Simplicity of scattering amplitudes

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(Partly) based on work with

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+ Paul Heslop, Valya Khoze (Durham)
+ Babis Anastasiou (ETH Zürich)
+ Panos Katsaroumpas, Dimitrios Korres, Daniel Koschade (QMUL)
+ Dung Nguyen & Mark Spradlin (Brown)

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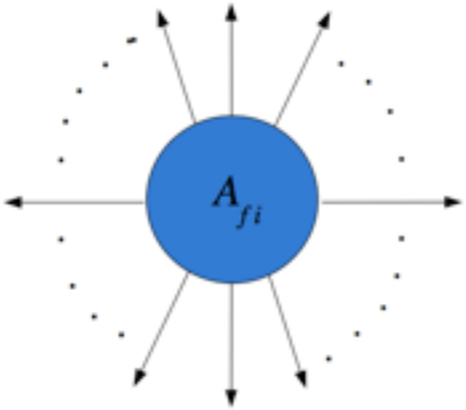
Plan

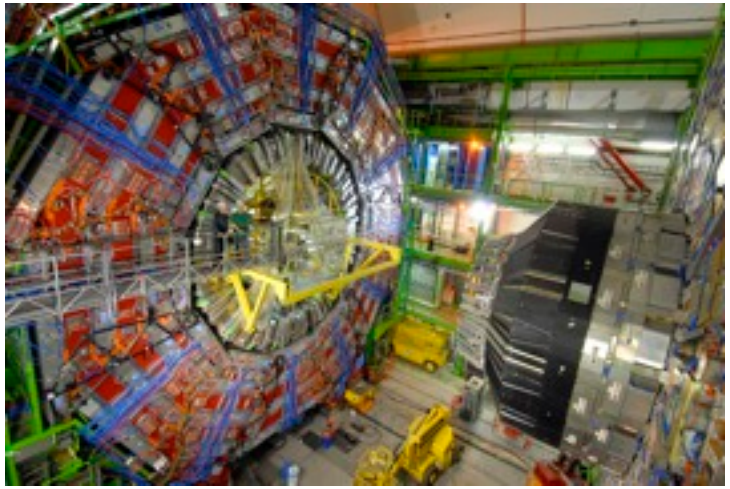
- Why are we interested in amplitudes ?

- Amplitudes and Wilson loops in N=4 SYM
  - iterative structure in N=4 scattering amplitudes
  - amplitude / Wilson loop duality
  - dual (super)conformal symmetry
  - form factors of 1/2 BPS operators in N=4 super Yang-Mills

What are we interested in:

- Scattering amplitudes of elementary particles (e.g. gluons, or gravitons)





  - collected in a unitary matrix - the S-matrix (Wheeler, 1937; Heisenberg, 1942)
  - (maximally) supersymmetric theories
  - theories with no supersymmetry, e.g. QCD

Why amplitudes ?

- Because they are simple
  - calculation with Feynman diagrams cumbersome, however final results often strikingly simple
- Gluon scattering is an important background for LHC
  - at tree level, gluon scattering can be equivalently calculated in any supersymmetric theory
  - one loop: $\mathcal{A}_g \;=\; (\mathcal{A}_g + 4\mathcal{A}_f + 3\mathcal{A}_s) \;-\; 4(\mathcal{A}_f + \mathcal{A}_s) \;+\; \mathcal{A}_s$

$\mathcal{N}=4$ $\quad$ $\mathcal{N}=1$ $\quad$ $\mathcal{N}=0$

one-loop amplitude in pure YM with a gluon running in the loop

gluon
4 Weyl fermions
6 real scalar fields

the most difficult piece, but simpler than $\mathcal{A}_g$

Textbook approach to amplitudes:

Calculate Feynman diagrams !

A typical Feynman diagram contains:

Propagators

Vertices

Gauge-dependent, off-shell internal states

Unwanted complexity (I)

Number of Feynman diagrams for $gg \longrightarrow n\, g$ scattering: (tree level)

☞

n	2	3	4	5	6	7	8
# of diagrams	4	25	220	2485	34300	559405	10525900

Gluon scattering

☞ Result is: $\mathcal{A}(1^{\pm}, 2^{+}, \ldots, n^{+}) = 0$

Why so simple ? Why zero?

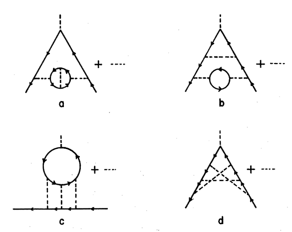
Unwanted complexity (II)

- Three-loop correction to electron g-2

Predrag Cvitanovic - acceptance speech, 1993 NKT Research Prize in Physics, Dansk Fysisk Selskab Årsmøde

“One day terror struck; I was invited to Caltech to give a talk. I could go to any other place and say that Kinoshita and I had calculated thousands of diagrams and that the answer was, well, the answer is:

72 diagrams like



$$= (1.181241456...)\,(\alpha_{\mathrm{e.m.}}/\pi)^3$$

(Cvitanovic & Kinoshita ’74)

(Laporta & Remiddi ’96)

But in front of Feynman? He is going to ask me why + and not - ? Why do 100 diagrams give a result of the order of unity, and not 10 or 100 or any other number? It might be the most precise agreement between fundamental theory and experiment in all of physics, but what does it mean ? ”

Predrag Cvitanovic again:

“So in fear of God I went into deep trance and after a month I came up with this: if gauge invariance of QED guarantees that all UV and IR divergences cancel, why not also the finite parts?

And indeed; when the diagrams we had computed were grouped into gauge-invariant subsets, a rather surprising thing happens: while the finite part of each Feynman diagram is of order 10 to 100, every subset adds up to approximatively

$$\pm 1/2\,(\alpha_{\rm e.m.}/\pi)^n$$

...For me, the above is the most intriguing hint that something deeper than what we know underlies quantum field theory... ”

Form Factors

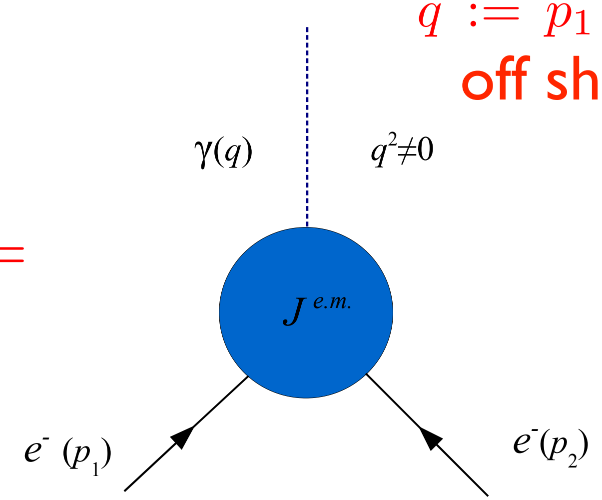
- Partially off-shell quantities

$$F = \int d^4x\, e^{iqx} \langle 0|\mathcal{O}(x)|state\rangle = \delta^{(4)}(q - p_{state}) \langle 0|\mathcal{O}(0)|state\rangle$$

$$q := p_1 + p_2$$

off shell

- $\langle 0|\, J_\mu^{e.m.}(0)\, |e^-(p_1)\, e^-(p_2)\rangle =$

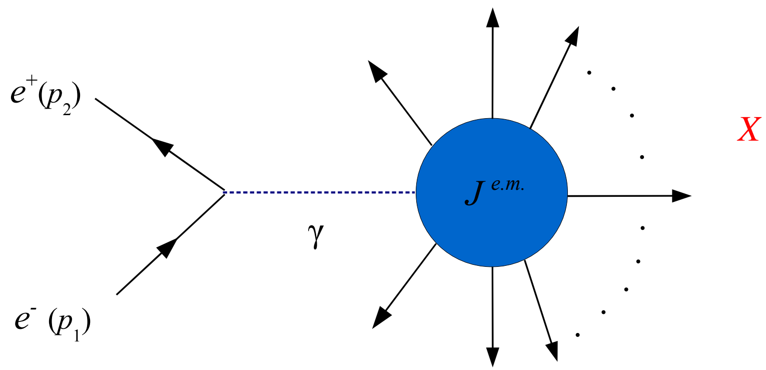


on shell

on shell

$$J_\mu^{e.m.} = \bar{\psi}\gamma_\mu\psi$$

- Appear in several interesting contexts:
  - deep inelastic scattering (e⁻ + p → e⁻ + hadrons)
  - e⁺e⁻ → hadrons :



$$= e\,\bar{v}(p_2)\gamma_\mu u(p_1)\,\frac{\eta^{\mu\nu}}{(p_1+p_2)^2}\,(-e)\langle X|\,J_\nu^{e.m.}(0)\,|0\rangle$$

e⁺e⁻ → hadrons (X)

all orders in α_{strong}, first order in $\alpha_{\text{e.m.}}$

hadronic electromagnetic current

- Total cross section: $\sigma = \frac{e^4}{2(q^2)^3} L^{\mu\nu} W_{\mu\nu}$

  - $L^{\mu\nu} = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \frac{q^2}{2}\eta^{\mu\nu}$ from LHS $(q = p_1 + p_2)$

  - $W_{\mu\nu} = \frac{1}{\pi}\mathrm{Im}\int d^4x\, e^{iqx}\,\langle 0|T\big[J_\mu(x)\,J_\nu(0)\big]|0\rangle$ from RHS
    - encodes our ignorance of QCD dynamics
    - usually evaluated using OPE / models

- Correlation functions appear in the picture

Typical "missing words" in an amplitude seminar:

- Correlation functions (& LSZ reduction)
- Path integral
- Action
- Off-shell

- Amplitudes  correlations functions

Can we bridge the two realms ?

Form factors sit in between...

Unwanted complexity (III)
General Relativity

corrections. For the Yang-Mills field it takes the form

$$V_{(\alpha\gamma''{}^{\sigma''})\beta'} \to -ic_{\alpha\beta\gamma}p'^{\sigma} = -ic_{\alpha\gamma\beta}(p^{\sigma}+p''^{\sigma}). \quad (2.3)$$

The propagators for the normal and fictitious quanta are, respectively,

$$G \to \gamma^{\alpha\beta}\eta_{\mu\nu}/p^2, \quad (2.4)$$

$$\hat{G} \to \gamma^{\alpha\beta}/p^2, \quad (2.5)$$

with p^2 being understood to have the usual small negative imaginary part.

The corresponding quantities for the gravitational field are much more complicated. In this case we shall employ the momentum-index combinations $p\mu\nu$, $p'\sigma'\tau'$, $p''\rho''\lambda''$, $p'''\iota'''\kappa'''$. The vertices must not only be symmetric in each index pair but must also remain unchanged under arbitrary permutations of the momentum-index triplets. At least 171 separate terms are required in the complete expression for S_3 in order to exhibit this full symmetry, and for S_4 the number is 2850. However, these numbers can be greatly reduced by counting only the combinatorially distinct terms[2] and leaving it understood that the appropriate symmetrizations are to be carried out. In this way S_3 is reduced to 11 terms and S_4 to 28 terms, as follows:

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \to$$

$$\begin{aligned}
&\mathrm{Sym}\big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^{\sigma}p^{\lambda}\eta^{\mu\nu}\eta^{\tau\rho})\\
&-\tfrac{1}{2}P_3(p^{\tau}p'^{\mu}\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^{\rho}p^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^{\sigma}p'^{\lambda}\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu})\\
&\qquad -P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\big], \quad (2.6)
\end{aligned}$$

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \to$$

$$\begin{aligned}
&\mathrm{Sym}\big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^{\sigma}p^{\tau}\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^{\sigma}p'^{\mu}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\
&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^{\sigma}p^{\tau}\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^{\rho}p'^{\lambda}\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\
&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa})+\tfrac{1}{2}P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa})-\tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\
&-P_{12}(p^{\sigma}p^{\tau}\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^{\rho}p'^{\lambda}\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_{\sigma}p'^{\rho}\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^{\rho}p'^{\iota}\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^{\sigma}p^{\rho}\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^{\sigma}p^{\rho}\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\
&\qquad -P_6(p^{\rho}p'^{\iota}\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^{\sigma}p'^{\rho}\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^{\sigma}p'^{\mu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\big]. \quad (2.7)
\end{aligned}$$

The "Sym" standing in front of these expressions indicates that a symmetrization is to be performed on each index pair $\mu\nu$, $\sigma\tau$, etc. The symbol P indicates that a summation is to be carried out over all distinct permutations of the momentum-index triplets, and the subscript gives the number of permutations required in each case.

Expressions (2.6) and (2.7) can be obtained in a straightforward manner by repeated functional differentiation of the Einstein action. This procedure, however, is exceedingly laborious. A more efficient (but still lengthy) method is to make use of the hierarchy of identities (II, 17.31). It is a remarkable fact that once $S_2{}^0$ is known all the higher vertex functions, and hence the complete action functional itself, are determined by the general coordinate invariance of the theory. It is convenient, in the actual computation of the vertices via (II, 17.31), to invent diagrammatic schemes for displaying the combinatorics of indices. Since each reader will devise the scheme which suits him best we shall not shackle him by describing one here. We also make no attempt to display S_5 or any higher vertices.

The vertex $V_{(\alpha i)\beta}$ has the following form for the gravitational field:

$$\begin{aligned}
&V_{(\mu^{\sigma''\tau''})\nu'} \to\\
&\tfrac{1}{2}\mathrm{Sym}\big[2p''{}_{\mu}p'^{\sigma}\delta_{\nu}{}^{\tau}-p''{}_{\mu}p'{}_{\nu}\eta^{\sigma\tau}\\
&\qquad +(p_{\nu}p'^{\sigma}-p'{}_{\nu}p^{\sigma})\delta_{\mu}{}^{\tau}+p\cdot p'\delta_{\mu}{}^{\sigma}\delta_{\nu}{}^{\tau}\big], \quad (2.8)
\end{aligned}$$

where the momentum-index combinations are $p\mu$, $p'\nu'$, $p''\sigma''\tau''$, and the symmetrization is to be performed on the index pair $\sigma\tau$. The propagators for the normal and fictitious quanta are given by

$$G \to (\eta_{\mu\sigma}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\sigma\tau})/p^2, \quad (2.9)$$

$$\hat{G} \to \eta^{\mu\nu}/p^2. \quad (2.10)$$

[2] The choice of terms is not completely unique since momentum conservation may be used to replace a given term by other terms. We give here what we believe (but have not proved) to be the expressions containing the smallest number of terms.

← 3-point vertex: 171 terms

← 4-point vertex: 2850 terms

Einstein-Hilbert Lagrangian and Yang-Mills Lagrangian give rise to very different-looking Feynman rules...

Bryce S. DeWitt , Phys. Rev. 162:1239-1256,1967.

- ....however:

$$\boxed{\begin{aligned} \mathcal{A}_{\mathrm{GR}}(1^+2^+3^-) &= [\mathcal{A}_{\mathrm{YM}}(1^+2^+3^-)]^2 \\ \mathcal{A}_{\mathrm{GR}}(1^-2^-3^+) &= [\mathcal{A}_{\mathrm{YM}}(1^-2^-3^+)]^2 \end{aligned}}$$

  - KLT relations
  - hint at further secret similarities between GR and YM amplitudes...

- three-point amplitudes are the smallest amplitudes
  - entirely determined by helicities + Lorentz invariance
  - appear only in complexified Minkowski

- EH Lagrangian (and Feynman rules) not needed !

Unexplained simplicity hints at...

- ...hidden structures in perturbative quantum field theory...
- ...which are not captured by Feynman diagrams
- Need new framework to calculate S-matrix directly

(Cambridge, 1966)

Key ideas in 

- On-shellness
  - "The fields themselves are of little interest. They are merely used to calculate transition amplitudes for interactions. These amplitudes are the elements of the S-matrix"
  - "One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, whether or not some underlying Lagrangian theory exists"
- Complexify
  - "One of the most remarkable discoveries in elementary particle physics has been that of the complex plane"

What was “missing” in 1966

- Massless particles
  - most of the beautiful structure uncovered so far appears in theories of massless particles

- New symmetries/concepts
  - supersymmetry, conformal symmetry, large-N limit, string theory, AdS/CFT correspondence...
  - simplest S-matrix (in 4D): N=4 SYM & N=8 supergravity (maximal supersymmetry)

$m = 0$: spinor helicity formalism

(Berends, Kleiss, De Causmaecker, Gastmans, Wu; De Causmaecker, Gastmans, Troost, Wu; Kleiss, Stirling; Xu, Zhang, Chang; Gunion, Kunszt)

- Define $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$ where $\sigma^\mu = (1, \vec{\sigma})$
- Massless particles: $p^2 = \det p = 0$
- Hence $\boxed{p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}}$ $\cdot$ $\lambda\ (\tilde{\lambda})$ negative (positive) helicity spinors

Key formula !

- Inner products $\langle 12 \rangle := \varepsilon_{ab} \lambda_1^a \lambda_2^b$ $[12] := \varepsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$

$$2(p_1 \cdot p_2) = \langle 12 \rangle [12]$$

Allows to expose (not explain!) simplicity

MHV amplitude

- First non-vanishing amplitude: Maximally Helicity Violating

$$\mathcal{A}_{\text{MHV}}(1^+ \ldots i^- \ldots j^- \ldots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

helicities are a permutation of −−++....+

(Parke & Taylor, 1986; Berends, Giele 1987; Mangano, Parke, Xu 1988)

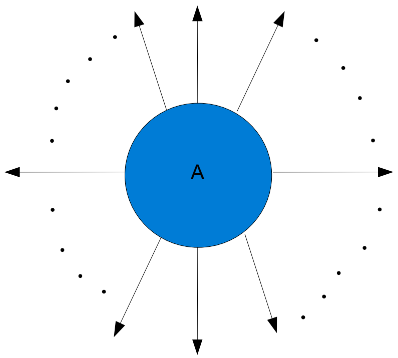
- Simple geometry in Penrose's twistor space
(Witten, 2003)

  - localised on a line in twistor space
  - holomorphic (only < > spinor products)
  - generic amplitudes (with more negative helicities) localise on unions of lines
  - first example of hidden structure

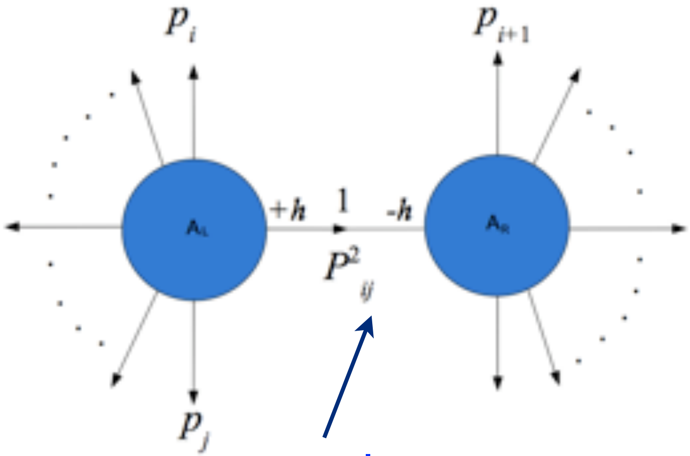
On-shell (BCF) recursion relations

(Britto, Cachazo, Feng; BCF + Witten, 2005)

- Exploit analytic structure of amplitudes
  - Singularities of tree amplitudes:



$$\xrightarrow[P_{ij}^2 \to 0]{} \quad \sum_h$$



simple pole

h = internal particles helicities

- Factorisation on multi-particle poles (simple poles, tree level)

- <u>idea</u>: physical singularities → poles in a single complex variable z

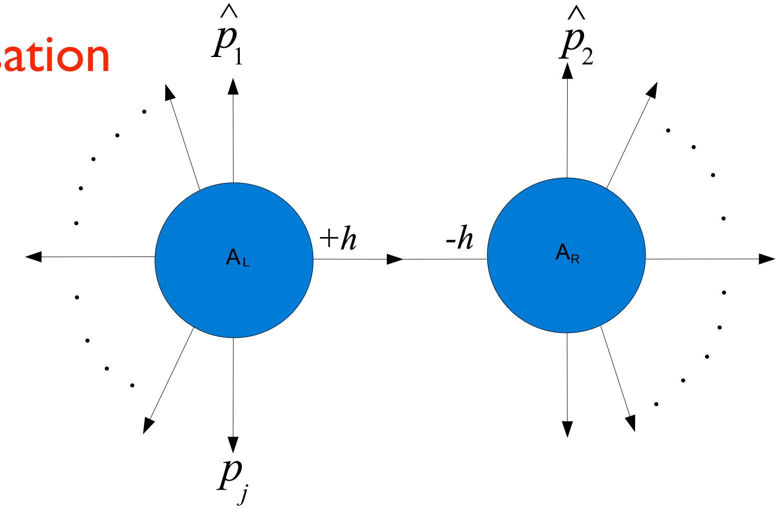
- Shift momenta: $\hat{p}_1(z) = p_1 + z\,\eta\,, \quad \hat{p}_2(z) = p_2 - z\,\eta$
  with $\hat{p}_1^2 = \hat{p}_2^2 = 0$ for all z and $\eta^2 = 0$

  - shifted momenta are complex!

- $\mathcal{A}(z) := \mathcal{A}(\hat{p}_1, \hat{p}_2, p_3, \ldots, p_n)$ $\quad$ $\mathcal{A}(0)$ is the amplitude

- $\mathcal{A}(z) = \sum_P \frac{c_P}{z - z_P}$ $\quad$ only simple poles

  - assume $\mathcal{A}(z) \to 0$ as $z \to \infty$ (depends on theory)
  - residues c_P from factorisation

Final result: $\quad \mathcal{A} = \sum_{j,h}$ 

- Results very simple!
  - building blocks are amplitudes, and 3-pt amplitudes "seed" the recursion
- Wide applicability:
  - General Relativity (Bedford, Brandhuber, Spence, GT '05; Cachazo, Svrcek'05; Benincasa, Boucher-Veronneau, Cachazo '07; Arkani-Hamed, Kaplan '08)
  - rational part of QCD amplitudes (Bern, Dixon, Kosower; "BLACKHAT" collaboration)
  - particles with masses (Badger, Glover, Khoze, Svrcek)
  - N=4/N=8 manifestly supersymmetric recursion relations (Brandhuber, Heslop, GT; Arkani-Hamed, Cachazo, Kaplan; Drummond, Henn)

On-shell methods

- Key ideas:
  - calculate (on-shell) amplitudes
  - ~~(off-shell) Green's functions, Lagrangians, fields...~~
  - not restricted to four dimensions
    - recent amplitude calculations in 6D supersymmetric theories
      (Cheung, O'Connell; Bern, Carrasco, Dennen, Huang, Ita; Brandhuber, Korres, Koschade, GT)
- Advantages:
  - gauge-invariant, on-shell data at each intermediate step of the calculation
  - ♻ amplitudes with fewer legs/fewer loops ♻

Hidden structures in planar N=4 SYM

i. Iterative structure at weak coupling

(Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov)

- $\mathcal{A}_{n,\mathrm{MHV}} = \mathcal{A}^{\mathrm{tree}}_{n,\mathrm{MHV}} \mathcal{M}_n$ $\mathcal{M}_n$ is a "helicity-blind" function

- All-loop MHV amplitude:

$$\mathcal{M}_n := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}_n^{(L)} \sim e^{\mathrm{BDS} + \mathcal{R}}$$

$a \sim g^2 N/(8\pi^2)$

  - $\mathrm{BDS} \sim \mathrm{div} + \gamma_K \mathrm{Finite}^{(1)}(p_1, \ldots, p_n)$ BDS ansatz
  - div = universal infrared-divergent part
  - γ_K is the cusp anomalous dimension BES equation → integrability
    (Beisert, Eden, Staudacher)
  - $\mathrm{Finite}^{(1)}(p_1, \ldots, p_n)$ = finite part of one-loop amplitude
  - $\mathcal{R}$ is the Remainder Function, $\mathcal{R} = 0$ for $n = 4, 5$ $\mathcal{R} \neq 0$ for $n \geq 6$

- Planar higher-loop amplitudes from lower loops!

⚠ ‣ Plus a remainder: BDS conjecture breaks down at two loops and $n = 6$ (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich)

  ‣ infrared divergences exponentiate
  (Giele, Glover; Kunszt, Signer, Trocsany; Sterman, Teyeda-Yeomans; Catani; Magnea, Sterman)

  ‣ exponentiation of finite parts: new and nontrivial

- Task: determine the remainder function

  ‣ hard to calculate, even numerically (one data point takes one week)

  ‣ will approach from the Wilson loop side......

ii. Wilson loop/amplitude duality

(Alday, Maldacena; Drummond, Korchemsky, Sokatchev + Henn; Brandhuber, Heslop, GT)

- MHV amplitudes in planar N=4 super Yang-Mills calculated by a Wilson loop

$$\langle W[C]\rangle \ := \ \mathrm{Tr}\, P \exp\left[ig \oint_C d\tau \Big(\dot{x}_\mu(\tau) A^\mu(x(\tau))\Big)\right]$$

  - Strong coupling (Alday & Maldacena)
  - Weak coupling (Drummond, Korchemsky, Sokatchev+Henn; Brandhuber, Heslop, GT)

- C determined by the momenta of the scattered particles

ii. Wilson loop/amplitude duality

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- MHV amplitudes in planar N=4 super Yang-Mills calculated by a Wilson loop

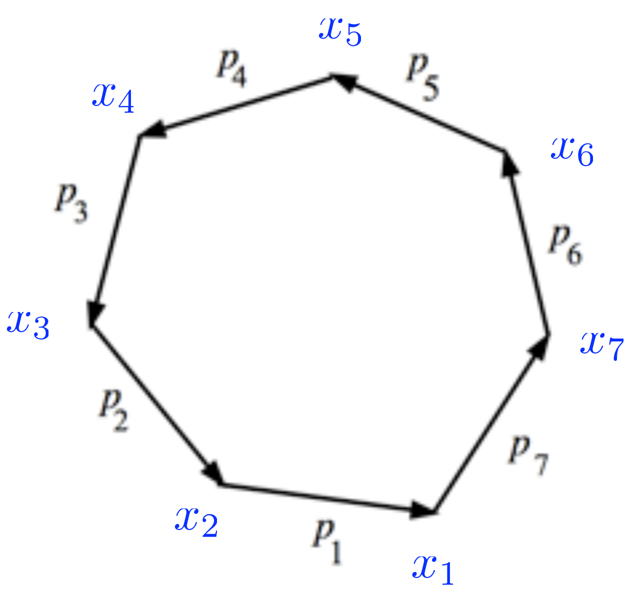
$$\langle W[C]\rangle \;:=\; \mathrm{Tr}\, P \exp\left[ig \oint_C d\tau \Big(\dot{x}_\mu(\tau) A^\mu\big(x(\tau)\big) \;\cancel{+\,\dot{y}_i(\tau)\phi^i\big(x(\tau)\big)}\Big)\right]$$

  - Strong coupling (Alday & Maldacena)
  - Weak coupling (Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, GT)

- C determined by the momenta of the scattered particles
- Purely gluonic; locally supersymmetric

$$\dot{x}^2 = \dot{y}^2\,, \qquad \text{solved by} \qquad \dot{x}^2 = 0 \qquad \dot{y} = 0$$

- The contour of the Wilson loop:
  - A particular polygonal contour, made of lightlike segments:
    - colour ordering $\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_7})$
  - $\sum_{i=1}^{n} p_i = 0$ momentum conservation
    closed contour



    - $p_i = x_i - x_{i+1}$, lightlike
    - x are T-dual (region) momenta

- At strong coupling: four-point amplitude is the same as BDS ! (Alday & Maldacena)

$$\mathcal{M}_4 \sim \exp\left[\mathrm{div} + \gamma_K^{\mathrm{strong}} \mathrm{Finite}^{(1)}(p_1, \ldots, p_4)\right]$$

- notice:
  - $\gamma_K^{\mathrm{strong}} \sim \sqrt{\lambda} + \cdots$ replaces
  - $\gamma_K^{\mathrm{weak}} \sim 4\lambda + 4\zeta_2\lambda^2 + \cdots$
  - leading-order AM calculation: $\gamma_K^{\mathrm{strong}} \to \sqrt{\lambda}$

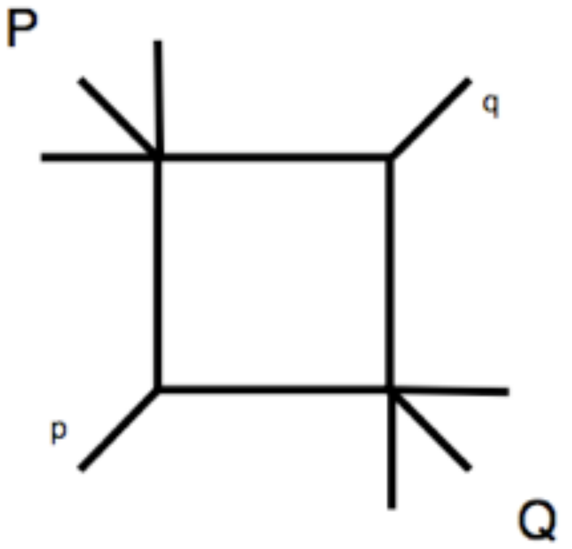
Lightlike Wilson loops at weak coupling

- Compare $< W[C] >$ to n-point MHV amplitude in N=4 SYM in perturbation theory
  - 4-point case at one loop (Drummond, Korchemsky, Sokatchev)
  - n-point case (Brandhuber, Heslop, GT)
- Results are in perfect agreement !
  - unexpected: eikonal approximation usually reproduces IR behaviour only; we also get finite parts

n-point, planar MHV amplitudes in N=4 SYM

- Simplest one-loop amplitude

$$\mathcal{A}^{1-\mathrm{loop}}_{\mathrm{MHV}} = \mathcal{A}^{\mathrm{tree}}_{\mathrm{MHV}} \sum$$



(Bern, Dixon, Dunbar, Kosower, 1994)

- Sum of two-mass easy box functions, all with coefficient 1

Diagrammatic interpretation

All-loop conjecture

(Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, GT)

- MHV Amplitude "=" Wilson loop

  - more precisely: Wilson loop calculates $\mathcal{M}$

    - $\mathcal{M}$ is the helicity-blind function in $\mathcal{A}_{\mathrm{MHV}}^{(L)} = \mathcal{A}_{\mathrm{MHV}}^{\mathrm{tree}} \mathcal{M}^{(L)}$
    - Subtlety in the infrared-divergent part

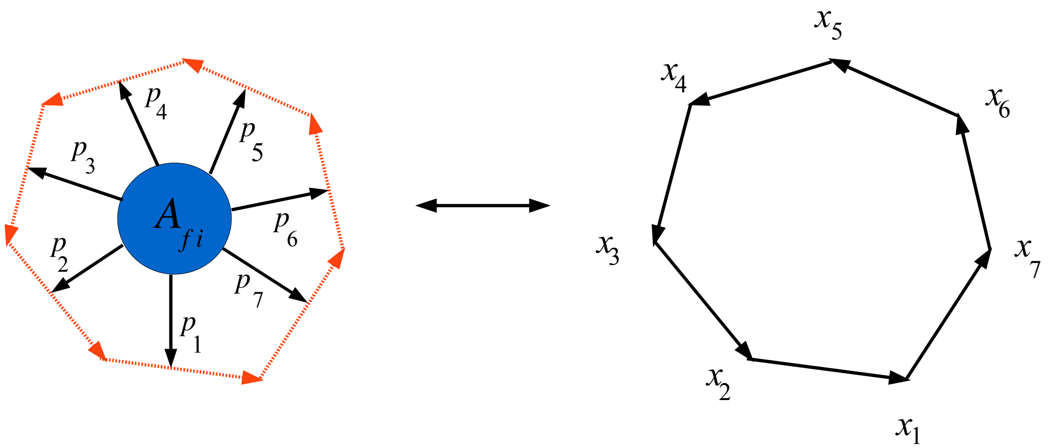
- Conjecture: (Log) < $W[C]$ > = (Log) $\mathcal{M}$ to all loops

In terms of the remainders:

$$\boxed{\mathcal{R}_{n,\mathrm{WL}} = \mathcal{R}_n}$$

Why is this interesting/useful ?

- New duality



- Remainder function is easier to compute

$$< W[C] > \;=\; \mathrm{Exp}\,(\;\; \mathrm{BDS} + \mathcal{R} \;\;)$$

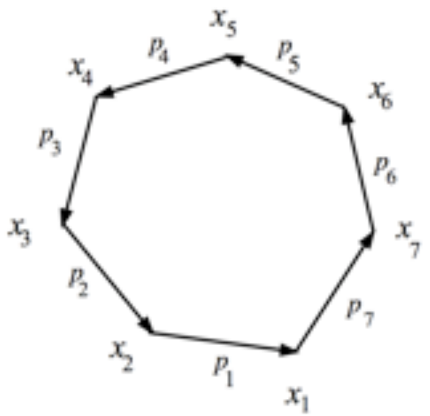
- Wilson loop: one hour. Amplitude: one week
  - (dimensionally regularised) Wilson loop integral functions much simpler to evaluate than corresponding amplitude integral functions
- Functional dependence of $\mathcal{R}$ constrained by dual conformal symmetry

iii. Dual conformal symmetry

(Drummond, Henn, Korchemsky, Sokatchev)

- Natural symmetry from Wilson loop perspective:

  - it is the standard conformal group acting on dual momenta x's

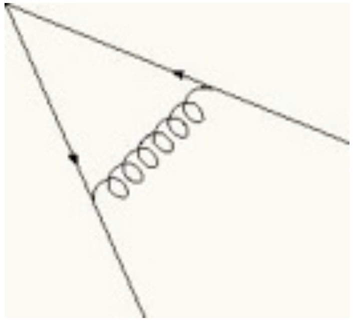


$$p_i = x_i - x_{i+1}$$

$$x_{n+1} = x_1$$

  - symmetry is anomalous

    - UV divergences from cusps in the contour
      (UV for the Wilson loop = IR for the amplitude)



- BDS Ansatz explained by dual conformal symmetry
  - a solution to the associated anomalous Ward identity
  - remainder $\mathcal{R}$ is a function of cross-ratios
    - $\dfrac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$ invariant under $x_i \to \dfrac{x_i}{x_i^2}$
  - solution is unique at four and five points (modulo constants)
    - lightlike condition forbids nontrivial cross ratios for $n < 6$
- For $n \geqslant 6$ points, cross ratios open up and $\mathcal{R} \neq 0$
  - e.g. at $n = 6$: $u_1 = \dfrac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2}, \quad u_2 = \dfrac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}, \quad u_3 = \dfrac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$
  - $\mathcal{R}_6 = \mathcal{R}_6(u_1, u_2, u_3)$ non-vanishing starting at 2 loops

- The hunt for new symmetries to constrain $\mathcal{R}$ is open !
  - goal: complete "algebraic" determination of amplitudes

- Remarkable series of recent strong-coupling calculations
  (Alday, Maldacena; Alday, Gaiotto Maldacena; Alday, Maldacena, Sever, Vieira)
  - integrability of worldsheet theory, Y-systems...

- Weak-coupling side:
  - n-point remainder integrals (Anastasiou, Brandhuber, Heslop, Khoze, Spence, GT)
  - evaluated for any n in $(1+1)$-dim kinematics (Heslop, Khoze)
  - 6-point integrals calculated by Del Duca, Duhr, Smirnov. 17-pages result, contains Goncharov polylogs
  - Goncharov, Spradlin, Vergu and Volovich eliminate Goncharov's polylogs. 2-line result, only classical polylogs!

$< W[C] >$ at two loops, n points

(Anastasiou, Brandhuber, Heslop, Khoze, Spence, GT)

- Remainder function for any n known in terms of a set of integral functions

  - # of independent topologies does not grow with n

    - n-point Wilson loop under numerical control

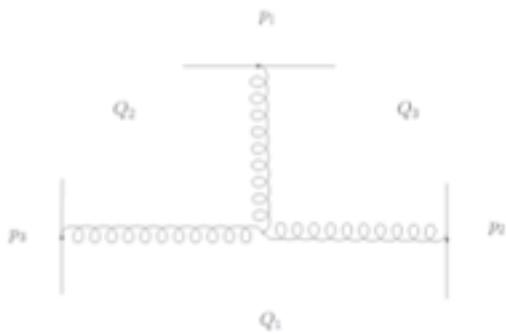
- Compare, where possible, to amplitude results

  - collinear limits of Wilson loops same as amplitude's

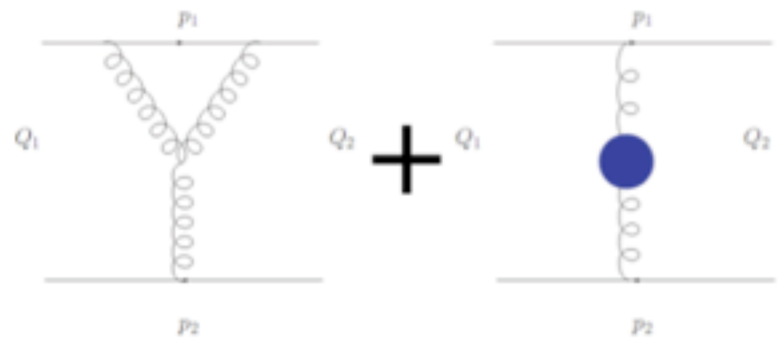
    - if it quacks as a duck, it's a duck!

  - check dual conformal symmetry

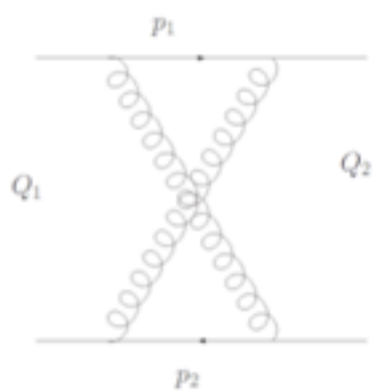
Wilson loop master integrals



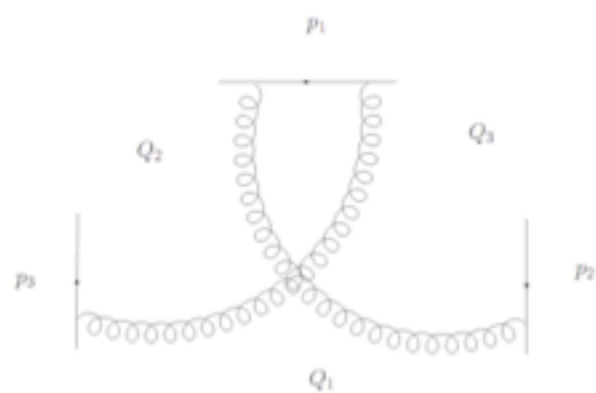
$f_H(p_1, p_2, p_3; Q_1, Q_2, Q_3)$



$f_Y(p_1, p_2; Q_1, Q_2)$

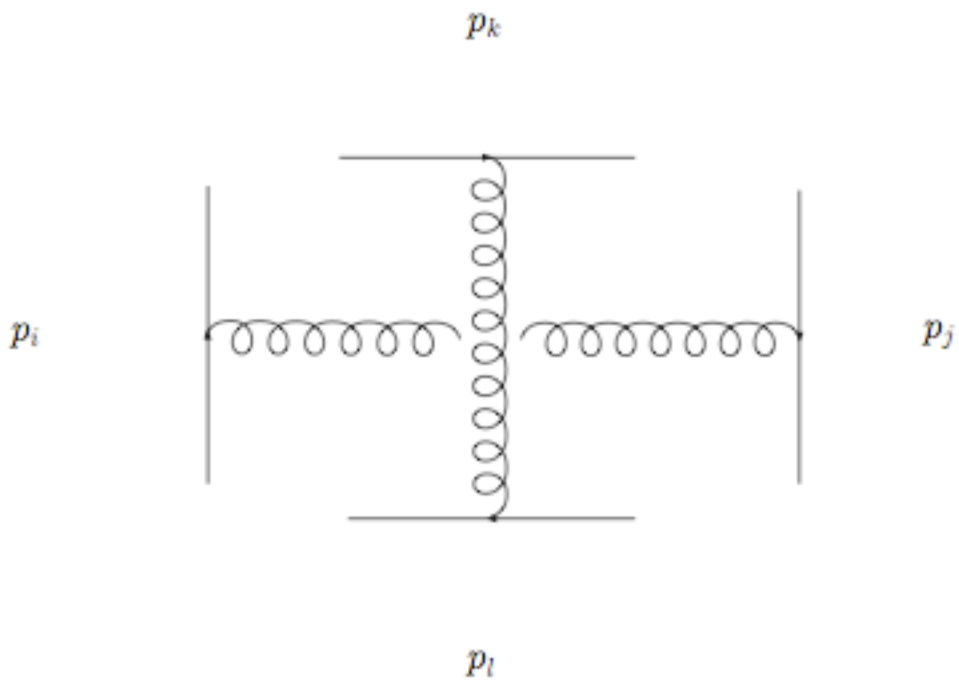


$f_X(p_1, p_2; Q_1, Q_2)$



$f_C(p_1, p_2, p_3; Q_1, Q_2, Q_3)$

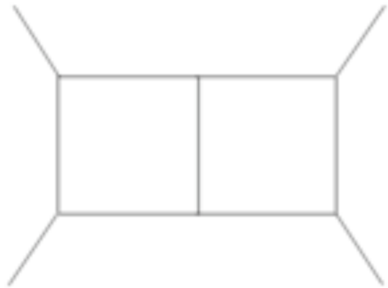
- four topologies: hard, Y (+ self-energy), cross, curtain



- factorised cross (product of two one-loop integrals)

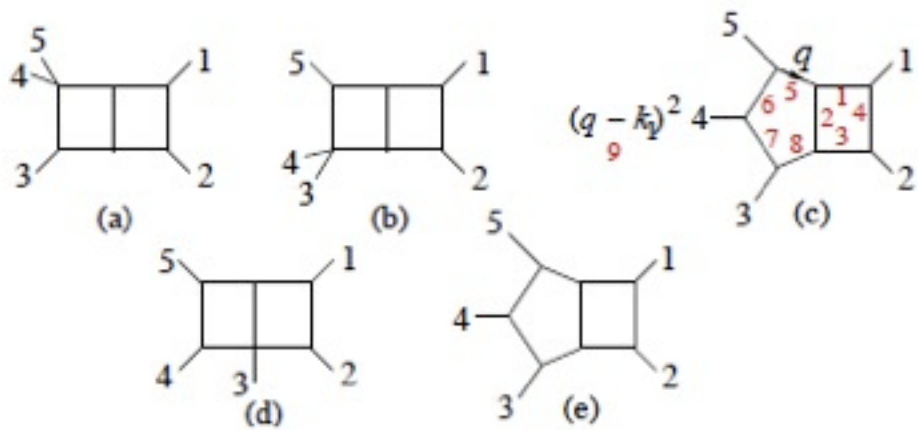
Amplitude master integrals

n=4



(Bern, Rozowsky, Yan)

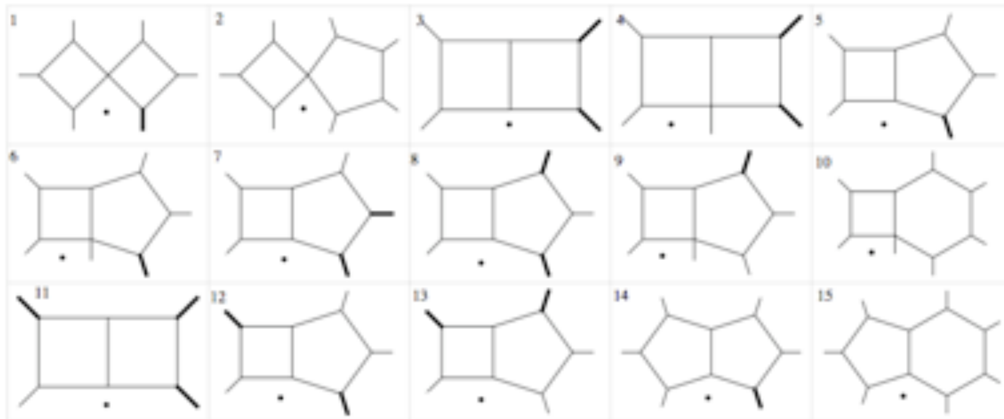
n=5



(Bern, Czakon, Kosower, Roiban, Smirnov)

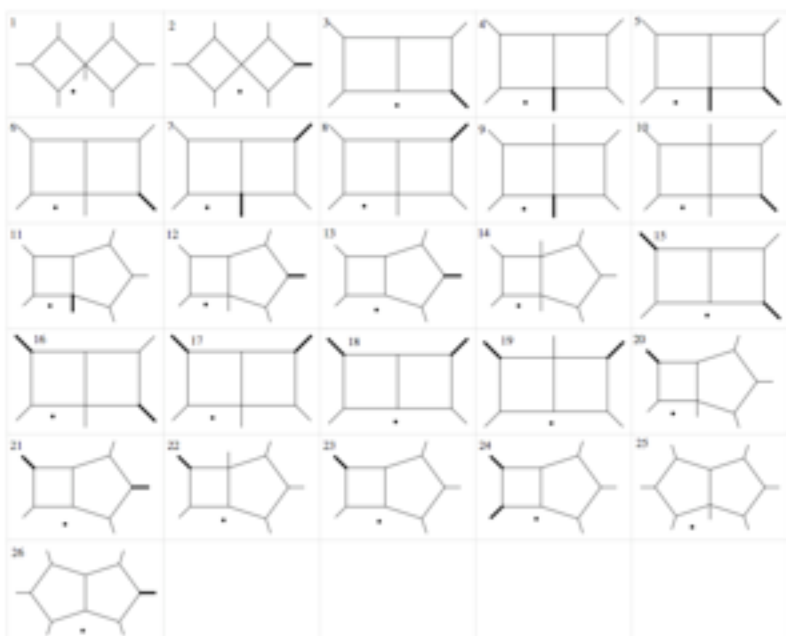
Amplitude master integrals (cont'd)

n=6



(Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich)

n=7



(Vergu)

all n : Vergu arXiv:0908.2394 [hep-th]; Arkani-Hamed et al arXiv:1008.2958 [hep-th]

- Wilson loop: no new integrals after 9 points
  - hard diagram with three masses

- Amplitude: no new integrals after 12 points

- Interesting mismatch between these two numbers...

Amplitude/Wilson loop duality at $\mathcal{O}(\epsilon)$

(Brandhuber, Heslop, Nguyen, Katsaroumpas, Spence, Spradlin, GT)

- Back to four- and five-point Wilson loops:
  - up to $\mathcal{O}(1)$ fully determined by the BDS Ansatz
  - D = 4 - 2 ϵ (ϵ <0) regularises divergences in the Wilson loop
  - what about $\mathcal{O}(\epsilon)$ corrections ? effectively away from four dimensions
- Main result: amplitude and WL are still identical up to and including $\mathcal{O}(\epsilon)$ terms
  - miraculous agreement that cannot be called a coincidence...
  - clearly beyond dual conformal symmetry !

- In practice, define an $O(\epsilon)$ remainder for amplitude and WL. For example,

$$\mathcal{R}_4^{(2)} = \mathcal{M}_4^{(2)} - \Big[\tfrac{1}{2}\Big(\mathcal{M}_4^{(1)}(\epsilon)\Big)^2 + f^{(2)}(\epsilon)\, \mathcal{M}_4^{(1)}(2\epsilon) + C^{(2)}\Big] + \mathcal{O}(\epsilon^2)$$

$$\mathcal{R}_{4,\mathrm{WL}}^{(2)} = w_4^{(2)} - \Big[\tfrac{1}{2}\Big(w_4^{(1)}(\epsilon)\Big)^2 + f_{\mathrm{WL}}^{(2)}(\epsilon)\, w_4^{(1)}(2\epsilon) + C^{(2)}\Big] + \mathcal{O}(\epsilon^2)$$

  - $f^{(2)}(\epsilon),\ f_{\mathrm{WL}}^{(2)},\ C^{(2)}$ are the same as in the $O(1)$ iteration

- Main results: $\mathcal{R}_4^{(2)} - \mathcal{R}_{4,\mathrm{WL}}^{(2)} = 3\,\zeta_5\,\epsilon + \mathcal{O}(\epsilon^2)$

$$\mathcal{R}_5^{(2)} - \mathcal{R}_{5,\mathrm{WL}}^{(2)} = -\tfrac{5}{2}\,\zeta_5\,\epsilon + \mathcal{O}(\epsilon^2)$$

parity-even part

- Four-point remainder at $O(\epsilon)$:

$$\begin{aligned}\mathcal{R}^{(4)}(x) \quad = \quad & -\frac{\epsilon}{360}\Big\{16\pi^4\log(x) - 15\pi^4\log(1+x) - 30\pi^2\log^2(x)\log(1+x) \\ & -15\log^4(x)\log(1+x) - 120\pi^2\log(x)\mathrm{Li}_2(-x) - 120\log^3(x)\mathrm{Li}_2(-x) \\ & +180\pi^2\mathrm{Li}_3(-x) + 540\log^2(x)\mathrm{Li}_3(-x) - 1440\log(x)\mathrm{Li}_4(-x) \\ & +1800\mathrm{Li}_5(-x) + 690\pi^2\zeta_3 - 5940\zeta_5\Big\}\,,\end{aligned}$$

$x := s/t$

- Transcendentality 5 function

- only classical polylogs appear

Comments:

1. Conjecture: dual (super)conformal symmetry lifted from Wilson loops to amplitudes
   (Drummond, Henn, Korchemsky, Sokatchev)

   - new hidden symmetry of planar N=4 amplitudes!
   - $\mathcal{A}_{\mathrm{MHV}} \to x_1^2 \cdots x_n^2 \, \mathcal{A}_{\mathrm{MHV}}$ under inversions $x_i \to \dfrac{x_i}{x_i^2}$
     - on-shellness, large-N limit, N=4 symmetry
   - tree-level S-matrix of N=4 SYM is dual superconformal covariant

     (Brandhuber, Heslop, GT)
   - one-loop dual conformal anomaly under control (DHKS, BHT)
   - at tree/loop level, it restricts considerably the form of amplitudes
     (Brandhuber, Heslop, GT; Drummond, Henn, Korchemsky, Sokatchev; Korchemsky, Sokatchev; Bargheer, Beisert, Galleas, Loebbert, McLoughlin)
     - loops without loops

2. Weak coupling: Yangian symmetry of tree-level scattering amplitudes (Drummond, Henn, Plefka)

- from commuting the generators of the two superconformal algebras
- it is still a matter of debate whether the predictive power of the Yangian symmetry exceeds that of the two superconformal symmetries
- However: Yangian symmetry might be easier to implement than e.g. ordinary conformal symmetry

iv. Form factors in N=4 SYM

- Studied at weak coupling in a pioneering letter by Willy van Neerven (van Neerven 1986)
  - simplest (Sudakov) form factor at one and two loops, with Feynman diagrams
  - exponentiation of finite parts in N=4 super Yang-Mills !
  - 4 citations on Spires, all after 2009 (2 last month....)
- Recent strong-coupling analysis (Alday, Maldacena; Maldacena, Zhiboedov)
  - independent of the particular operator as long as anomalous dimension is small compared to $\sqrt{\lambda}$
  - calculated by a periodic Wilson loop
- Recent series of weak-coupling calculations

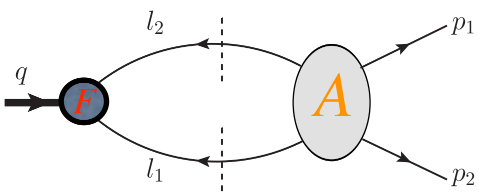
(Brandhuber, Spence, GT, Yang;   Bork, Kazakov, Vartanov)

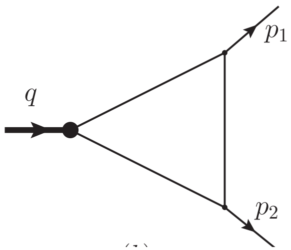
- Perturbative questions on $< 0 \mid \mathcal{O}(0) \mid \textit{state} >$:
  - dependence on external state
  - dependence on operator
  - which integral functions can appear
  - duality with Wilson lines

- Consider scalar 1/2 BPS operators
  - e.g. $\mathcal{O}(x) = \mathrm{Tr}\,(\phi_{12}\,\phi_{12})(x)$ where $\phi_{AB} = \frac{1}{2}\,\epsilon_{ABCD}\,\bar{\phi}^{CD}$
  - Sudakov form factor: $< 0 \mid \mathcal{O}(0) \mid \phi_{12}(p_1)\;\phi_{12}(p_2) >$

    Note: $\mathcal{O}$ is a colour singlet
  - MHV: $< 0 \mid \mathcal{O}(0) \mid g^+(p_1) \ldots \phi_{12}(p_i) \ldots g^+ g^+ \ldots \phi_{12}(p_j) \ldots g^+(p_n) >$

- On-shell methods can be successfully applied to form factors
(Brandhuber, Spence, GT, Yang)
  - off-shellness limited to part of the diagram
  - tree-level form factors derived using BCFW recursion relations
  - unitarity: at one loop, glue form factors and amplitudes
- Sudakov: $F(q^2) \ := \ \langle 0|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\,|\phi_{12}(p_1)\phi_{12}(p_2)\rangle$





$q \ := \ p_1 + p_2$

$$\Big[F(q^2)\Big]^{1\,\mathrm{loop}} \ = \ 2\,(-q^2)^{-\epsilon}\Big[-\frac{1}{\epsilon^2}+\frac{\zeta_2}{2}+\mathcal{O}(\epsilon)\Big]$$

$$\Big[\mathrm{Log}\ F(q^2)\Big]^{2\,\mathrm{loop}} \ = \ (-q^2)^{-2\epsilon}\Big[\frac{\zeta_2}{\epsilon^2}+\frac{\zeta_3}{\epsilon}+\mathcal{O}(\epsilon)\Big]$$

• “MHV” form factors: add arbitrary number of g+’s in external state

$$F(1,\ldots,n) \;=\; \langle 0|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\;|g^+(p_1)\cdots\phi_{12}(p_i)\cdots\phi_{12}(p_j)\cdots g^+(p_n)\rangle$$

- structure very similar to that of MHV amplitudes in N=4

<u>tree</u>

$$F^{(0)} \;=\; \frac{\langle i\, j\rangle^2}{\langle 1\,2\rangle\cdots\langle n\,1\rangle}$$

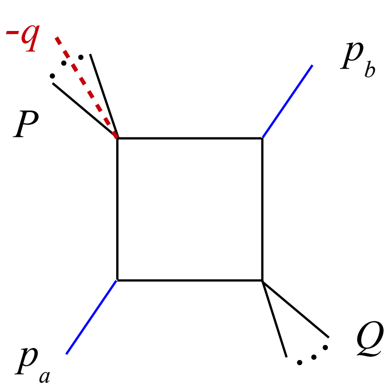
<u>Recall:</u>

$$p_{\alpha\dot\alpha} \;:=\; \lambda_\alpha\tilde\lambda_{\dot\alpha}$$
$$\langle l\, m\rangle \;:=\; \epsilon_{\alpha\beta}\,\lambda_l^\alpha\,\lambda_m^\beta$$

- holomorphic function of spinor variables
- localises on a line in twistor space

<u>one loop</u>

$$F^{(1)} \;=\; F^{(0)}\Big[\sum_{i=1}^{n}\frac{(-s_{ii+1})^{-\epsilon}}{\epsilon^2} + \sum_{a,b}\mathrm{Fin}^{\mathrm{2me}}(p_a,p_b,P,Q)\Big]$$



- one-loop result proportional to tree level
- sum of finite two-mass easy box functions

Summary

- Hidden structures in scattering amplitudes
    - on-shell recursion
- Focused on N=4 amplitudes
    - iteration at higher loops
    - amplitude/Wilson loop duality
    - beyond four dimensions
    - dual conformal symmetry
- Form factors from unitarity
- Plenty of questions to ask!