ_1, Φ_2, Φ_3.

Or, consider the spin-averaged B meson $\Rightarrow \omega_{\mathrm{spin}}$ cancels

$$m_{\mathrm{B}}^{\mathrm{av}} \equiv \frac{1}{4} m_{\mathrm{B}} + \frac{3}{4} m_{\mathrm{B}}^{*} = E^{\mathrm{stat}} + m_{\mathrm{bare}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}}$$

$\Rightarrow$ Need two observables Φ_1, Φ_2, and the spin splitting term becomes a separate issue.

b-quark mass at the $1/m$ order

- At the NLO of HQET $m_{\mathrm{B}} = E^{\mathrm{stat}} + m_{\mathrm{bare}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}}$
- Matching 1 $\Phi_{\mathrm{kin}}^{\mathrm{QCD}}(L_1) = \omega_{\mathrm{kin}} R_1^{\mathrm{kin}}(L_1)$
- Matching 2 $\Phi^{\mathrm{QCD}}(L_1, M) = L_1 \Gamma^{\mathrm{QCD}}(L_1, M) = L_1 \left[\Gamma^{\mathrm{stat}}(L_1) + m_{\mathrm{bare}} + \omega_{\mathrm{kin}} \Gamma^{\mathrm{kin}}(L_1)\right]$
- The meson mass in the volume L_2 is then given by

$$\Gamma(L_2, M) = \frac{\Phi^{\mathrm{QCD}}(L_1, M)}{L_1} + \left[\Gamma^{\mathrm{stat}}(L_2) - \Gamma^{\mathrm{stat}}(L_1)\right]$$

$$+ \left[\frac{\Phi_{\mathrm{kin}}^{\mathrm{QCD}}(L_1, M)}{R_1^{\mathrm{kin}}(L_1)} (\Gamma^{\mathrm{kin}}(L_2) - \Gamma^{\mathrm{kin}}(L_2))\right]$$

- Repeat this for the step $L_2 \to L_\infty$
- Solve (in the continuum) $m_{\mathrm{B}}(M_b) = m_{\mathrm{B}}^{exp}$.

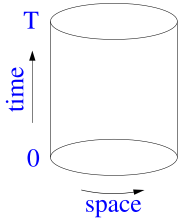
Implementation details

Correlators on the lattice

Implementation : Schrödinger functional [Lüscher '92, Sint '94]

Implementation: Schrödinger functional of size $T \times L^3$

- Dirichlet boundary conditions in time (at $x_0 = 0$ and $x_0 = T$)
- Periodic boundary conditions in space, up to a phase $\Psi(x + \hat{k}L) = e^{i\theta}\Psi(x)$.



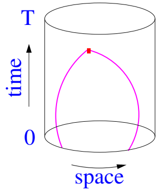
Transition amplitude for $C(x_0 = 0) \to C'(x_o = T)$

$$\begin{aligned} \mathcal{Z}[C', C] &= \langle C'|\mathrm{e}^{-\mathbb{H}\mathrm{T}}\mathbb{P}|\mathrm{C}\rangle \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-E_n T} \psi_n[C'] \psi_n[C]^* \end{aligned}$$

Implementation: 2 pts functions in QCD

Boundary to current correlators

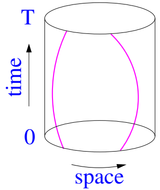
$$f_{\rm A}(x_0) = -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}}\left\langle (A_{\rm I})_0(x)\,(\bar\zeta_{\rm b}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z}))\right\rangle$$



and boundary to boundary correlator

$$f_1 = -\frac{a^{12}}{2L^6}\sum_{\mathbf{y},\mathbf{z},\mathbf{y}',\mathbf{z}'}\left\langle (\bar\zeta'_{\rm b}(\mathbf{y}')\gamma_5\zeta'_{\rm l}(\mathbf{z}'))\,(\bar\zeta_{\rm b}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z}))\right\rangle$$

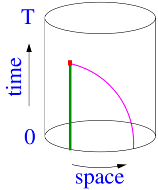
$$k_1 = -\frac{a^{12}}{2L^6}\sum_{\mathbf{y},\mathbf{z},\mathbf{y}',\mathbf{z}'}\left\langle (\bar\zeta'_{\rm b}(\mathbf{y}')\gamma_{\rm k}\zeta'_{\rm l}(\mathbf{z}'))\,(\bar\zeta_{\rm b}(\mathbf{y})\gamma_{\rm k}\zeta_{\rm l}(\mathbf{z}))\right\rangle$$



Implementation: 2 pts functions in the static theory

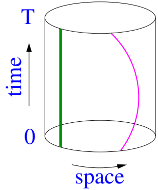
Boundary to current correlators

$$f_{\rm A}^{\rm stat}(x_0) \quad = \quad -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}} \Big\langle (A_{\rm I}^{\rm stat})_0(x)\,(\bar{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z}))\Big\rangle$$



and boundary to boundary correlator

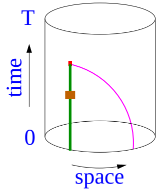
$$f_1^{\rm stat} = -\frac{a^{12}}{2L^6}\sum_{\mathbf{y},\mathbf{z},\mathbf{y}',\mathbf{z}'} \Big\langle (\bar{\zeta}'_{\rm h}(\mathbf{y}')\gamma_5\zeta'_{\rm l}(\mathbf{z}'))\,(\bar{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z}))\Big\rangle$$



Implementation: 2 pts functions at the $1/m$ order

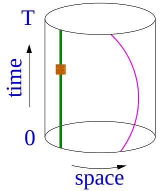
Boundary to current correlators

$$f_{\rm A}^{\rm kin}(x_0) = -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z},u}\left\langle\, {\rm A}_0^{\rm stat}(x)\, O^{\rm kin}(u)\,(\bar\zeta_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z}))\right\rangle$$



Boundary to boundary correlator

$$f_1^{\rm kin} = -\frac{a^{12}}{2L^6}\sum_{\mathbf{y},\mathbf{z},\mathbf{y}',\mathbf{z}',u}\left\langle(\bar\zeta'_{\rm h}(\mathbf{y}')\gamma_5\zeta'_{\rm l}(\mathbf{z}'))\, O^{\rm kin}(u)\,(\bar\zeta_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z}))\right\rangle$$



And the same for $f_{\rm A}^{\rm spin}$, $f_1^{\rm spin}$

Heavy quark expansion

At the NLO of heavy quark effective theory

current to boundary correlators

$$f_{\mathrm{A}} \propto Z_{\mathrm{A}}^{\mathrm{HQET}} \left\{ f_{\mathrm{A}}^{\mathrm{stat}} + c_{\mathrm{A}}^{\mathrm{HQET}} f_{\delta\mathrm{A}}^{\mathrm{stat}} + \omega_{\mathrm{kin}} f_{\mathrm{A}}^{\mathrm{kin}} + \omega_{\mathrm{spin}} f_{\mathrm{A}}^{\mathrm{spin}} \right\}$$

boundary to boundary correlators

$$f_1 \propto f_1^{\mathrm{stat}} + \omega_{\mathrm{kin}} f_1^{\mathrm{kin}} + \omega_{\mathrm{spin}} f_1^{\mathrm{spin}}$$

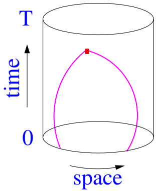
$$k_1 \propto f_1^{\mathrm{stat}} + \omega_{\mathrm{kin}} f_1^{\mathrm{kin}} - \frac{1}{3}\omega_{\mathrm{spin}} f_1^{\mathrm{spin}}$$

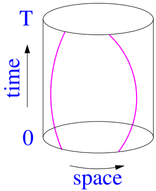
Second Application

The heavy-light decay constant

Observable(s) for the decay constant

Correlators: current-to-boundary f_A and boundary-to-boundary f_1





Build an observables related to the decay constant :

$$\Phi_B^{\mathrm{QCD}} = \ln\left(\frac{-f_{\mathrm{A}}(x_0)}{\sqrt{f_1}}\right) \quad \xrightarrow{L\gg 1} \quad \ln\left(\frac{1}{2}F_{\mathrm{B}}\sqrt{m_{\mathrm{B}}L^3}\right)$$

and in the effective theory (at the leading order):

$$\Phi_B^{\mathrm{hqet}} = \ln Z_{\mathrm{A}}^{\mathrm{stat}} + \ln\left(\frac{-f_{\mathrm{A}}^{\mathrm{stat}}}{\sqrt{f_1^{\mathrm{stat}}}}\right) + \mathcal{O}(1/m)$$

F_B, including $1/m$ corrections

At the LO of HQET

$$\Phi_B^{\mathrm{hqet}} = \ln Z_{\mathrm{A}}^{\mathrm{stat}} + \ln\left(\frac{-f_{\mathrm{A}}^{\mathrm{stat}}}{\sqrt{f_1^{\mathrm{stat}}}}\right)$$

$\Rightarrow$ Need 1 observable

F_B, including $1/m$ corrections

At the NLO of HQET

$$\Phi_B^{\rm hqet} = \ln Z_{\rm A}^{\rm hqet} + \ln\left(\frac{-f_{\rm A}^{\rm stat}}{\sqrt{f_1^{\rm stat}}}\right) + \underbrace{c_{\rm A}^{hqet}\frac{f_{\delta{\rm A}}^{\rm stat}}{f_{\rm A}^{\rm stat}} + \omega_{\rm kin}\left(\frac{f_{\rm A}^{\rm kin}}{f_{\rm A}^{\rm stat}} + \frac{1}{2}\frac{f_1^{\rm kin}}{f_1^{\rm stat}}\right) + \omega_{\rm spin}\left(\frac{f_{\rm A}^{\rm spin}}{f_{\rm A}^{\rm stat}} - \frac{1}{2}\frac{f_1^{\rm spin}}{f_1^{\rm stat}}\right)}_{1/m}$$

$\Rightarrow$ Need 4 observables (+1 for the mass)

Generalization of the strategy

Define a vector of observables Φ , vector of matching parameters ω, ...
and extract $\omega(m_\mathrm{q}, a)$ from the matching in L_1,

$$\Phi^{\mathrm{QCD}}(L_1, m_\mathrm{q}) = \phi(L_1, a)\,\omega(m_\mathrm{q}, a) + \eta(L_1, a)$$

then compute the observables in L_2 :

$$\Phi(L_2, m_\mathrm{q}) = \lim_{a\to 0}\Big[\phi(L_2, a)\,\Big(\phi^{-1}(L_1, a)\,\Phi^{\mathrm{QCD}}(L_1, m_\mathrm{q}) - \eta(L_1, a)\Big) + \eta(L_2, a)\Big]$$

The HQET parameters needed for a large volume simultation are given by

$$\omega(m_\mathrm{q}, a) = \phi^{-1}(L_2, a)\Phi(L_2, m_\mathrm{q}) - \eta(L_2, a)$$

Choice of observables

$\Phi_1 =$	$L\Gamma^{\rm P} =$	$L\Gamma^{\rm stat}$	$+Lm_{\rm bare}^{\rm HQET}$	$+c_{\rm A}^{\rm HQET} L\Gamma_{\delta \rm A}^{\rm stat}$	$+\omega_{\rm kin} LE^{\rm kin}$	$+\omega_{\rm spin} LE^{\rm spin}$
$\Phi_2 =$	$\ln\left(\frac{-f_{\rm A}}{\sqrt{f_1}}\right) =$	$\zeta_{\rm A}$	$+\ln Z_{\rm A}^{\rm HQET}$	$+c_{\rm A}^{\rm HQET}\rho_{\delta A}$	$+\omega_{\rm kin}\Psi^{\rm kin}$	$+\omega_{\rm spin}\Psi^{\rm spin}$
$\Phi_3 =$	$R_{\rm A} =$	$R_{\rm A}^{\rm stat}$		$+c_{\rm A}^{\rm HQET} R_{\delta \rm A}$	$+\omega_{\rm kin} R_{\rm A}^{\rm kin}$	$+\omega_{\rm spin} R_{\rm A}^{\rm spin}$
$\Phi_4 =$	$\frac{1}{4}(R_1^{\rm P} + 3R_1^{\rm V}) =$	$R_1^{\rm stat}$			$+\omega_{\rm kin} R_1^{\rm kin}$	
$\Phi_5 =$	$\frac{3}{4}\ln\left(\frac{f_1}{k_1}\right) =$					$\omega_{\rm spin}\frac{f_1^{\rm spin}}{f_1^{\rm stat}}$

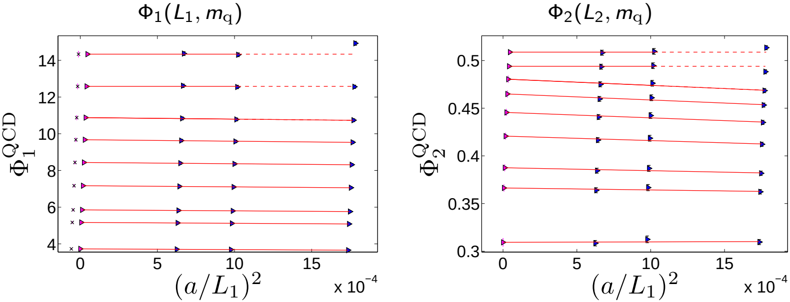
We define the 5 dimensional vectors Φ, η, ω and a 5 by 5 matrix ϕ

$$\Phi(L, m_{\rm q}) = \lim_{a\to 0}\left[\phi(L,a)\,\omega(m_{\rm q},a) + \eta(L,a)\right]$$

Numerical results

Results

Continuum extrapolation of the QCD observables



The RGI quark masses M are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

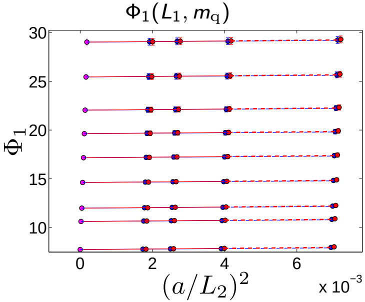
$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45 \text{ fm}$

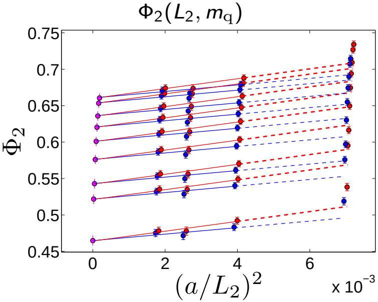
$L_1/a = 40, 32, 24(20)$

$\beta = 6.638, 6.4574, 6.2483$

Results

Continuum extrapolation of the static observables in L_2





The RGI quark masses M are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45$ fm

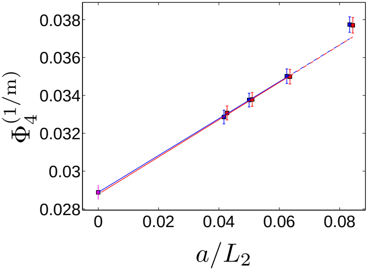
$L_1/a = 12, 10, 8$ $\qquad$ $L_1/a = 24, 20, 16$

$\beta = 5.758, 5.619, 5.4689$ $\qquad$ Point with $L_2/a = 32, L_1/a = 16$ will be added in the near future

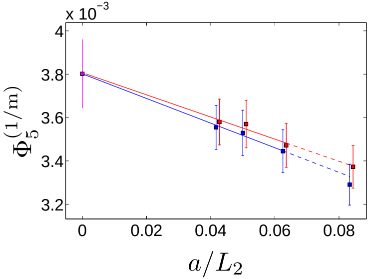
Results

Continuum extrapolation of the $1/m$ observables in L_2

Kinetic term



Magnetic term



The RGI quark masses M are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

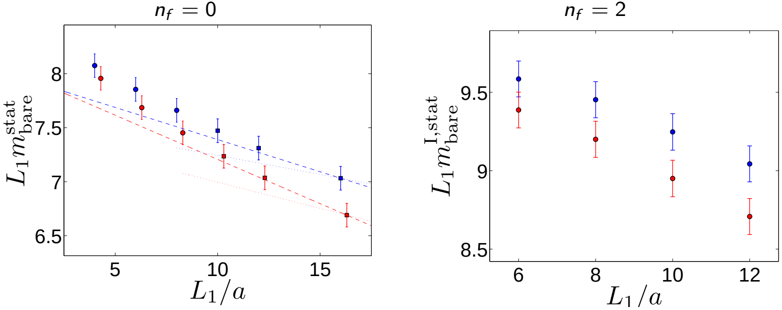
$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45\ \mathrm{fm}$

$L_1/a = 12, 10, 8$ $\quad$ $L_1/a = 24, 20, 16$

$\beta = ..., ...$ $\quad$ Point with $L_2/a = 32, L_1/a = 16$ will be added in the near future

Results

Example of static parameter: $m_{\mathrm{bare}}^{\mathrm{stat}}$



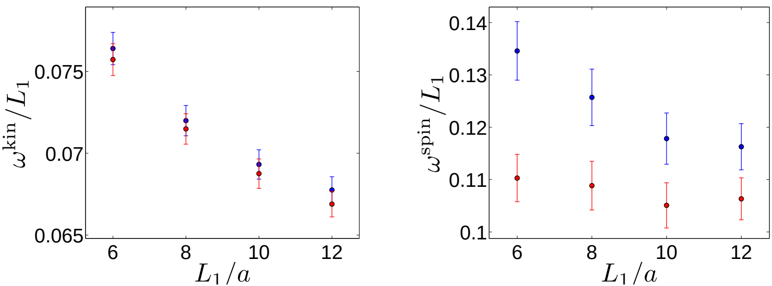
The RGI quark masses M are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45\ \mathrm{fm}$

$L_1/a = 12, 10, 8$ $L_1/a = 24, 20, 16$

$\beta = ..., ...$ Point with $L_2/a = 32, L_1/a = 16$ will be added in the near future

Example of $1/m$ parameters



The RGI quark masses M are such that $z = L_1 M \in (4, 6, 7, 9, 11, 13, 15, 18, 21)$

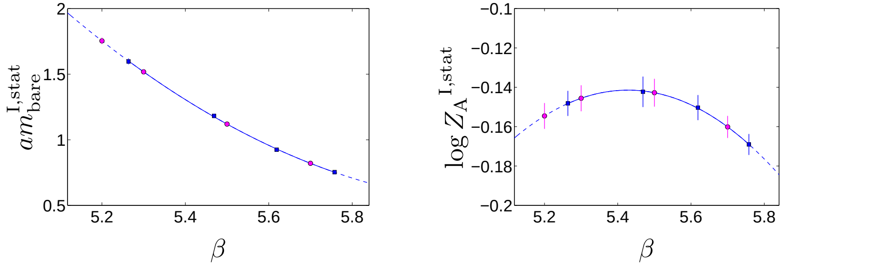
$L_1/r_0 = 0.9 \Rightarrow L_1 \sim 0.45$ fm

$L_1/a = 12, 10, 8$ $\quad$ $L_1/a = 24, 20, 16$

$\beta = ..., ...$ $\quad$ Point with $L_2/a = 32, L_1/a = 16$ will be added in the near future

Results

Interpolation at the desired β values



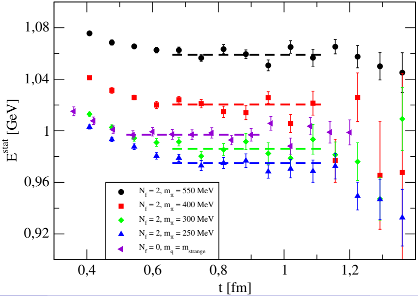
from $\beta = \ldots$ to $\beta = 5.2, 5.3, 5.5, 5.7$

Results

Extraction of $E^{\rm stat}$ from large volume simulation

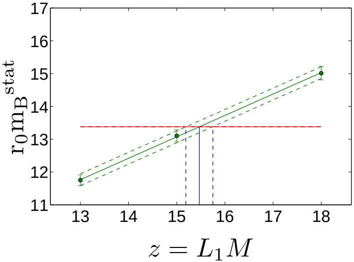
β	a (fm)	$L^3 \times T$	m_π (MeV)	#	traj. sep.
5.3	0.07	$32^3 \times 64$	550	152	32
		$32^3 \times 64$	400	600	32
		$48^3 \times 96$	300	192	16
		$48^3 \times 96$	250	350	16

$E^{\rm stat}$ extracted at finite β by solving an GEVP, for different pion masses.



Results

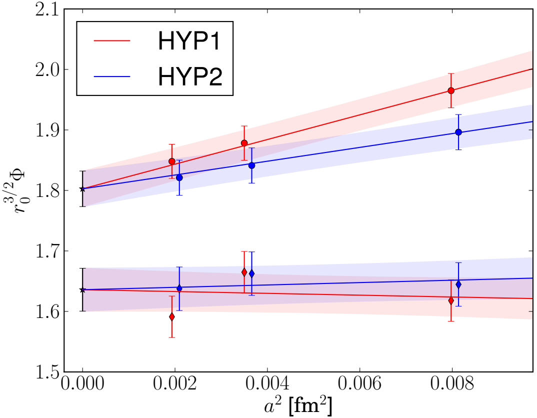
Interpolation at the physical mass, in the static approximation



The RGI quark masses M are such that $z = L_1 M \in (13, 15, 18)$

Example: f_{B_s} for $n_f = 0$

- Large volume matrix element extracted by solving a GEVP
- HQET parameters determined non-perturbatively



Status of the project

- $n_f = 0$
  - ▷ b-quark mass [Alpha '06]

$$m_b(m_b) = \underbrace{4.350(64)}_{\text{static}}\ \text{GeV}\ \underbrace{-0.049(29)}_{O(\Lambda^2/m_b)}\ \text{GeV} + O(\Lambda^3/m_b^2)$$

  - ▷ I. HQET parameters [Alpha '10]
  - ▷ II. Spectoscopy [Alpha '10]
  - ▷ III. Decay constant [Alpha submitted in June]

$$F_{B_s}^{\text{stat}} = 229 \pm 6\ \text{MeV} \qquad F_{B_s}^{\text{stat}+1/\text{m}} = 219 \pm 8\ \text{MeV}$$

- $n_f = 2$
  - ▷ HQET parameter : almost finished
  - ▷ Large volume part: very preliminary results (1 lattice spacing)

$$m_b(m_b)^{\text{stat}} = 4.255(25)(50)(??) \qquad m_b(m_b)^{\text{HQET}} = 4.276(25)(50)(??)$$

Summary and outlook

Summary and outlook

- HQET is theoretically sound (continuum limit, renormalizable)
- Non perturbative renormalization is possible.
- The $1/m$ terms are accessible
- The large volume matrix element and energies can be accurately computed with the GEVP method
- Is has been tested in the quenched approximation
- For $n_f = 2$, the renormalization is almost done
  but a lot has to be done for the large volume part.

Backup

Third Application

The fine splitting $m^*_{\mathrm{B}} - m_{\mathrm{B}}$

The Mass splitting

As a consequence of spin flavor symmetry

$$\begin{aligned} m_{\mathrm{B}} &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} + \omega_{\mathrm{spin}} E^{\mathrm{spin}} \\ m_{\mathrm{B}^*} &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} - \frac{1}{3}\omega_{\mathrm{spin}} E^{\mathrm{spin}} \end{aligned}$$

$$\Rightarrow m_{\mathrm{B}} - m_{\mathrm{B}^*} = \frac{4}{3}\omega_{\mathrm{spin}} E^{\mathrm{spin}}$$

The Mass splitting

As a consequence of spin flavor symmetry

$$\begin{aligned} m_{\mathrm{B}} &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} + \omega_{\mathrm{spin}} E^{\mathrm{spin}} \\ m_{\mathrm{B}^*} &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} - \frac{1}{3}\omega_{\mathrm{spin}} E^{\mathrm{spin}} \end{aligned}$$

$$\Rightarrow m_{\mathrm{B}} - m_{\mathrm{B}^*} = \frac{4}{3}\omega_{\mathrm{spin}} E^{\mathrm{spin}}$$

Define on observable to extract ω_{spin}, eg

$$\Phi^{\mathrm{QCD}} = \frac{3}{4}\ln\left(\frac{f_1}{k_1}\right) \leftrightarrow \Phi^{\mathrm{HQET}} = \omega_{\mathrm{spin}} \frac{f_1^{\mathrm{spin}}}{f_1^{\mathrm{stat}}} \equiv \omega_{\mathrm{spin}} \rho^{\mathrm{spin}}$$

The Mass splitting

As a consequence of spin flavor symmetry

$$
\begin{aligned}
m_{\mathrm{B}} &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} + \omega_{\mathrm{spin}} E^{\mathrm{spin}} \\
m_{\mathrm{B}^*} &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} - \frac{1}{3}\omega_{\mathrm{spin}} E^{\mathrm{spin}}
\end{aligned}
$$

$$
\Rightarrow m_{\mathrm{B}} - m_{\mathrm{B}^*} = \frac{4}{3}\omega_{\mathrm{spin}} E^{\mathrm{spin}}
$$

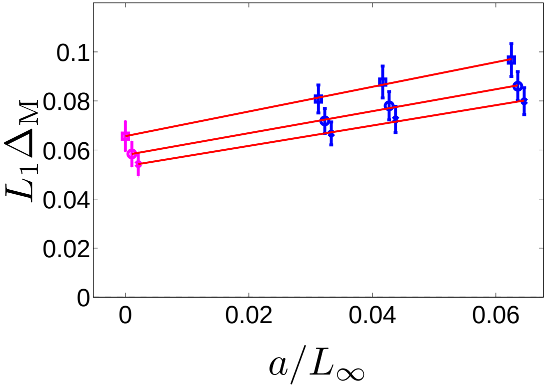
Define on observable to extract ω_{spin}, eg

$$
\Phi^{\mathrm{QCD}} = \frac{3}{4}\ln\left(\frac{f_1}{k_1}\right) \leftrightarrow \Phi^{\mathrm{HQET}} = \omega_{\mathrm{spin}} \frac{f_1^{\mathrm{spin}}}{f_1^{\mathrm{stat}}} \equiv \omega_{\mathrm{spin}} \rho^{\mathrm{spin}}
$$

And compute

$$
m_{\mathrm{B}} - m_{\mathrm{B}^*} = \frac{4}{3}\left[\frac{E^{\mathrm{spin}}}{\rho^{\mathrm{spin}}(L_2)}\right]\left[\frac{\rho^{\mathrm{spin}}(L_2)}{\rho^{\mathrm{spin}}(L_1)}\right]\Phi^{\mathrm{QCD}}(L_1)
$$

Quenched results for the Mass splitting



$\Rightarrow m_{\mathrm{B}^*} - m_{\mathrm{B}} = \Delta_M = 32 \pm 3 \pm ??\ \mathrm{MeV}$ Prel.!!

and experimentally $46.1 \pm 1.5\ \mathrm{MeV}$ [PDG]

too small $\Rightarrow$ quenched effect ?

Green functions and renormalization

Under the path integral: expand in $1/m$ $\Rightarrow \mathcal{L}^{(\nu)}(x)$ only as insertions

$$\mathcal{L}(x) = \overline{\psi}_{\rm h}(x) D_0 \psi_{\rm h}(x) + \sum_\nu \mathcal{L}^{(\nu)}(x)$$

$$\begin{aligned}
\langle \mathcal{O} \rangle &\propto \int {\rm D}\phi \, {\rm e}^{-S_{\rm l} - a^4 \sum_x \overline{\psi}_{\rm h}(x) D_0 \psi_{\rm h}(x)} \left(1 - a^4 \sum_{x,\nu} \mathcal{L}^{(\nu)}(x) \right) \mathcal{O} \\
&\equiv \langle [1 - a^4 \sum_{x,\nu} \mathcal{L}^{(\nu)}(x)] \, \mathcal{O} \, \rangle^{\rm stat}
\end{aligned}$$

At the $1/m$ order

$$\begin{aligned}
\langle \mathcal{O} \rangle &= \langle \mathcal{O} \rangle^{\rm stat} + \omega_{\rm kin} \sum_x \langle \mathcal{O} \mathcal{O}_{\rm kin}(x) \rangle^{\rm stat} + \omega_{\rm spin} \sum_x \langle \mathcal{O} \mathcal{O}_{\rm spin}(x) \rangle^{\rm stat} \\
&= \langle \mathcal{O} \rangle^{\rm stat} + \omega_{\rm kin} \, \langle \mathcal{O} \rangle^{\rm kin} + \omega_{\rm spin} \, \langle \mathcal{O} \rangle^{\rm spin}
\end{aligned}$$

$\omega_{\rm kin}, \omega_{\rm spin}$ cancel divergences $\leftrightarrow$ Continuum limit, renormalizability