Introduction to Event Generators

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Topics of the lectures

1. Lecture 1: *The Monte Carlo Principle*
2. Lecture 2: *Parton level event generation*
3. Lecture 3: *Dressing the Partons*
4. Lecture 4: *Modelling beyond Perturbation Theory*

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Menu of lecture 1

- Prelude: Selecting from a distribution
- Standard textbook numerical integration (quadratures)
- Monte Carlo integration
- A basic simulation example

Prelude: Selecting from a distribution

The problem

- A typical Monte Carlo/simulation problem:
  Distribution of "usual" random numbers #:
  "flat" in [0, 1].
- But: Want random numbers $x \in [x_{\rm min}, x_{\rm max}]$,
  distributed according to (probability) density $f(x)$.

The exact solution

- The first method applies if both the integral of the density $f(x)$ and its inverse are known (i.e. practically never).
- To see how it works realise that the

  diff. probability $\mathcal{P}(x \in [x', x' + \mathrm{d}x']) = f(x')\mathrm{d}x'$.
- Therefore: x given by

$$\int\limits_{x_{\min}}^{x} \mathrm{d}x' f(x') = \# \int\limits_{x_{\min}}^{x_{\max}} \mathrm{d}x' f(x').$$

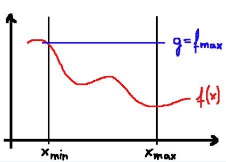
- Since everything known:

$$x = F^{-1}\left[F(x_{\min}) + \#\left(F(x_{\max}) - F(x_{\min})\right)\right].$$

The work-around solution: "Hit-or-miss"

(Solution, if exact case does not work.)

- Builds on "over-estimator" $g(x)$ (G and G^{-1} known):
$$g(x) > f(x) \quad \forall x \in [x_{\rm min},\, x_{\rm max}].$$
- Select an x according to g (with exact algorithm);
- Accept with probability $f(x)/g(x)$ (with another random number);
- Obvious fall-back choice for $g(x)$:
$$g(x) = \mathrm{Max}_{[x_{\rm min},\, x_{\rm max}]}\{f(x)\}.$$



Quadratures: standard numerical integration

Reminder: Basic techniques

- Typical problem: Need to evaluate an integral, cannot do it in closed form.
- Example: nonlinear pendulum.
  Can calculate period T from E.o.M. $\ddot{\theta} = -g/l \sin\theta$:

$$T = \sqrt{\frac{8l}{g}} \int\limits_0^{\theta_{\max}} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_{\max}}}$$

Elliptic integral, no closed solution known
$\Longrightarrow$ entering (again) the realm of numerical solutions.

Numerical integration: Newton-Cotes method

- Nomenclature now: Want to evaluate $I_f^{(a,b)} = \int\limits_a^b \mathrm{d}x f(x)$.
- Basic idea: Divide interval $[a, b]$ in N subintervals of size $\Delta x = (b-a)/N$ and approximate

$$I_f^{(a,b)} = \int\limits_a^b \mathrm{d}x f(x) \approx \sum_{i=0}^{N-1} f(x_i)\Delta x = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x,$$

  i.e. replace integration by sum over rectangular panels.
- Obvious issue: What is the error? How does it scale parametrically with "step-size" (or, better, number of function calls)? Answer: It is linear in Δx.

Improving on the error: Trapezoid, Simpson and all that

- A careful error estimate suggests that by replacing rectangles with trapezoids the error can be reduced to quadratic in Δx.
- This boils down to including a term $[f(b) - f(a)]/2$:

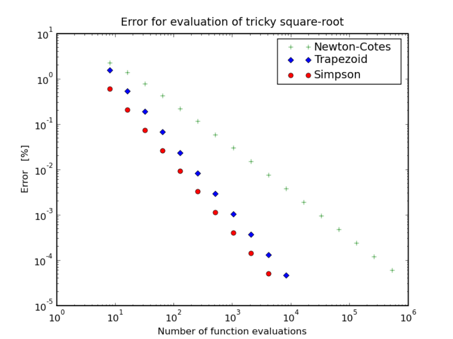
$$I_f^{(a,b)} \approx \sum_{i=1}^{N-1} f(x_i)\Delta x + \frac{\Delta x}{2}\left[f(a) + f(b)\right]$$

- Repeating the error-reducing exercise replaces the trapezoids by parabola: The Simpson rule. In so doing, the error decreases to $(\Delta x)^4$.

Numerical integration: Results

- Consider test function $f(x) = \sqrt{4 - x^2}$ in $[0, 2]$.

$$(I_f^{(0,2)} = \int\limits_0^2 \mathrm{d}x \sqrt{4 - x^2} = \pi).$$



Convergence of numerical integration: Summary

- First observation: Numerical integrations only yield estimators of the integral, with an estimated accuracy given by the error.

(Proviso: the function is sufficiently well behaved.)

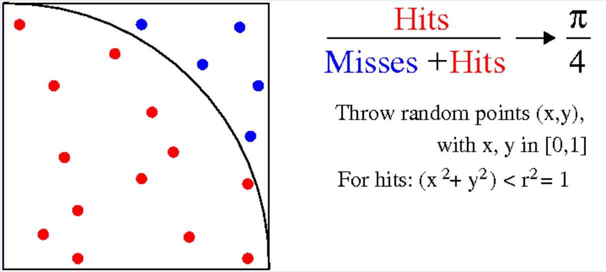
- Scaling behaviour of the error translates into scaling behaviour for the number of function calls necessary to achieve a certain precision.
- In one dimension/per dimension, therefore, the convergence scales like
  - Trapezium rule: $\simeq 1/N^2$
  - Simpson's rule $\simeq 1/N^4$

  with the number N of function calls.

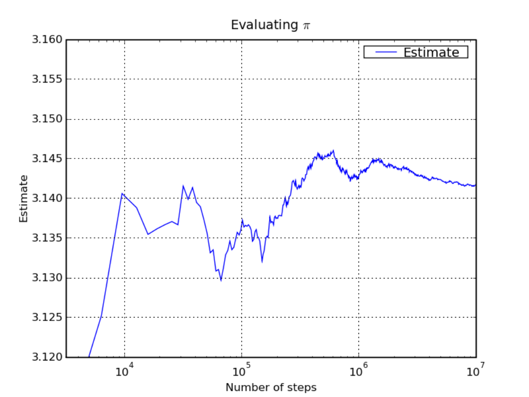
Monte Carlo integration

The underlying idea: Determination of π

- Use random number generator!



Determination of π



Error estimate in Monte Carlo integration

- MC integration: Estimate integral by N probes

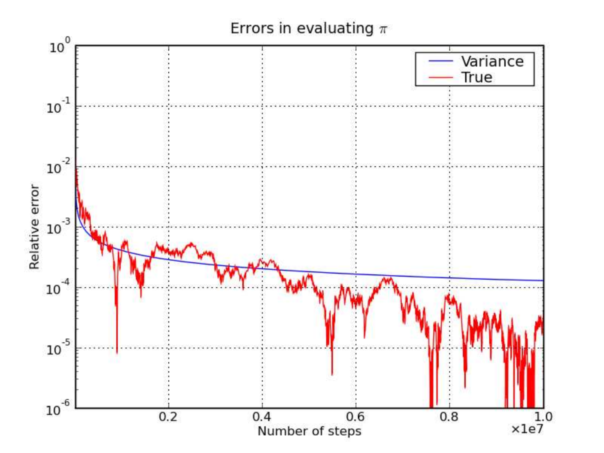
$$
\begin{aligned}
I_f^{(a,b)} &= \int\limits_a^b \mathrm{d}x f(x) \\
\longrightarrow \quad \langle I_f^{(a,b)} \rangle &= \frac{b-a}{N}\sum_{i=1}^{N} f(x_i) = \langle f \rangle_{a,b},
\end{aligned}
$$

  where x_i homogeneously distributed in $[a,\, b]$
- Basic idea for error estimate: statistical sample
  $\Longrightarrow$ use standard deviation as error estimate

$$
\langle E_f^{(a,b)}(N) \rangle = \sigma = \left[\frac{\langle f^2 \rangle_{a,b} - \langle f \rangle_{a,b}^2}{N}\right]^{1/2}.
$$

- Independent of the number of integration dimensions!
  $\Longrightarrow$ Method of choice for high-dimensional integrals.

Determination of π: Errors



Improve convergence: Importance sampling

- Want to minimise number of function calls.

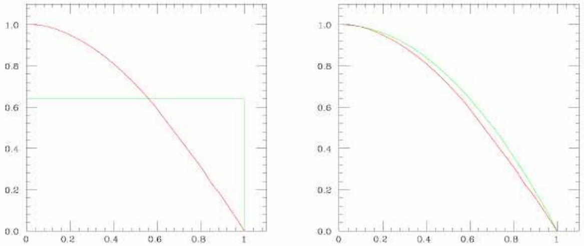
  (They are potentially CPU-expensive.)

  $\Longrightarrow$ Need to improve convergence of MC integration.
- First basic idea: Samples in regions, where f largest

  ($\Longrightarrow$ corresponds to a Jacobian transformation of integral.)
- Algorithm:
  - Assume a function $g(x)$ similar to $f(x)$.
  - Obviously $f(x)/g(x)$ is smooth $\Longrightarrow$ $\langle E(f/g)\rangle$ is small.
  - Must sample according to $\mathrm{d}x\, g(x)$ rather than $\mathrm{d}x$: $g(x)$ plays role of probability distribution; we know already how to deal with this!
- Works, if $f(x)$ is well-known. Hard to generalise.

Importance sampling: Example results

- Consider $f(x) = \cos\frac{\pi x}{2}$ and $g(x) = 1 - x^2$:



$$
\begin{aligned}
I &= \int_0^1 dx \cos\tfrac{\pi}{2}x \\
&= 0.637 \pm 0.308/\sqrt{N}
\end{aligned}
$$

$$
\begin{aligned}
I &= \int_0^1 dx (1-x^2)\, \frac{\cos\frac{\pi}{2}x}{1-x^2} \\
&= \int d\rho\, \frac{\cos\frac{\pi}{2}x}{1-x^2}[x(\rho)] \\
&= 0.637 \pm 0.032/\sqrt{N}
\end{aligned}
$$

Improve convergence: Stratified sampling

- Want to minimise number of function calls.

  (They are potentially CPU-expensive.)

  $\Longrightarrow$ Need to improve convergence of MC integration.

- Basic idea here: Decompose integral in M sub-integrals

$$\langle I(f)\rangle = \sum_{j=1}^{M}\langle I_j(f)\rangle, \quad \langle E(f)\rangle^2 = \sum_{j=1}^{M}\langle E_j(f)\rangle^2$$

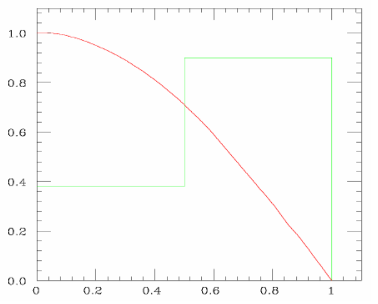
- Then: Overall variance smallest, if "equally distributed".

  ($\Longrightarrow$ Sample, where the fluctuations are.)

- Algorithm:
  - Divide interval in bins (variable bin-size or weight);
  - adjust such that variance identical in all bins.

Stratified sampling: Example results

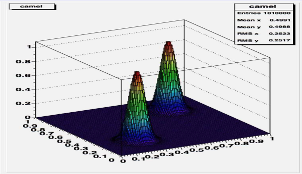
- Consider $f(x) = \cos\frac{\pi x}{2}$ and $g(x) = 1 - x^2$:



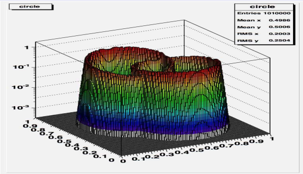
$$\langle I \rangle = 0.637 \pm 0.147/\sqrt{N}$$

Example for stratified sampling: VEGAS

- Good for Vegas: Singularity "parallel" to integration axes



- Bad for Vegas: Singularity forms ridge along integration axes



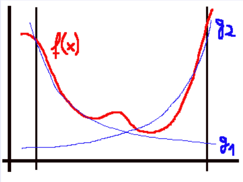
Improve convergence: Multichannel sampling

- Want to minimise number of function calls.

  (They are potentially CPU-expensive.)

  $\Longrightarrow$ Need to improve convergence of MC integration.

- Basic idea: Best of both worlds: Hybrid between importance and stratified sampling.



- Have "bins" – weight α_i – of "eigenfunctions" – $g_i(x)$:

  $\Longrightarrow \quad g(\vec{x}) = \sum_{i=1}^{N} \alpha_i g_i(\vec{x})$.

- In particle physics, this is the method of choice for parton level event generation!

Basic simulation paradigm

An example from thermodynamics

- Consider two-dimensional Ising model:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j$$

(Spins fixed on 2-D lattice with nearest neighbour interactions.)

- Traditional model to understand (spontaneous) magnetisation & phase transitions.
- To evaluate an observable $\mathcal{O}$, sum over all micro states $\phi_{\{i\}}$, given by the individual spins. (Similar to path integral in QFT.)

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi_{\{i\}} \,\mathrm{Tr}\, \left\{ \mathcal{O}(\phi_{\{i\}}) \exp\left[-\frac{\mathcal{H}(\phi_{\{i\}})}{k_B T} \right] \right\}$$

- Typical problem in such calculations (integrations!): Phase space too large $\Longrightarrow$ need to sample.

Metropolis-Algorithm

- Metropolis algorithm simulates the canonical ensemble, summing/integrating over micro-states with MC method.
- Necessary ingredient: Interactions among spins in probabilistic language (will come back to us.)
- Algorithm will look like: Go over the spins, check whether they flip (compare $\mathcal{P}_{\rm flip}$ with random number), repeat to equilibrate.
- To calculate $\mathcal{P}_{\rm flip}$: Use energy of the two micro-states (before and after flip) and Boltzmann factors.
- While running, evaluate observables directly and take thermal average (average over many steps).

Why Metropolis is correct: Detailed balance

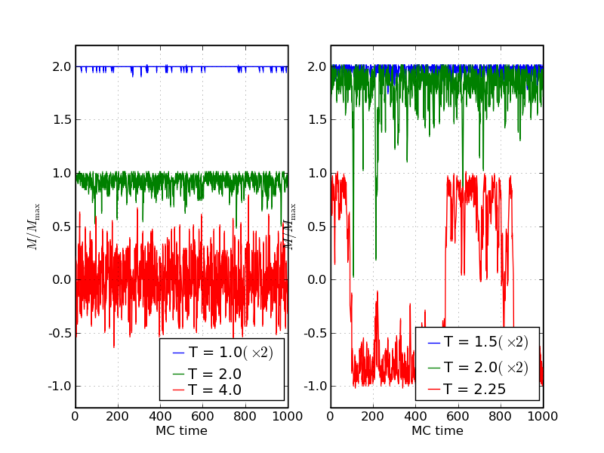
- Consider one spin flip, connecting micro-states 1 and 2.
- Rate of transitions given by the transition probabilities $\mathcal{W}$
- If $E_1 > E_2$ then $\mathcal{W}_{1\to 2} = 1$ and $\mathcal{W}_{2\to 1} = \exp\left(-\frac{E_1-E_2}{k_B T}\right)$
- In thermal equilibrium, both transitions equally often:
$$\mathcal{P}_2\mathcal{W}_{2\to 1} = \mathcal{P}_1\mathcal{W}_{1\to 2}$$
This takes into account that the respective states are occupied according to their Boltzmann factors.

$(\mathcal{P}_i \sim \exp(-E_i/k_B T))$

- In principle, all systems in thermal equilibrium can be studied with Metropolis - just need to write transition probabilities in accordance with detailed balance, as above $\Longrightarrow$ general simulation strategy in thermodynamics.

Some example results

- Fix temperature, use a 10×10 lattice



Summary of lecture 1

- Discussed some basic numerical techniques.
- Introduced Monte Carlo integration as the method of choice for high-dimensional integration space (phase space).
- Introduced some standard improvement strategies to the convergence of Monte Carlo integration.
- Discussed connections between simulations and Monte Carlo integration with the example of the Ising model.