Numerical evaluation of loop corrections

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Together with Sebastian Becker and Stefan Weinzierl
[New in the group: Daniel Götz and Christopher Schwan]

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QCD @ LHC 2011, August 25th 2011
St Andrews, Scotland

Introduction - Physics at the LHC

- ”Higgs hunting” and ”Hunt for NP” is accompanied by large QCD background!
- Experimenter’s wish list: [Les Houches wish list]
  $pp \to VV + jets$, $H + 2jets$, $t\bar{t}b\bar{b}$, $t\bar{t} + 2jets$, $VVb\bar{b}$, $VV + 2jets$, $V + 3jets$, VVV.
- Due to the large QCD background we need to find the famous “needle in the haystack”.

$\Rightarrow$ A detailed understanding of multi-parton QCD final states is unavoidable!

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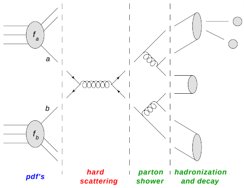

$$d\sigma_{pp\to x_{had.}}(s,\mu_F^2,\mu_R^2) \propto f_a(x_a;\mu_F^2) f_b(x_b;\mu_F^2) \otimes d\hat{\sigma}_{ab\to x_{partonic}}(\hat{s};\{p_x\},\mu_F^2,\mu_R^2) \otimes [''PS/Had.'']$$

- During the workshop we saw that for a reliable prediction a good description of all the parts is necessary, where we will focus on the hard scattering.
- $d\hat{\sigma}_{ab\to x_{partonic}}(\hat{s};\{p_x\},\mu_F^2,\mu_R^2) \propto |\mathcal{A}_{ab\to x_{partonic}}(\hat{s};\{p_x\},\mu_F^2,\mu_R^2)|^2$
- Many jets at the LHC: Want $\mathcal{A}_{ab\to x_{partonic}} \equiv \mathcal{A}_n$ for large n, w/ $n = 2 + \#x$!
- Want $|\mathcal{A}_n|^2$ at NLO in α_s due to a large scale dependence at LO and more accurate jet desriptions! See talk by Joey Huston on Tuesday.

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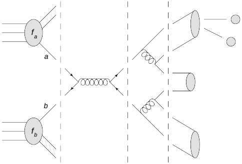

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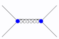
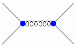
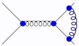
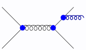
The NLO pieces to a LO contribution $\mathcal{A}_n^{(0)}$

$|\mathcal{A}_n|^2 = |\mathcal{A}_n^{(0)}|^2 + 2Re(\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}) \sim \Big|$  $\Big|^2 + 2\,\mathbf{Re}\Big($  **x**  $\Big)$ $\Big|$ $|\mathcal{A}_{n+1}|^2 = |\mathcal{A}_{n+1}^{(0)}|^2 \sim \Big|$  $\Big|^{\mathbf{2}}$

Our goal

Evaluate $|\mathcal{A}_n|^2$ for large n to NLO accuracy in α_s [especially the virtual piece] in a fully numerically Monte Carlo (MC) framework! [With the intention for real application, at the moment: $e^+e^- \to jets$ and $pp \to V + jets$]

Introduction - Current methods to solve $|\mathcal{A}_n|^2$ at NLO

Problem

$2Re(\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)})$ and $|\mathcal{A}_{n+1}^{(0)}|^2$ contain singularities:

- In $\mathcal{A}_{n+1}^{(0)}$ due to the unresolved 1-particle phase space integration (soft and collinear).
- In $\mathcal{A}_n^{(1)}$ due to the loop integration (soft, collinear and ultraviolet).

As we saw already in the talks by Zoltan Kunszt ["NLO revolution"] and Nigel Glover on Tuesday there exists a collection of methods by now to tackle these problems.

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Tackle the real emission: $\mathcal{A}_{n+1}^{(0)}$

- Subtract suitably chosen dipole terms $\mathcal{D}_{[ij,k]}$ in order to get a finite integrand [Catani, Seymour].
- This procedure is well known and exists in various improvements [Dittmaier et al., Czakon et al., Gehrmann et al., ...] and variations like residue subtraction [Frixione et al., ...], antenna subtraction [Kosower, Glover et al., ...], ...

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Tackle the virtual contributions: $\mathcal{A}_n^{(1)}$

- Feynman graph approach: [Passarino & Veltman, Denner & Dittmaier, ...]
  - Each single Feynman diagram has to be considered! The complexity grows factorially with the number of legs!
- Unitarity based methods: [BDK, OPP, Anastasiou et al., Berger et al., Ellis, Giele, Kunszt, Melnikov, Zanderighi, ...]
  - Can write any one-loop amplitude as linear combination of a [small] set of master integrals: $\mathcal{A}_n^{(1)} = \sum_j c_j I_j + \mathcal{R}$.
  - The c_j are rather involved functions of external momenta and helicities. Rational terms $\mathcal{R}$ have to be considered.
  - In practice the c_j are determined numerically for each phase space point. Several evaluations are needed for a certain precision, which takes time!

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However, I was missing the following: An alternative, fully numerical method to evaluate the virtual piece. Based on ideas which were first investigated by [Nagy & Soper et al.] we would like to explore this alternative.

Introduction - Fully numerical method to solve $\mathcal{A}_n^{(1)}$

Fully numerical MC solution

- The error of a MC does not depend on the dimensionality of the integration region and grows just about as $1/\sqrt{N}$, with N the number of integrand evaluations.
- Advantage: The $(3n-4)$-dimensional phase-space integral and the 4-dimensional loop integral can be performed together in one single MC evaluation at $(\{p_1, p_2, ..., p_n\}, k)$, where the p_j are the external momenta and k the loop momentum. No need to evaluate the inner loop integral separately per phase-space point! No extra cost!
- However: A fully numerical [MC] integration has to be performed in $D=4$. Instabilities in the integrand due to infrared (IR) and ultraviolet (UV) divergences have to be taken care of first.

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Numerical roadmap to $\mathcal{A}_n^{(1)}$

[Becker, Reuschle, Weinzierl - JHEP 1012:013,2010]

1) For a numerically stable loop integration of $\mathcal{A}_n^{(1)}$ in $D = 4$ we need to subtract the IR and UV divergences first.

$\Rightarrow$ Extend the subtraction method to the virtual part of the NLO calculation! — Slides 5, 6; Backup Slides

$\Rightarrow$ Devise [local] virtual subtraction terms! — Slides (8), 9 - 12

2) The construction of these subtraction terms depends on a fixed cyclic ordering of the external legs.

$\Rightarrow$ Work with color decomposition and color ordered Feynman rules! — Slides (6), 7

$\Rightarrow$ Use partial amplitudes rather than a pure Feynman diagrammatic approach! — Slides (6), 7

This reduces the complexity down to about exponential growth with the number of external legs.

3) The loop integrand [as well as the total UV subtraction term] may be constructed recursively.

$\Rightarrow$ Use Berends-Giele type recursion relations on color ordered one-loop off-shell currents! — Slides 13, (14, 15)

4) Some of the loop propagators still go on-shell for certain values of the loop momentum.

$\Rightarrow$ Find a suitable and numerically stable deformation of the integration contour into the complex plane!

Won't go into detail here. — Backup Slides

Final destination: A scheme in order to combine subtraction terms and contour deformation into one compatible method!

Subtraction method I

Observables to LO and NLO

... in a rather condensed notation

$$\langle O\rangle^{LO}+\langle O\rangle^{NLO}=\int_n O_n d\sigma^B+\int_{n+1} O_{n+1}d\sigma^R+\int_{n[+loop]} O_n d\sigma^V+\int_n O_n d\sigma^C$$

- $d\sigma^B$: Born level; $d\sigma^R$: Real emission; $d\sigma^V$: Virtual contribution; $d\sigma^C$: Initial state collinear subtraction term.
- Each of the NLO terms is separately divergent and only their sum is finite.
- However, for a numerical integration each term needs to be finite.
- Introduce additional terms to subtract the divergencies.
  - Real emission: Subtraction of suitable [dipole] terms $d\sigma^A$. Already known.
  - Virtual contribution: Subtract suitably chosen virtual subtraction terms $d\sigma^{A'}$ at the loop integrand level.
  - Enables a fully numerical loop integration!

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Wish list for $d\sigma^{A'}$

$$\langle O\rangle^{NLO}=\int_{n+1}\left(O_{n+1}d\sigma^R-O_n d\sigma^A\right)+\int_{n+loop}\left(O_n d\sigma^V-O_n d\sigma^{A'}\right)+\int_n\left(O_n\int_{loop} d\sigma^{A'}+O_n d\sigma^C+O_n\int_1 d\sigma^A\right)$$

- Introduce additional terms $d\sigma^{A'}$ to render $\left(d\sigma^V-d\sigma^{A'}\right)$ finite at the loop level.
- Integration of $d\sigma^{A'}$ yields simple analytic results, which cancel the poles of $d\sigma^C+\int_1 d\sigma^A$.
- The terms in brackets are finite:
  - The subtracted real term and the subtracted virtual term(!) are finite and can be integrated numerically!
  - The finite remaining term exhibits a simple analytical structure!

Subtraction method II

On the amplitude level $d\sigma^V \propto 2Re(\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)})d\phi_n$

- $\mathcal{A}_n^{(1)} = \mathcal{A}_{bare}^{(1)} + \mathcal{A}_{CT}^{(1)} = (\mathcal{A}_{bare}^{(1)} - \mathcal{A}_{UV}^{(1)} - \mathcal{A}_{IR,soft}^{(1)} - \mathcal{A}_{IR,coll}^{(1)}) + (\mathcal{A}_{CT}^{(1)} + \mathcal{A}_{UV}^{(1)} + \mathcal{A}_{IR,soft}^{(1)} + \mathcal{A}_{IR,coll}^{(1)})$.
- $\mathcal{A}_n^{(1)}$ is the finite renormalized amplitude. All IR and UV singularities are contained in the bare amplitude $\mathcal{A}_{bare}^{(1)}$.
- Define integrands $\mathcal{G}_x^{(1)}$ inside the amplitudes via $\mathcal{A}_x^{(1)} \equiv \int \frac{d^D k}{(2\pi)^D} \mathcal{G}_x^{(1)}$, $x = bare, uv, soft, coll$.
- They match exactly the singular behavior of $\mathcal{G}_{bare}^{(1)}$ in the divergent points of the integration region and are easily integrable analytically.
- $\mathcal{A}_n^{(1)} = \left(\int \{ \mathcal{G}_{bare}^{(1)} - \mathcal{G}_{UV}^{(1)} - \mathcal{G}_{IR,soft}^{(1)} - \mathcal{G}_{IR,coll}^{(1)} \} \right) + (\mathcal{A}_{CT}^{(1)} + \mathcal{A}_{UV}^{(1)} + \mathcal{A}_{IR,soft}^{(1)} + \mathcal{A}_{IR,coll}^{(1)})$.

The first bracket is finite and can be integrated numerically.
In the second bracket the UV subtraction term cancels [analytically] against the UV counterterm from renormalization, whereas the soft and collinear subtraction terms cancel [analytically] against the dipole contributions from real radiation.

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Cyclic ordering

Our virtual subtraction terms [as well as the contour deformation] depend on a fixed cyclic ordering of the external legs in the amplitude!

- Work with color ordered partial amplitudes. [Next slide]
- Define [partial] integrands $G_x^{(1)}$ inside the partial amplitudes via $A_x^{(1)} \equiv \int \frac{d^Dk}{(2\pi)^D} G_x^{(1)}$, $x = bare, uv, soft, coll$, henceforth be denoted as "subtraction terms". [Note non-caligraphic letters]
- $G_{bare}^{(1)}$ and $G_{uv}^{(1)}$ will be constructed recursively.
- $G_{soft}^{(1)}$ and $G_{coll}^{(1)}$ will be formulated directly on the amplitude level.

Intermezzo: Color decomposition

Factorizing color information and kinematic information yields [simpler] color stripped amplitudes with a fixed cyclic ordering of the external legs. Gaining color information is simply a combinatorial issue.

Example: N-gluon amplitude in the color-flow decomposition

... $g^2 = 4\pi\alpha_s$

$$\mathcal{A}_n^{(1)}(1,\ldots,n) = \left(\frac{g}{\sqrt{2}}\right)^{n-2}\Bigg[\sum_{\sigma\in S_n/Z_n} N_c\,\delta_{i_{\sigma_1} j_{\sigma_2}}\ldots\delta_{i_{\sigma_n} j_{\sigma_1}}\,A_{n,0}^{(1)}(g_{\sigma_1},\ldots,g_{\sigma_n})$$

$$+\sum_{\substack{\sigma\in S_n/(Z_m\times Z_{n-m})\\ \text{For all partitions } m>0}} \delta_{i_{\sigma_1} j_{\sigma_2}}\ldots\delta_{i_{\sigma_m} j_{\sigma_1}}\,\delta_{i_{\sigma_{m+1}} j_{\sigma_{m+2}}}\ldots\delta_{i_{\sigma_n} j_{\sigma_{m+1}}}\,A_{n,m}^{(1)}(g_{\sigma_1},\ldots,g_{\sigma_m};g_{\sigma_{m+1}},\ldots,g_{\sigma_n})\Bigg]$$

- Description of $\mathcal{A}_n^{(1)}$ in terms of color ordered one-loop partial amplitudes $A_{n,m}^{(1)}$. No color information to be considered.
- Subleading [in color] one-loop partial amplitudes $A_{n,m\neq 0}^{(1)}$ can be related to leading one-loop partial amplitudes $A_{n,0}^{(1)}$.
- Similar decompositions available for processes with m quark-pairs and n gluons. One has to classify all possible color structures in the permutation sum.

Properties

- Color ordering: Representation of all graphs with the same number and the same fixed cyclic ordering of the external legs by a partial amplitude. This ensures a fixed [fully ordered] structure of the loop propagators.
- Particle content: One-loop partial amplitudes may be further decomposed into primitive amplitudes, classified by the quark or gluon content in the loop: $A_{n,0}^{(1)} = A_{n,0,lc}^{(1)} + \frac{n_f}{N_c} A_{n,0,nf}^{(1)}$.

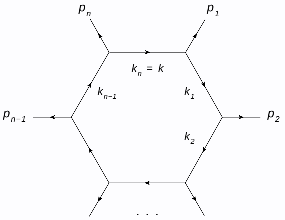
  $\rightarrow$In amplitudes with mixed quark/gluon content in the loop the routing of the fermion lines through the loop matters.
- Partial/primitive amplitudes are gauge invariant, which is important for our method, and obey momentum conservation.
- [Mangano & Parke, Maltoni et al., Weinzierl, Bern, Dixon, Kosower, ...]

Intermezzo: One-loop integration

In order to construct [local] virtual subtraction terms we have to understand the singular structure of our one-loop integrand.

Using partial amplitudes ensures a fixed sequential propagator structure, with only n different loop propagators to consider. This enables a meaningful classification of the divergent regions in regard to automatization.

Typical one-loop diagram



$$\int \frac{d^4k}{(2\pi)^4}\, P_a(k) \prod_{i=1}^{n} \frac{1}{k_i^2 - m_i^2 + i\delta},\ k_i \equiv k - q_i.$$

$P_a(k)$ a polynomial of degree a in k and $q_i \equiv p_1 + \ldots + p_i$.

Divergent regions

... with m_i the masses in the loop and p_i the external momenta

Soft infrared divergencies for $k \sim q_i$, if $p_i^2 = m_{i-1}^2$, $m_i = 0$, $p_{i+1}^2 = m_{i+1}^2$.
Massless particle exchanged between two on-shell particles & $k_i \to 0$

$$k \to q_i \Rightarrow \begin{cases} k_{i-1}^2 - m_{i-1}^2 & \to & p_i^2 - m_{i-1}^2 & = & 0 \\ k_i^2 - m_i^2 & \to & 0 - m_i^2 & = & 0 \\ k_{i+1}^2 - m_{i+1}^2 & \to & p_{i+1}^2 - m_{i+1}^2 & = & 0 \end{cases}$$

Collinear infrared divergencies for $k \sim q_i - xp_i$, if $p_i^2 = 0$, $m_{i-1} = 0$, $m_i = 0$.
$x \in [0,1]$ Massless external on-shell particle attached to two massless propagators & $k_i || p_i$

$$k \to q_i - xp_i \Rightarrow \begin{cases} k_{i-1}^2 - m_{i-1}^2 & \to & (1-x)^2 p_i^2 - m_{i-1}^2 & = & 0 \\ k_i^2 - m_i^2 & \to & x^2 p_i^2 - m_i^2 & = & 0 \end{cases}$$

Ultraviolet divergencies for $k \to \infty$, if $4 + a - 2n \geq 0$.

Regularization

- In dimensional regularization we use a D-dimensional integral, with $D = 4 - 2\varepsilon$ and $|\varepsilon| \ll 1$. The result is analytically known and can be expanded around ε, which yields terms $\propto 1/\varepsilon$ and $\propto 1/\varepsilon^2$.
- For integrals with large n the traditional analytic calculation is cumbersome. We choose a numerical method which, however, has to be applied in $D = 4$.

Subtraction terms - IR subtraction I

As we saw the infrared divergences are related to soft and collinear partons in the loop. An amplitude with soft or collinear divergences must have at least one gluon line in the loop.

The soft and collinear contributions to a given one-loop partial amplitude are the soft and collinear contributions to all associated one-loop graphs G with the same number and fixed cyclic ordering of external legs.

Consider $m_i = 0$ and let k_i^2 be a propagtor related to a gluon in the loop of graph G. The single one-loop graphs G are not to be confused with the [total] subtraction terms $G_{uv,coll,soft}$.

Soft subtraction functions $\sim$ soft source terms

There is a soft singularity when the loop momentum k approximates q_i, or in other words when $k_i \to 0$. We define a soft subtraction function [Nagy & Soper, ...]:

$$S_i^{soft}(G, p_1, \ldots, p_n) \quad = \quad \frac{\lim_{k \to q_i} \left\{ k_{i-1}^2 k_i^2 k_{i+1}^2 G(k, p_1, \ldots, p_n) \right\}}{k_{i-1}^2 k_i^2 k_{i+1}^2}$$

How to: 1) "Multiplying out" the dangerous propagators. 2) Taking the soft limit. 3) "Dividing" the propper structure "back in".

Collinear subtraction functions $\sim$ collinear source terms

There is a collinear singularity when the loop momentum k approximates $q_i - xp_i$, w/ $x \in [0,1]$, or in other words $k_i || p_i$. We define a collinear subtraction function [Nagy & Soper, ...]:

$$S_i^{coll}(G, p_1, \ldots, p_n) \quad = \quad \frac{\lim_{k \to q_i - xp_i} \left\{ k_{i-1}^2 k_i^2 G(k, p_1, \ldots, p_n) \right\}}{k_{i-1}^2 k_i^2} g_{UV}(k_{j-1}^2, k_j^2)$$

- Introduce a factor $g_{UV}(k_{j-1}^2, k_j^2)$ to avoid possible UV divergences in the collinear subtraction term.
- $g_{UV}(k_{j-1}^2, k_j^2) = 1$ in the collinear region and suppresses our collinear term with additional $\sim \mathcal{O}(1/k)$ in the UV limit.
- For $x = 0, 1$ the collinear singularity runs into the soft singularity. To avoid double counting we have to subtract these singularities in a suitable way.

Subtraction terms - IR subtraction II

As we saw on the previous slide, each subtraction term $S_i^{soft}(G, p_1, \ldots, p_n)$ or $S_i^{coll}(G, p_1, \ldots, p_n)$ depends on a single one-loop graph G. [Not to be confused with the subtraction terms $G_{uv,coll,soft}$]

Now, in simple terms: Summing up the subtraction terms for all graphs with a gluon at position i in the loop, further summing over all conceivable positions i in the loop and using gauge invariance to simplify the results in the end yields simple total subtraction terms in local form, proportional to the Born level.

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Total [local] IR subtraction terms

New!

$$G_{soft}^{(1)} = 4i \sum_{j\in I_g} \frac{p_j p_{j+1}}{k_{j-1}^2 k_j^2 k_{j+1}^2} A_j^{(0)} \quad \text{and} \quad G_{coll}^{(1)} = -2i \sum_{j\in I_g} \left(\frac{S_j g_{UV}(k_{j-1}^2,k_j^2)}{k_{j-1}^2 k_j^2} + \frac{S_{j+1} g_{UV}(k_j^2,k_{j+1}^2)}{k_j^2 k_{j+1}^2} \right) A_j^{(0)}.$$

I_g is the set of gluons in the loop and the propagator corresponding to $j \in I_g$ in the loop belongs to a gluon.

$S_j = 1$ if the outgoing line j corresponds to a quark, $S_j = 1/2$ if it corresponds to a gluon.

- Formulated directly on the amplitude level! Match the soft and collinear limit of the amplitude on integrand level!
- Simple and fast! Ideal for numerical implementation!
- Yield simple analytical results upon integration! Proportional to the Born level amplitudes!
  [Assadsolimani, Becker, Weinzierl - Phys.Rev.D81:094002,2010]
  [Assadsolimani, Becker, Reuschle, Weinzierl - Nucl.Phys.Proc.Suppl.205-206:224-229,2010]
  [Becker, Reuschle, Weinzierl - JHEP 1012:013,2010]

$$S_\varepsilon^{-1}\mu_S^{2\varepsilon}\int \frac{d^Dk}{(2\pi)^D} G_{soft}^{(1)} = \frac{-1}{(4\pi)^2}\frac{\exp(\varepsilon\gamma_E)}{\Gamma(1-\varepsilon)} \sum_{j\in I_g} \frac{2}{\varepsilon^2}\left(\frac{-2p_jp_{j+1}}{\mu_S^2}\right)^{-\varepsilon} A_j^{(0)} \;\; \& \;\; S_\varepsilon^{-1}\mu_S^{2\varepsilon}\int \frac{d^Dk}{(2\pi)^D} G_{coll}^{(1)} = \frac{-1}{(4\pi)^2}\frac{\exp(\varepsilon\gamma_E)}{\Gamma(1-\varepsilon)} \sum_{j\in I_g} \frac{2}{\varepsilon}(S_j+S_{j+1})\left(\frac{\mu_{UV}^2}{\mu_S^2}\right)^{-\varepsilon} A_j^{(0)}$$

$S_\varepsilon \equiv (4\pi)^\varepsilon \exp(-\varepsilon\gamma_E)$ the typical volume factor in dimensional regularization.

Remark: These are the subtraction terms for the case $m_i = 0$. The IR subtraction terms for the massive case $m_i \neq 0$ have also been derived and are only slightly more involved.

Subtraction terms - UV subtraction I

Consider again, for $m_i = 0$, our one-loop integrand $G(k,n) \equiv P_a(k) \prod_{j=1}^{n} \frac{1}{k_j^2}$, with $k_j = k - \sum_i p_i$ and $P_a(k)$ is again a polynomial of degree a in k.

It can be shown that the UV divergent diagrams are only those which contribute to propagator or vertex corrections.

So, G contains all one-loop diagrams that contribute to a given correction. Still, refer to it representatively as "Graph".

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We want a term that mimics the [exact] local UV behavior

- Expand around the inverse loop propagator in the UV limit, more precisely around $(\bar{k}^2 - \mu_{UV}^2)^{-1}$, where $\bar{k} \equiv k - Q$.
- $G(k,n) \approx \frac{P_a(\bar{k})}{(\bar{k}^2-\mu_{UV}^2)^n}\left(1 + \sum_{m=1}^{\ell} \frac{X_m(\bar{k})}{(\bar{k}^2-\mu_{UV}^2)^m}\right) \equiv G(\bar{k},n,\ell)$, with $X_m(\bar{k})$ polynomial of order m in $\bar{k}$. ... μ_{UV} and Q see below
- The cut on ℓ depends on the degree of divergence of the graph G: $\ell = 0$ logarithmic; $\ell = 1$ linear; $\ell = 2$ quadratic. E.g. for the gluon self energy use $\ell = 2$, since quadratically divergent. Ultimately: Count only UV divergent powers of $\bar{k}$.
- Small example: $\frac{1}{(k-p)^2-m^2} = \frac{1}{\bar{k}^2-\mu_{UV}^2}\left\{1 + \frac{2\bar{k}.(p-Q)}{\bar{k}^2-\mu_{UV}^2} - \frac{(p-Q)^2-m^2+\mu_{UV}^2}{\bar{k}^2-\mu_{UV}^2} + \frac{(2\bar{k}.(p-Q))^2}{(\bar{k}^2-\mu_{UV}^2)^2}\right\} + \mathcal{O}(1/|\bar{k}|^5)$

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Upon integration

... omit a factor $\mu_S^{2\varepsilon}(2\pi)^{-D}$ in the integration measure

- $\int d^Dk\, G(\bar{k},n,\ell) = C\left(\frac{1}{\varepsilon} - \log\left(\frac{\mu_{uv}^2}{\mu_S^2}\right)\right)A_n^{(0)} + R$, with C a constant factor of proportionality and R a finite [rational] term.
- Re-define $G_{uv}(\bar{k},n,\ell) = G(\bar{k},n,\ell) - \frac{-2\mu_{uv}^2}{(\bar{k}^2-\mu_{uv}^2)^3}R$, to absorb the finite term.
- Then: $\int d^Dk G_{uv}(\bar{k},n,\ell) = C\left(\frac{1}{\varepsilon} - \log\left(\frac{\mu_{uv}^2}{\mu_S^2}\right)\right)A_n^{(0)} \propto$ "common pole part" $\times$ "Born amplitude", exactly as we want it!

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Use μ_{uv}^2 and Q to control the quality of the contour deformation in the UV region

- $\mathrm{Re}(\mu_{uv}^2) = 0$ and $\mathrm{Im}(\mu_{uv}^2) < 0$ ensures that the integration contour in the UV region never approaches the singular surface defined by $(\bar{k}^2 - \mu_{uv}^2) = 0$.
- The integrated UV subtraction terms are independent of the four-vector Q. We can choose Q to our will in order to enhance the numerical stability of the loop integrand upon contour deformation.
- The results of our calculation [of the one-loop amplitude] are in the end independent of μ_{uv}^2 and Q!

Subtraction terms - UV subtraction II

Total [local] UV subtraction terms

New!

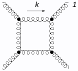
- Only propagator and vertex corrections are UV divergent, quadratically at maximum!
- We can derive a relatively small set of simple local subtraction terms!
- Simple analytic result upon integration, proporational to a common pole part and the respective Born level term!
- These subtraction terms can be used as [local] "counter" terms to recursively construct the total UV subtraction term! The total UV subtraction term will also be proportional to the common pole part and the Born level amplitude!

  [Assadsolimani, Becker, Reuschle, Weinzierl - Nucl.Phys.Proc.Suppl.205-206:224-229,2010]

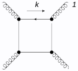
  [Becker, Reuschle, Weinzierl - JHEP 1012:013,2010]

Example: 4-gluon vertex

... omit a factor $\mu_S^{2\varepsilon}(2\pi)^{-D}$ in the integration measure



$\oplus$ (closed ghost loops) $\oplus$ (pinched diagrams) $\oplus$



(Use color ordered Feynman rules!)

$$G^{(1)\mu\nu\lambda\kappa}_{uv,gggg,lc} = \left[\frac{32(1-\varepsilon)\bar{k}^\mu \bar{k}^\nu \bar{k}^\lambda \bar{k}^\kappa}{(\bar{k}^2-\mu_{UV}^2)^4} + \frac{-8(1-\varepsilon)W(\bar{k})^{\mu\nu\lambda\kappa} - \frac{4}{3}\mu_{UV}^2 V4^{\mu\nu\lambda\kappa}}{(\bar{k}^2-\mu_{UV}^2)^3} + \frac{2(1-\varepsilon)(\eta^{\mu\nu}\eta^{\lambda\kappa}+\eta^{\mu\kappa}\eta^{\nu\lambda})}{(\bar{k}^2-\mu_{UV}^2)^2} \right],$$

$$G^{(1)\mu\nu\lambda\kappa}_{uv,gggg,nf} = \left[\frac{-32\bar{k}^\mu \bar{k}^\nu \bar{k}^\lambda \bar{k}^\kappa}{(\bar{k}^2-\mu_{UV}^2)^4} + \frac{8W(\bar{k})^{\mu\nu\lambda\kappa}}{(\bar{k}^2-\mu_{UV}^2)^3} + \frac{4(V4^{\mu\nu\lambda\kappa}-\eta^{\mu\lambda}\eta^{\nu\kappa})}{(\bar{k}^2-\mu_{UV}^2)^2} \right],$$

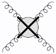
where $W(\bar{k})^{\mu\nu\lambda\kappa} \equiv \eta^{\mu\nu}\bar{k}^\lambda\bar{k}^\kappa + \eta^{\mu\kappa}\bar{k}^\nu\bar{k}^\lambda + \eta^{\nu\lambda}\bar{k}^\mu\bar{k}^\kappa + \eta^{\lambda\kappa}\bar{k}^\mu\bar{k}^\nu$

$$\int d^Dk\, G^{(1)\mu\nu\lambda\kappa}_{uv,gggg,lc} = \frac{i}{(4\pi)^2}\left(\frac{2}{3}V_4^{\mu\nu\kappa\lambda}\left(\frac{1}{\varepsilon} - \log\left(\frac{\mu_{uv}^2}{\mu_S^2}\right)\right)\right) \quad \text{and} \quad \int d^Dk\, G^{(1)\mu\nu\lambda\kappa}_{uv,gggg,nf} = \frac{i}{(4\pi)^2}\left(\frac{4}{3}V_4^{\mu\nu\kappa\lambda}\left(\frac{1}{\varepsilon} - \log\left(\frac{\mu_{uv}^2}{\mu_S^2}\right)\right)\right)$$

Check against renormalized 4-gluon vertex $\propto 1 - \frac{\alpha_S}{4\pi}\frac{N_C}{2}\left(\frac{2}{3} + \frac{4}{3}\frac{n_f}{N_C}\right)\frac{1}{\varepsilon_{UV}}$

$V_4^{\mu\nu\kappa\lambda}$ = color ordered 4-gluon-vertex

Local counter terms (lc $\oplus$ nf):



$\hat{=}\ G^{(1)}_{uv,gggg,lc} \oplus G^{(1)}_{uv,gggg,nf}$

Intermezzo: Recursive methods

- Utilizing Berends-Giele type recursion relations, based on color ordered [one-loop] off-shell currents [Berends & Giele, v. Hameren, ...], to construct the [total] bare one-loop integrand $G^{(1)}_{bare}$ and the [total] UV subtraction term $G^{(1)}_{UV}$!

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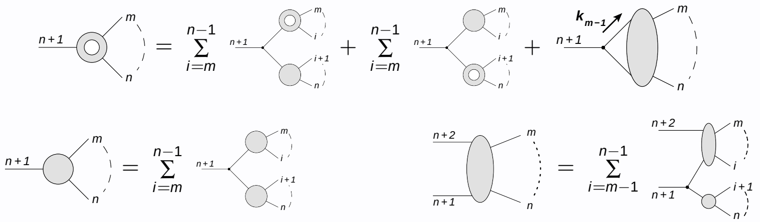
Example: One-loop off-shell gluon current in a 3-valent toy-model



- The recursive construction of $G^{(1)}_{bare}$ ensures the correct incorporation of all necessary one-loop diagrams to a given partial/primitve amplitude.

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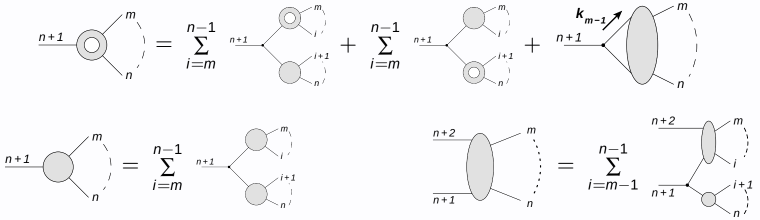

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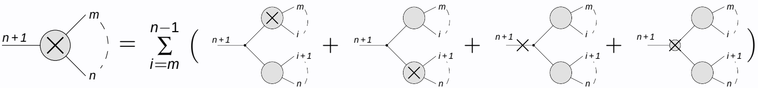
Example: Total UV subtraction term to the one-loop off-shell gluon current in this toy-model



- The recursive construction of $G^{(1)}_{uv}$ ensures the correct incorporation of all necessary UV counterterms to a given partial/primitve amplitude.

Implemented the recursive algorithms for the full theory [so far at leading color] in several C++ libraries! In spinor formalism!

UV subtraction - Consistency check I

Check whether the implemented recursive constructions of the one-loop integrand and the UV subtraction term play along well for large values of the loop momentum.

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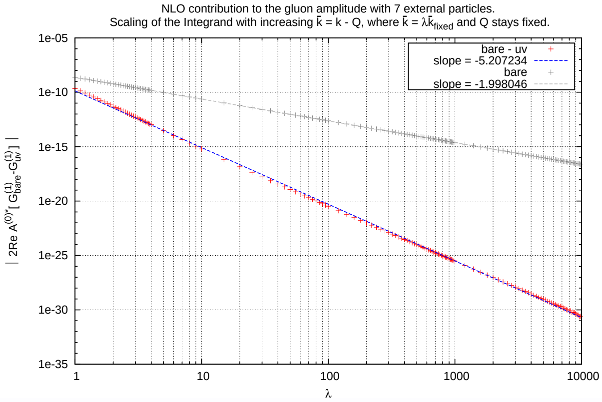
In the plot we show $|2Re(A^{(0)*}G^{(1)})|$ vs. a UV scaling parameter λ for an n-gluon amplitude with $n = 7$:
[preliminary, S. Becker, CR]



Summing over all helicities!

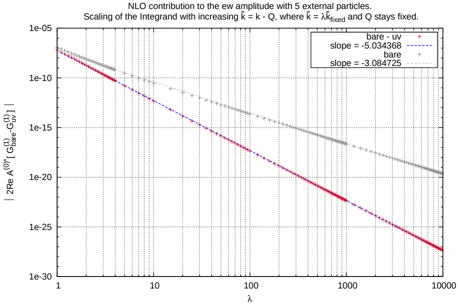
The unsubtracted integrand grows like $\int \frac{d^4k}{k^2}$: Quadr. UV divergent!

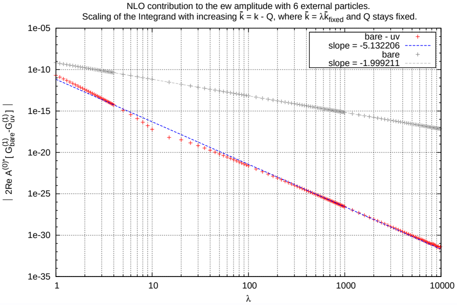
The UV subtracted integrand grows like $\int \frac{d^4k}{k^5}$: UV finite!

In gray we see the unsubtracted [total] integrand, which is obviously quadratically UV divergent. In red we see the fully UV subtracted [total] integrand, which shows clearly a finite behavior!

UV subtraction - Consistency check II

In these plots we show $|2Re(A^{(0)*}G^{(1)})|$ vs. a UV scaling parameter λ for $e^+e^- \to$ 3/4 *jets* [i.e. $q\bar{q}$ + 1/2 g's]:
[preliminary, S. Becker, CR]





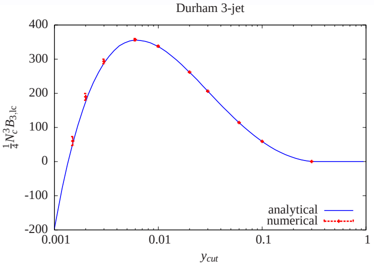
In gray we see again the unsubtracted [total] integrand, which is obviously UV divergent. In red we see the fully UV subtracted [total] integrand, which shows again clearly a finite behavior!

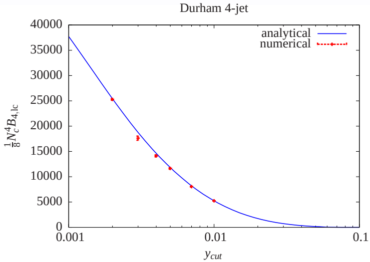
Something to learn: We notice that in the unsubtracted 3 jet event there is only a total linear divergence $\sim d^4k/k^3$ in contrast to the total quadratic divergence $\sim d^4k/k^2$ in the unsubtracted 4 jet event.
Due to the fact that in the 4 jet event the gluon propagator, and hence the gluon self energy, appears off-shell for the first time.

Proof of principle - Computing jet rates in $e^+e^- \to jets$

Use the whole method in a calculation of jet rates in e^+e^-, together with a simple phase space generating algorithm [RAMBO] and the Durham jet algorithm.

The plots show $e^+e^- \to$ 3 and 4 jets at leading color: $1/8N_c^{3/4}B_{3/4,lc}$ vs. y_{cut}, where $B_{3/4,lc}$ is the NLO coefficient in the perturbative expansion for the 3/4-jet rate and y_{cut} the jet resolution parameter. [S. Weinzierl, preliminary]





Blue shows the results from previous analytic calculations. Red shows the numerical results of our method. Good agreement!

Brief reminder on jet rates

- The production rate for n-jet events, or short the 'n-jet rate', is given by the ratio of the cross section for n-jet events divided by the total hadronic cross section.
- In e^+e^- annihilation:

$$R_n = \frac{\sigma_{n-jet}}{\sigma_{tot}} = \left(\frac{\alpha_s}{2\pi}\right)^{n-2}\bar{A}_n + \left(\frac{\alpha_s}{2\pi}\right)^{n-1}\bar{B}_n + O(\alpha_s^n)$$

- In practice we calculate:

$$\frac{\sigma_{n-jet}}{\sigma_0} = \left(\frac{\alpha_s}{2\pi}\right)^{n-2}A_n + \left(\frac{\alpha_s}{2\pi}\right)^{n-1}B_n + O(\alpha_s^n)$$

with σ_0 the leading order cross section for $e^+e^- \to hadrons$.
The relation between $\bar{A}_n$ ($\bar{B}_n$) and A_n (B_n) can be determined from the perturbative expansion of the total hadronic cross section.

Summary and Outlook

There is need for fast high accuracy tools in LHC physics!

- We present a fully numerical algorithm to compute NLO QCD amplitudes with many legs in the final state for fixed order of α_s in perturbation theory! [Becker, Reuschle, Weinzierl - JHEP 1012:013,2010]

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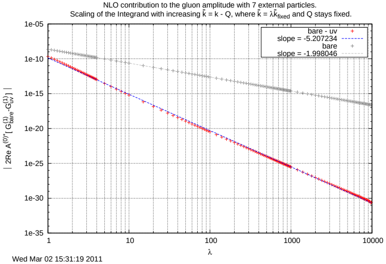

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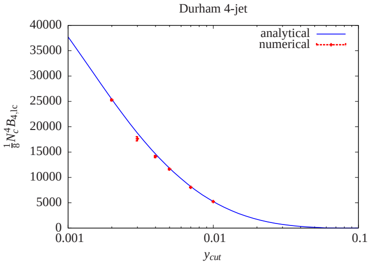

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In essence

- Our subtraction terms are simple and fast!
- They are tailored such that they go hand in hand with the contour deformation and the real emission!
- Proof of principle: Possible to reproduce the results for $e^+e^- \to$ up to 4 jets with a purely numerical approach! 5 and 6 jets is in preparation.
- So far we computed our results for leading color! Extension to full color is work in progress.
- The next bigger step would be to apply our method to Z production plus jets at the LHC.

Thank you for your attention!

Backup: Pole structure / Insertion operators L and I

After integration, the soft and collinear poles of a primitive one-loop amplitude with massless partons are given by [Becher & Neubert, Magnea]:

$$S_\varepsilon^{-1}\mu_S^{2\varepsilon}A_{bare}^{(1)} = \frac{\alpha_S}{4\pi}\frac{\exp(\varepsilon\gamma_E)}{\Gamma(1-\varepsilon)}\sum_{i\in I_g}\left[\frac{2}{\varepsilon^2}\left(\frac{-2p_i\cdot p_{i+1}}{\mu_S^2}\right)^{-\varepsilon}+\frac{2}{\varepsilon}(S_i+S_{i+1})\right]A_i^{(0)},$$

where $S_\varepsilon = (4\pi)^\varepsilon\exp(-\varepsilon\gamma_E)$ the typical volume factor in dimensional regularization. The index i in S_i refers to the external particle: $S_q = S_{\bar{q}} = 1$, $S_g = 1/2$. This is exactly the pole structure also reproduced by our IR subtraction terms. A more familiar [and also involving the UV poles] formulation of this may be:

$$S_\varepsilon^{-1}\mu_S^{2\varepsilon}A_{bare}^{(1)} = \frac{\alpha_S}{4\pi}\frac{\exp(\varepsilon\gamma_E)}{\Gamma(1-\varepsilon)}\left[\frac{(n-2)}{2}\frac{\beta_0}{\varepsilon}+\sum_i\sum_{j\neq i}\mathbf{T}_i\mathbf{T}_j\left(\frac{1}{\varepsilon^2}+\frac{\gamma_i}{\mathbf{T}_i^2}\frac{1}{\varepsilon}\right)\left(\frac{-2p_i\cdot p_j}{\mu_S^2}\right)^{-\varepsilon}\right]A^{(0)},$$

where the first part in the squared brackets is exactly the negative of the counterterm contribution $S_\varepsilon^{-1}\mu_S^{2\varepsilon}A_{CT}^{(1)}$ from UV renormalization.

The sum of the collinear subtraction part for the intial state plus the one-particle phase-space integration over the real subtraction part can be written as:

$$d\sigma^C + \int_1 d\sigma^A = \mathbf{I}\otimes d\sigma^B + \mathbf{K}\otimes d\sigma^B + \mathbf{P}\otimes d\sigma^B,$$

where color correlations still remain. The insertion operators **K** and **P** pose no problem for the numerical evaluation. The term $\mathbf{I}\otimes d\sigma^B$ has the appropriate pole structure to cancel the IR divergences coming from the loop. Hence, $d\sigma^V + \mathbf{I}\otimes d\sigma^B$ is IR finite.

Remember now that $d\sigma^V \propto 2Re(\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)})d\phi_n$, where further $\mathcal{A}_n^{(1)} = \mathcal{A}_{bare}^{(1)} + \mathcal{A}_{CT}^{(1)}$. We make $\mathcal{A}_{bare}^{(1)}$ finite by introducing our subtraction terms locally in the "first bracket" and are in the "second bracket" left with an analytical structure of the form $\mathcal{A}_{CT}^{(1)} + \mathcal{A}_{soft}^{(1)} + \mathcal{A}_{coll}^{(1)} + \mathcal{A}_{UV}^{(1)}$. This structure defines us a new insertion operator **L** via:

$$2Re(\mathcal{A}_n^{(0)*}\mathcal{A}_{CT}^{(1)} + \mathcal{A}_{soft}^{(1)} + \mathcal{A}_{coll}^{(1)} + \mathcal{A}_{UV}^{(1)})d\phi_n = \mathbf{L}\otimes d\sigma^B.$$

The insertion operator **L** contains the explicit poles in the dimensional regularization parameter related to the IR singularities of the one-loop amplitude. These poles cancel when combined with the insertion operator **I**:

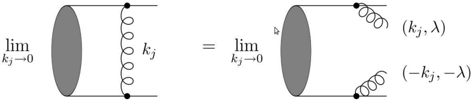
$$(\mathbf{I}+\mathbf{L})\otimes d\sigma^B = \text{finite}.$$

Soft subtraction term

When gluon i is soft the corresponding propagator goes on-shell and we may replace:

$$\frac{-ig^{\mu\nu}}{k_j^2} \to \frac{i}{k_j^2}\left(d^{\mu\nu}(k_j^b, n) - 2\,\frac{k_j^{b\mu} n^\nu + n^\mu k_j^{b\nu}}{2k_j^b.n}\right),$$

with k_j^b the on-shell limit of k_j, n a light-like reference vector and $d^{\mu\nu}$ the sum over physical polarizations. Adding self-energy diagrams will not change the soft limit. With this inclusion and a similar replacement as above the contribution from the polarizatin sum makes a partial tree-level amplitude, where two gluons with momenta k_j^b and $-k_j^b$ have been inserted between the legs j and $j+1$:



In the soft limit this tree-level partial amplitude is given by two eikonal factors times the tree-level amplitude without these two additional gluons:

$$\left(\frac{p_j^\mu}{p_j.k_j^b}\right) g^{\mu\nu} \left(\frac{p_{j+1}^\nu}{p_{j+1}.(-k_j^b)}\right) A_j^{(0)}.$$

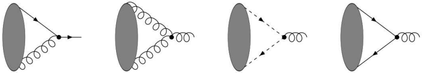
In the soft limit we may replace $2p_j.k_j^b$ by k_{j-1}^2 and $2p_{j+1}.(-k_j^b)$ by k_{j+1}^2, which then leads to the form of our local soft subtraction terms.

The terms with $k_j^{b\mu} n^\nu + n^\mu k_j^{b\nu}$ in our replacement at the beginning vanish for the sum of all diagrams due to gauge invariance.

Backup: Derivation of the IR subtraction terms - More rigorously II

<u>Collinear subtraction term</u>

We have to consider configurations where two adjacent propagators go on-shell with a massless leg in between:

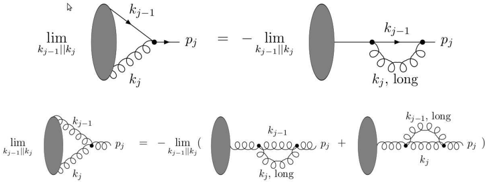


The diagrams where an external gluon splits into two ghosts or a $q\bar{q}$-pair are in the collinear limit not singular enough to yield a divergence after integration. We are left with the $q \to qg$- and the $g \to gg$-splittings.

In $q \to qg$ one can show that only the longitudinal polarization of the gluon contributes to the collinear limit. The same holds for $g \to gg$, here the collinear limit receives contributions when one of the two gluons in the loop carries carries a longitudinal polarization (not both). The external gluon has of course physical transverse polarization.

We use the fact that contraction of a longitudinal polarization into a gauge invariant set of diagrams yields zero. Now the "blobs" of the two cases we just discussed consist almost of a gauge invariant set of diagrams. There is only one missing, where the longitudinal polarized gluon couples directly to the other parton connected to the "blob". This is a self-energy insertion on an external line, by definition absent from the amputated one-loop amplitude.

We turn the argument around and replace the sum of collinear singular diagrams by the negative of the respective self-energy insertions on the external line:



Collinear subtraction term ctd.

As parametrization for the collinear limit we use the same as is usually used in the real emssion case. The singular part of the self-energies is then proportional to:

$$P^{long}_{q\to qg} = -\frac{2}{2k_{j-1}\cdot k_j}\left(-\frac{2}{1-x}+2\right)\not{p},$$

$$P^{long}_{g\to gg} = -\frac{2}{2k_{j-1}\cdot k_j}\left(-\frac{2}{x}-\frac{2}{1-x}+2\right)\left(-g^{\mu\nu}+2\frac{p^\mu n^\nu+n^\mu p^\nu}{2p.n}\right).$$

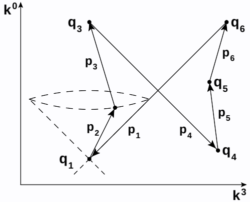
The soft singularities for $x=0,1$ must not be double counted. We therefore just have to consider the terms which are non-singular in the soft limit. These terms are independent of x and lead to the form of our local collinear subtraction terms.

Remark: The self-energy insertions introduce a spurious $1/p_j^2$-singularity. In order to calculate the singular part of the self-eneries we regulate this singularity by putting p_j^2 slightly off-shell, but keeping k_{j-1} and k_j on-shell and imposing momentum conservation.

Backup: Integration contour

Consider $m_i = 0$. The denominator in the loop integral becomes singular for $k_i \equiv (k - q_i)^2 = 0$.

- In a diagram where we plot the 0- and 3-components of the loop momentum, $(k - q_i)^2 = 0$ describes a light cone centered on the point q_i.
- Whenever $(k - q_i)^2 = 0$ holds for k, we need to deform the integration contour away from the light cone and into the complex plane.
- We choose for example $-p_1$ and $-p_4$ as our incoming momenta and in such a way that they have components only in the 0-3-plane.
- All other lines p_i are projections onto this plane. They connect the events at q_i, where $p_{i+1} = q_{i+1} - q_i$.



We deform the integration contour into the complex plane, without changing the value of the integral, by deforming the loop momentum into the complex plane:

$$\frac{1}{\left(k-q_i\right)^2} \xrightarrow{l(k)=k+i\kappa(k)} \frac{1}{\left(l(k)-q_i\right)^2} \xrightarrow{\kappa \text{ “small“}} \frac{1}{\left(k-q_i\right)^2 + 2i\kappa.\left(k-q_i\right)}$$

We have to choose κ such that whenever $(k - q_i)^2 = 0$ we will get $\kappa.(k - q_i) > 0$, where the numerical stability depends strongly on κ.

The deformation is not possible whenever $k = q_i$ or $k = q_i - xp_i$. These ”pinch” singularities are taken care of by the IR subtraction terms.

In practice

First subtract all IR and UV subtraction terms, which yields a UV- and IR-finite integrand. With $P(k)$ and $P_{uv}(k)$ Polynomials in k we have generically:

$$G^{(1)}_{bare} - G^{(1)}_{soft} - G^{(1)}_{coll} - G^{(1)}_{uv} = \frac{P(k)}{\prod_{j=1}^{n}(k_j^2 - m_j^2)} - \frac{P_{uv}(k)}{(\bar{k}^2 - \mu_{uv}^2)^{n_{uv}}}$$

We deform the integration contour whenever one of the propagators $1/(k_j^2 - m_j^2)$ or $1/(\bar{k}^2 - \mu_{uv}^2)$ goes on-shell. At the moment we use an algorithm following the idea of Gong, Nagy and Soper [GNS, 2009].