Amplitudes, Wilson loops and Correlators in $\mathcal{N}$=4 SYM and beyond

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Annual theory meeting

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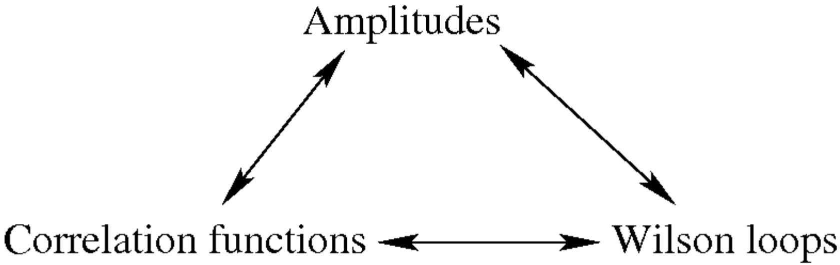
Summary of talk:

- Three objects of interest in $N = 4$ SYM: Amplitudes (S-matrix), Correlation functions of gauge invariant operators, Wilson loops
- Increasing evidence of a remarkable triality between all three objects



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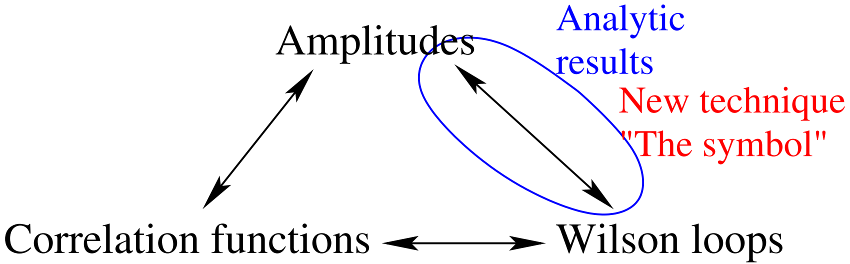

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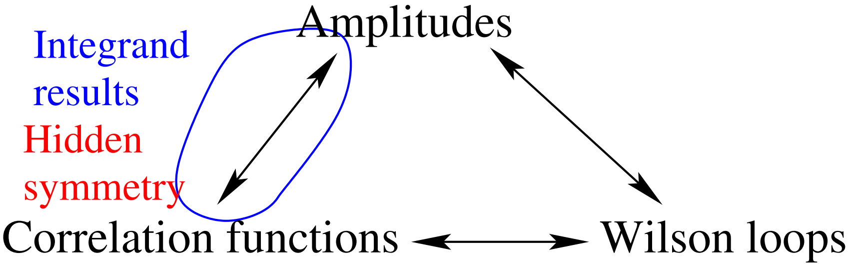
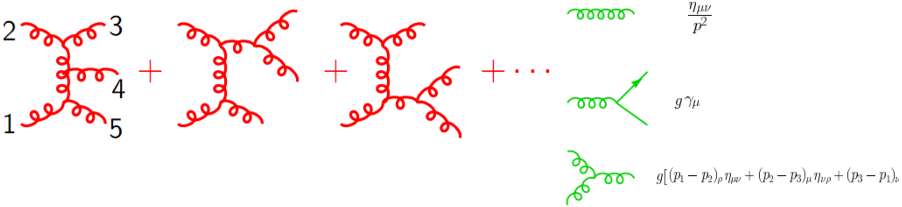


Why amplitudes?

- Problem with quantum field theory: standard approach is too complicated
- eg 5-point tree-level gluon scattering amplitude

Described by following Feynman diagrams:



If you evaluate these using textbook methods you will only discover that this is a very disgusting mess.

5-point tree-level amplitude

(courtesy of Zwi Bern - see also new edition of Zee QFT text book)

Result of a brute force calculation (actually only a small part of it):

$k_1 \cdot k_4 \, \varepsilon_2 \cdot k_1 \, \varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5$

Perhaps this is just the way it is?

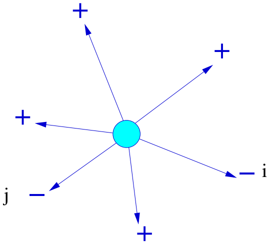
Simplifying scattering amplitudes

(see Zee text book)

- Colour-stripped gluon amplitudes
- spinor helicity
- Gluon amplitudes with all positive (negative) helicities vanish
- Similarly for the '1 negative' ('1 positive') helicity amplitudes
- 2 negative helicities only called 'maximally helicity violating' or MHV amplitudes (simplest)
- (3 negative helicities $\Rightarrow$ NMHV etc.)

MHV amplitudes remarkably simple

- Colour-stripped planar tree-level MHV amplitudes



$$\mathcal{A}_n^{\text{tree}}(i,j) := \frac{\langle i\, j\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}$$

[Parke Taylor (1986)]

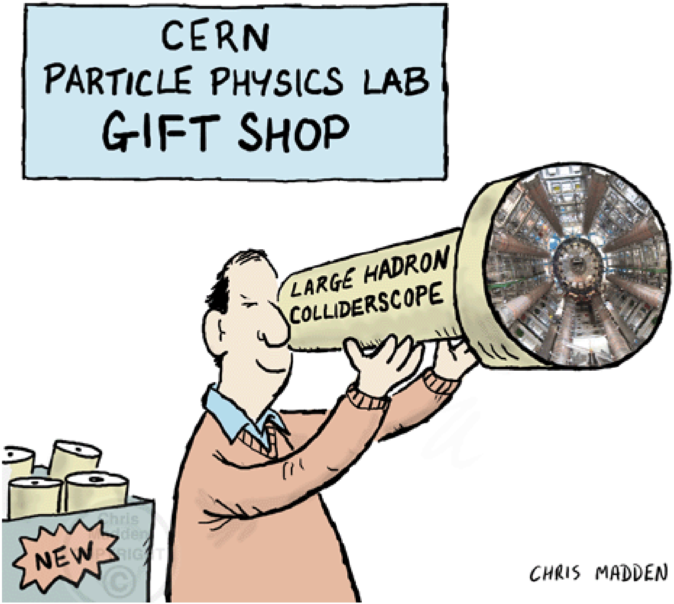
  - 5 gluon amplitude requires 25 diagrams (mess shown earlier) entirely given by this result
  - 10 gluon amplitude requires 10525900 diagrams!

Huge simplification

  - Simplest example of unexpected hidden structure in amplitudes

New way to do QFT

The other motivation for amplitudes...



Motivation for $\mathcal{N} = 4$ SYM. Prototype gauge theory

$SU(N)$ 4d gauge theory, 't Hooft coupling 'a':

- Gauge field
- 6 massless scalar fields (adjoint rep) ϕ_{AB}
- 4 massless fermions (adjoint rep)

- Finite, conformally invariant ("a" a freely tunable parameter)
- Despite apparent complication, amplitudes are the simplest.

The "hydrogen atom" of $d = 4$ quantum field theory.

Starting point in our quest to properly understand 4d QFT.

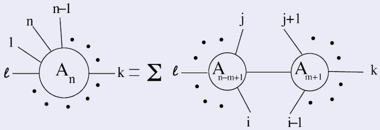
- Interests people from wide backgrounds, from string theory to pheno (breeding ground for new practical techniques)
- many simplifications and hidden structures, both for amplitudes and correlators: solvable?
- AdS/CFT correspondence

Scattering amplitudes from QCD to $\mathcal{N}$=4 SYM

- major area of research for many years
- many new structures/ insights including...

tree-level:

- onshell recursion (any gauge theory - but first proved using $\mathcal{N} = 4$) amplitudes in terms of amplitudes



[Britto Cachazo Feng Witten 2005]

- dual superconformal symmetry ($\mathcal{N}$=4) → Yangian symmetry, momentum twistors, Grassmanian, ... [Drummond Henn Korchemsky Sokatchev, Brandhuber Travaglini PH, Drummond Henn, Hodges, Arkani-Hamed Cachazo Kaplan, ...]

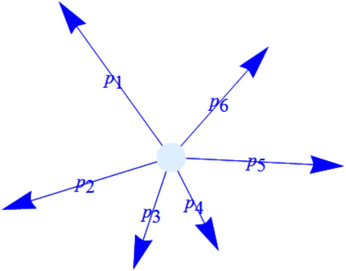
loop-level analytic amplitude

- Wilson loop/amplitude duality [Alday Maldacena 2007, Drummond Henn Korchemsky Sokatchev, Brandhuber Travaglini PH]
- Symbol [Goncharov Spradlin Vergu Volovich]
- OPE [Alday Gaiotto Maldacena Sever Vieira]
- (last week) differential equations for all loop amplitudes [Caron-Huot Song He, Bullimore Skinner]

Later I will discuss developments in understanding integrand of the amplitude

MHV Amplitude/Wilson loop duality

[Alday Maldacena 2007, Drummond Korchemsky Sokatchev, Brandhuber Travaglini PH]



planar *MHV* amplitude $\mathcal{M}_n$
$(D = 4 - 2\epsilon)$

MHV Amplitude/Wilson loop duality

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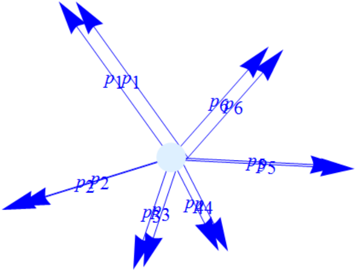

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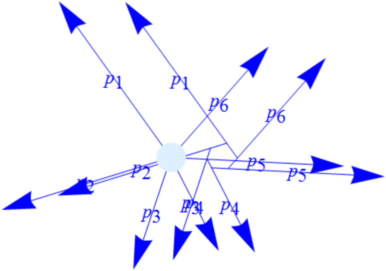

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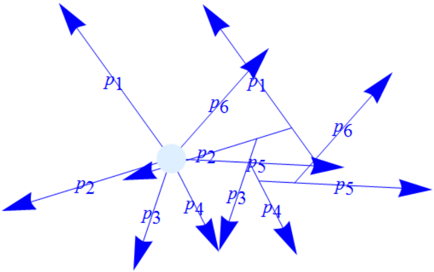

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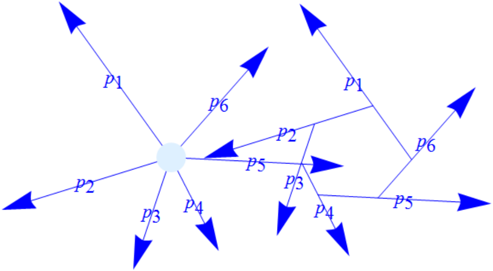

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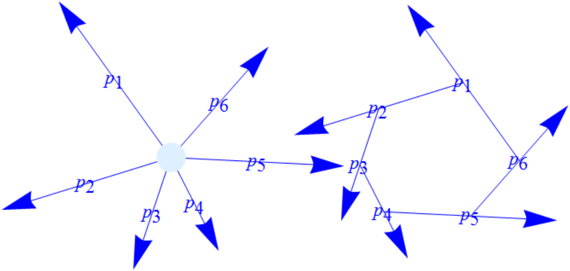

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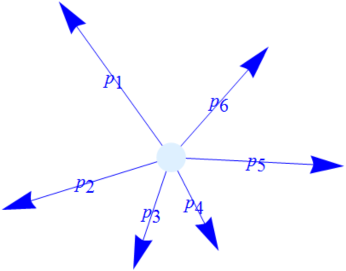

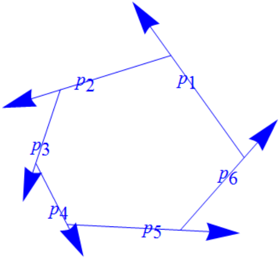

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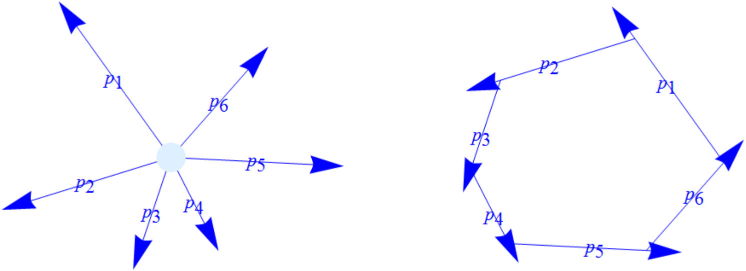

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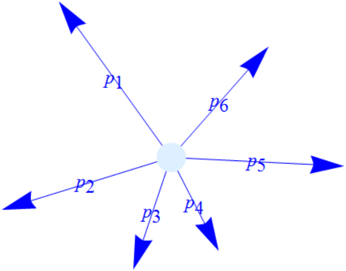

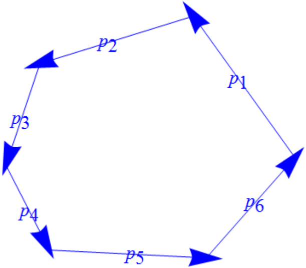

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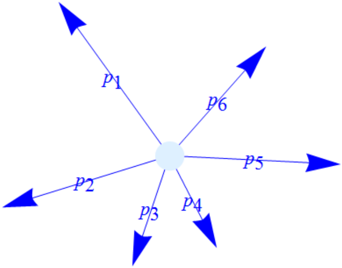

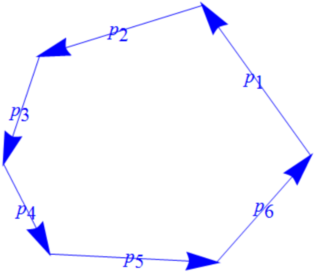

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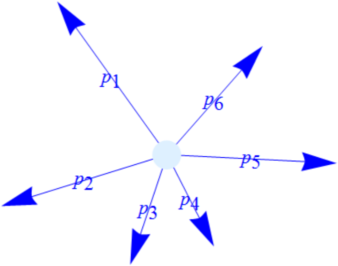

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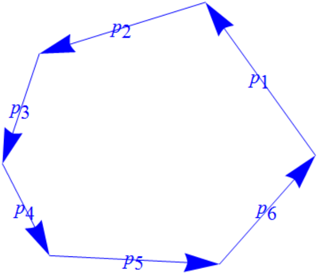
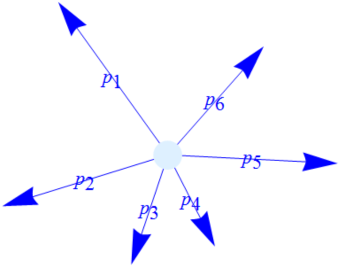


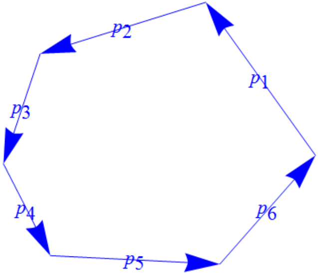
$\langle W[\mathcal{C}_n]\rangle$
$(D = 4 + 2\epsilon)$

- Wilson loop over the polygonal contour $\mathcal{C}_n$

MHV Amplitude/Wilson loop duality

[Alday Maldacena 2007, Drummond Korchemsky Sokatchev, Brandhuber Travaglini PH]





planar *MHV* amplitude $\mathcal{M}_n$ ‘=’ $\langle W[\mathcal{C}_n]\rangle$

$(D = 4 - 2\epsilon)$ $(D = 4 + 2\epsilon)$

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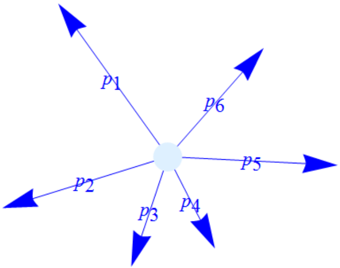
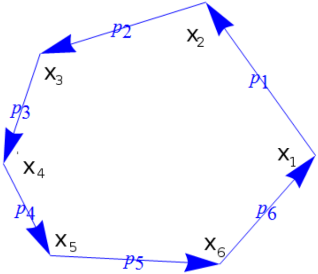




planar *MHV* amplitude $\mathcal{M}_n$ ‘$=$’ $\langle W[\mathcal{C}_n]\rangle$

$(D = 4 - 2\epsilon)$ $(D = 4 + 2\epsilon)$

- Wilson loop over the polygonal contour $\mathcal{C}_n$
- new variables: vertices, region momenta $p_i = x_{i\,i+1} = x_i - x_{i+1}$

Consequences/uses of the duality

New hidden symmetry of amplitudes

[Drummond Henn Korchemsky Sokatchev, Brandhuber Travaglini PH]

- Wilson loops conformally invariant $\rightarrow$ amplitudes have dual conformal symmetry

New results for perturbative amplitudes

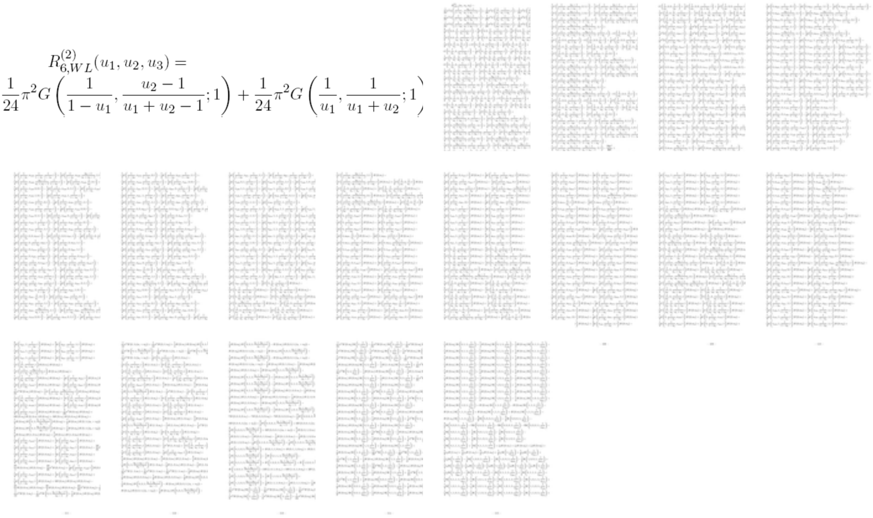
- dual conformal sym $\Rightarrow$ 4,5 point MHV (log of) amplitude “trivial”
- BDS [Bern Dixon Smirnov 2005, Drummond Henn Korchemsky Sokatchev]
- Wilson loop at two loops yields n-point integrands (completely different - easier - than the amplitude integrands)
    - 2 loop n-point integrals written down and done numerically[Anastasiou Brandhuber Khoze Spence Travaglini PH]

Analytic results (6-points 2-loops)

[Del Duca Duhr Smirnov]

Wilson loop integrals are easier, but still not easy!

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) =$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)$$



Analytic results simplified (6-points 2-loops)

[Goncharov Spradlin Vergu Volovich]

$$R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1 - 1/u_i) \right) - \frac{1}{8} \left(\sum_{i=1}^{3} \mathrm{Li}_2(1 - 1/u_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}$$

- 2 line result simplified down from 15 pages of Goncharov polylogs using the "Symbol"
- remarkable algebraic method, developed by pure mathematicians for trivialising polylog identities...

Polylogs and the symbol

[Goncharov Spradlin Vergu Volovich]

Polylogarithms: iterated integrals

$$\mathrm{Li}_1(x) = -\log(1-x) = \int^x \frac{1}{1-t}dt = -\int^x d\log(1-t)$$

$$\mathrm{Li}_n(x) = -\int^x \int^{t_1} \ldots \int^{t_{n-1}} d\log(1-t_n)d\log(t_{n-1})\ldots d\log(t_1)$$

Symbol maps weight n polylog to an n-tensor of rational functions.

$$\begin{aligned}\mathcal{S}\left[Li_n(x)\right] &= -d\log(1-x)\otimes d\log x\otimes\cdots\otimes d\log x,\\ &\to -(1-x)\otimes x\otimes\cdots\otimes x\end{aligned}$$

Nice interpretation in terms of cuts: discontinuity across branch cut indicated by first entry is given by the remaining entries

Properties of the symbol

Symbol of products = shuffle product of symbols

Example

$$\mathcal{S}\Big[\log(x)\Big] = x \qquad \mathcal{S}\Big[\log(y)\Big] = y$$

$$\mathcal{S}\Big[\log(x)\log(y)\Big] = x \otimes y + y \otimes x$$

$$\mathcal{S}\Big[\log(x)\log(y)\log(z)\Big] = x \otimes y \otimes z + \ldots (\text{permutations}) \, etc.$$

Linearity of the symbol:

$$\cdots \otimes x^n y \otimes \cdots = n(\cdots \otimes x \otimes \ldots) + (\cdots \otimes y \otimes \ldots)$$

(Recall $x \to d\log x$: this is just standard linearity of the tensor product)

Integrability constraint

- Not all tensors arise as symbols of a function
- Must satisfy a constraint: integrability constraint

Why the symbol?

Symbol trivialialises polylog identities (up to lower weight functions):

Example

$$Li_2(x) + Li_2(1-x) = -\log(1-x)\log(x) + \pi^2/6$$
$$\downarrow$$
$$-(1-x)\otimes x - x\otimes(1-x) = -(1-x)\otimes x - x\otimes(1-x)$$

However, inverting the symbol to obtain the function is extremely difficult in general.

- "symbolising" the 15 page formula, produced a long tensor
- this is the symbol of the much simpler 2 line expression

symbol is a purely mathematical technique for dealing with polylogs (nested integrals) (nothing specifically to do with $\mathcal{N} = 4$)

- useful for dealing with integrals in any (massless) quantum field theory

- open problems:
  - obtaining the symbol directly from a Feynman-type integral?
  - inverting the symbol to obtain the function (see [Duhr Gangl])

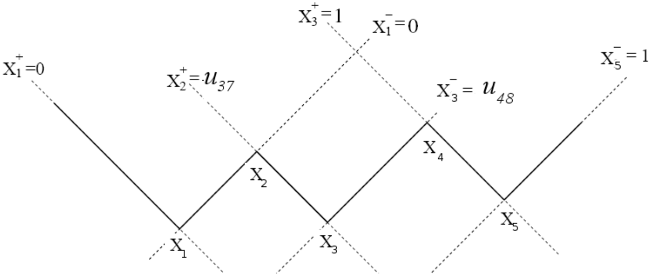
- symbol useful for simplifying complicated expressions, but

- also useful for constructing analytic results

- write ansatz for the symbol and use properties (collinear/soft limits etc) to constrain it

Example: 8 point special kinematics

- special kinematics, external momenta lying in $1+1$ dimensions

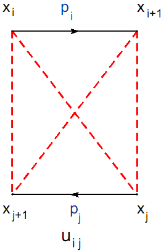


- depends on only two parameters (up to conformal transformations)
- $u_{15},\ u_{26}$ with $u_{37}=1-u_{15}, u_{48}=1-u_{26}$
- Simplifies kinematics considerably, but leaves non-trivial results. Good place to examine integrability questions etc.

Weak coupling

- The corresponding weak coupling result was obtained by [Del Duca Duhr Smirnov]
- It has the amazingly simple form (simplified from longer expression than 6-points):

$$R_8^{\mathrm{DDS}} = -\frac{1}{2}\log(u_{15})\log(u_{26})\log(u_{37})\log(u_{48}) - \frac{\pi^4}{18}\,.$$



$$u_{ij} = \frac{x_{ij+1}^2 x_{ji+1}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

- Can we generalise this result for more points and/or loops?

Using the symbol in $\mathcal{N} = 4$: ansatz for amplitudes in special kinematics

[Khoze PH]

Assume the entries of the symbol are all u's

Consistent with all known results so far. Complicated momentum twistor expressions all reduce to u's in 2d

With this single simple assumption we can:

- Derive the 8-point 2-loop result $a(\log u_1 \log u_2 \log u_3 \log u_4) + b$

(Here to simplify notation at 8-points we define $u_i := u_{i\,i+4}$)

8-point 2-loops derivation

Collinear limits $u_1 \to 1$, $u_3 \to 0$.

Three independent symbols (ℓ loops $\Rightarrow$ weight 2ℓ polylogs)

$$\mathcal{R}_8^{(2)} = a\mathcal{R}_{8;a}^{(2)} + b\mathcal{R}_{8;b}^{(2)} + c\mathcal{R}_{8;c}^{(2)} + 2\mathcal{R}_6^{(2)}$$

$$\mathcal{S}\left(\mathcal{R}_{8;a}^{(2)}\right) = u_1 \otimes u_2 \otimes u_3 \otimes u_4 \ + \ \text{7 terms related by cyclic symmetry}$$

$$\mathcal{S}\left(\mathcal{R}_{8;b}^{(2)}\right) = u_1 \otimes u_2 \otimes u_4 \otimes u_3 \ + \ \text{7 terms related by cyclic symmetry}$$

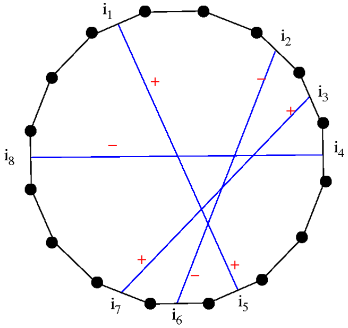
$$\mathcal{S}\left(\mathcal{R}_{8;c}^{(2)}\right) = u_1 \otimes u_3 \otimes u_2 \otimes u_4 \ + \ \text{7 terms related by cyclic symmetry}\,.$$

(Collinear vanishing function needs all four u's.)

- Requiring this to be the symbol of a function (integrability constraint) puts $a = b = c$ giving the two loop result.

2 loops n-point MHV amplitudes in 1+1 kinematics

[Khoze PH]



$$R_n = -\frac{1}{2}\Big(\sum_{\mathcal{S}} \log(u_{i_1 i_5}) \log(u_{i_2 i_6}) \log(u_{i_3 i_7}) \log(u_{i_4 i_8})\Big) - \frac{\pi^4}{72}(n-4)\ ,$$

$$\mathcal{S} = \Big\{ i_1, \ldots i_8 : 1 \le i_1 < i_2 < \cdots < i_8 \le n, \qquad i_k - i_{k-1} = \mathrm{odd} \Big\}$$

8-point 3-loop [Khoze PH]

Similar procedure for 3-loops:

Ansatz for symbol at 3-loops

$$\sum_{i_1\ldots i_6} \text{const}_{i_1\ldots i_6} \cdot u_{i_1} \otimes u_{i_2} \otimes u_{i_3} \otimes u_{i_4} \otimes u_{i_5} \otimes u_{i_6} .$$

- Symbol must vanish in the collinear limit
- Symbol should respect cyclic and parity symmetry
- Leads to 195 free constants!
- Next impose that the symbol must be the symbol of a function (integrability constraint)
- remarkably this fixes 182 of these, leaving only 13
- Fix 6 more using OPE [Alday Gaiotto Maldacena Sever Vieira]

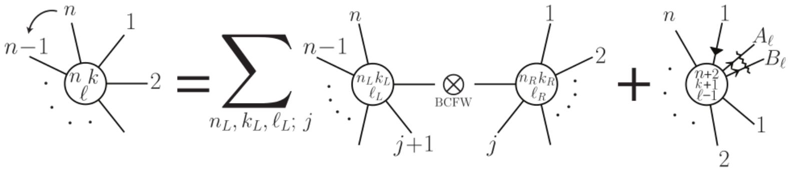
Summary: integrals

State of the art for analytic results:

- special kinematics:
  - 8-points 3 loops (10-points also) [Khoze PH]
- related work for general kinematics
  - 2 loop *n*-point (symbol only) [Caron-Huot]
  - 3 loop MHV 6-point (symbol only) [Dixon Drummond Henn]
  - 2-loop 6-point NMHV using similar techniques [Dixon Drummond Henn]

Integrands

- For rest of the talk we now mainly consider integrands of amplitudes/correlation functions
- A big recent focus of interest in amplitudes involves the 4 dim integrand of loop level amplitudes [Arkani-Hamed Bourjaily Cachazo Trnka + Caron-Huot]
- Physical object in planar limit?? (Unique in dual momenta)
- bcfw-like recursion for the integrand, integrands in terms of integrands [Arkani-Hamed Bourjaily Cachazo Trnka + Caron-Huot]



- Results for integrands go much further: but the integrals themselves are hard to evaluate
- Dualities/trialities simply phrased in terms of integrands...

Correlation functions of gauge invariant operators

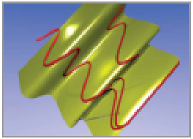
AdS/CFT

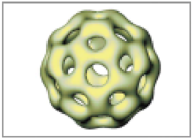
Supergravity/String theory on $AdS_5 \times S^5 \quad = \quad \mathcal{N}$=4 super Yang-Mills

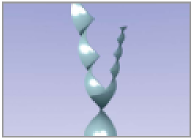
- Correlation functions of gauge invariant operators in SYM $\leftrightarrow$ string scattering in *AdS*

Centre for Particle Theory

$$\langle e^{\int_{\mathcal{B}} \varphi_0(\vec{x})\,\mathcal{O}(\vec{x})}\rangle_{\mathrm{CFT}} = \mathcal{Z}_{string}[\varphi(z,\vec{x})|_{\mathcal{B}} = \varphi_0(\vec{x})]$$







Triality

Amplitudes

$\mathcal{M}_n(p_1, \ldots p_n)$

(MHV)

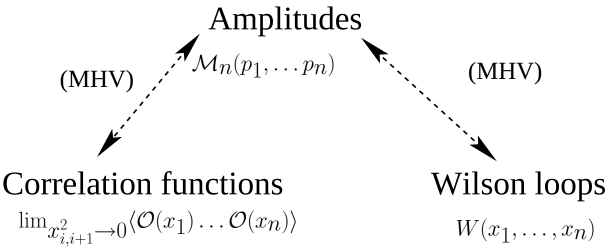
Correlation functions

Wilson loops

$W(x_1, \ldots, x_n)$

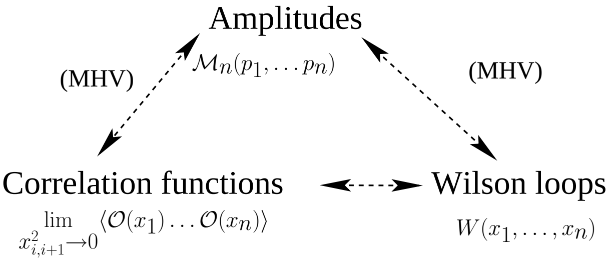
Planar

Triality



- Previous summer, [Eden Korchemsky Sokatchev].
- Integrand identity.
- Planar

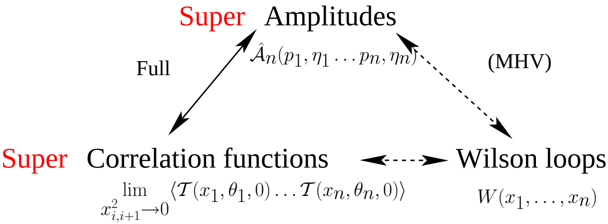
Triality



[Alday Eden Korchemsky Maldacena Sokatchev]

Non-planar

Triality



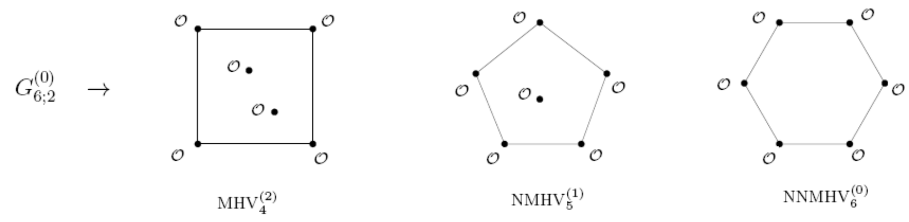
(super) correlation function/ (super) amplitude duality

[Eden Korchemsky Sokatchev PH]

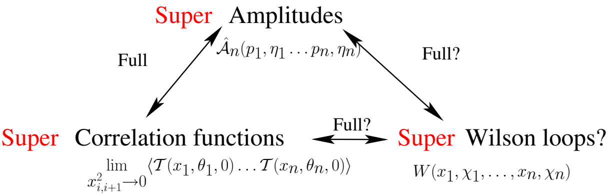
Triality

- Correlator integrands = Tree level correlators
- Entire planar $\mathcal{N} = 4$ S-matrix (integrand) as tree-level correlators of energy-momentum multiplets

Many amplitudes from a single tree-level correlator eg:



Triality



Super Wilson loop, [Mason Skinner, Caron-Huot]

- The simplest non-trivial correlation function is

$$G_4 := \langle \mathcal{O}(x_1)\bar{\mathcal{O}}(x_2)\mathcal{O}(x_3)\bar{\mathcal{O}}(x_4)\rangle \qquad \mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})$$

- $\mathcal{O} \in$ energy momentum multiplet
- strong coupling ("a"$\to \infty$) correlator computed from supergravity action ['d Hoker Freedman, Arutyunov Frolov]
- much studied at weak coupling ($a \to 0$) (1- and 2-loops). Many attempts at 3-loops, abandoned until recently...

Amplitude/correlator duality (simplest case)

Conjectured amplitude/correlator duality

$$\lim_{x_{i,i+1}^2 \to 0} \left(G_4 / G_4^{(0)} \right) = \left(A_4 / A_4^{(0)} \right)^2 := \mathcal{M}_4^2$$

- View as an integrand identity (everything is then finite, well-defined and rational even in the light-like limit)

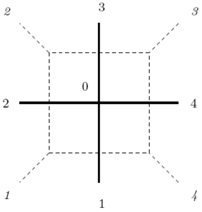
4-point 1 loop amplitude [Green Schwarz Brink]

- $A_4^{(1)}/A_4^{(0)} = x_{13}^2 x_{24}^2 \,\mathrm{Box}(p_1, p_2, p_3, p_4)$

$$\mathrm{Box}(p_1, p_2, p_3, p_4) := \int d^D\ell \frac{1}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$

$$= g(x_1, x_2, x_3, x_4) := \int d^D x_0 \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}$$

where $x_i - x_{i+1} = p_i$ and $\ell = x_0 - x_1$ (dual momenta)



4-point 1 loop correlator[Eden Howe Schubert Sokatchev West]

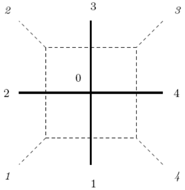
$$G_4^{(1)} := \langle \mathcal{O}\bar{\mathcal{O}}\mathcal{O}\bar{\mathcal{O}}\rangle \sim \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}(1-u-v)g(1,2,3,4)$$

$\left(u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\right)$

- limit: $\lim_{x_{i,i+1}^2 \to 0} G_4^{(1)} \sim \frac{x_{13}^2 x_{24}^2 g(1,2,3,4)}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}$ $\lim_{x_{i,i+1}^2 \to 0} G_4^{\text{tree}} \sim \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}$

$$\lim_{x_{i,i+1}^2 \to 0} \frac{G_4^{(1)}}{G_4^{\text{tree}}} = 2x_{13}^2 x_{24}^2 g(1,2,3,4)$$

$g(1,2,3,4) = 1$ loop box $= \int d^4 x_0 \, \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} =$



Finite for generic x_i but divergent in the limit $\Rightarrow$ consider integrand

Lots of checks...

- All n-point 1 loop (reproduce 2me box functions, but also parity odd part
- 4,5,6 point 2 loop
- beyond MHV agrees precisely with the integrand amplitude results derived in [Arkani-Hamed Bourjaily Cachazo Trnka + Caron-Huot]
  eg 6 point 1 loop NMHV (including highly complicated parity odd sector - numerical check) [Eden Korchemsky Sokatchev PH]

Simple derivation of 3-, 4-, 5-, 6-loop 4-point correlator/amplitude

[Eden Korchemsky Sokatchev PH+ Smirnov]

Original motivation:

- More is known about amplitudes than correlators.
- Can we use the duality to obtain new results about correlators from amplitudes?

In fact: we derive high loop correlators without using the duality

New input is the discovery of a new integrand symmetry for four-point correlators $\Rightarrow$ new insight into amplitude structure

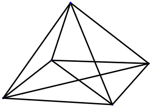
1- and 2- loop integrands: discovered symmetry

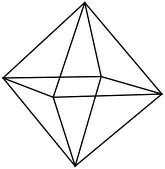
We can write (new form)

$$G_4^{(1)}/G_4^{(0)} = C \times x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times \left[\frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2} \right],$$

$$G_4^{(2)}/G_4^{(0)} = C \times x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times \left[\frac{\frac{1}{48}\sum_{\sigma\in S_6} x_{\sigma_1\sigma_2}^2 x_{\sigma_3\sigma_4}^2 x_{\sigma_5\sigma_6}^2}{\prod_{1\le i<j\le 6} x_{ij}^2} \right].$$

- notice: terms in brackets, $f^{(\ell)}$ are $S_{4+\ell}$ symmetric
- We prove this is true in general: long story!

$f^{(1)} =$ 

$f^{(2)} =$ 

Four-point correlator at arbitrary loops

General expression for 4-pnt function at any loop order ℓ

$$G_4^{(\ell)}/G_4^{(0)} = C \times x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times \frac{P^{(\ell)}(x_1, \ldots, x_{4+\ell})}{\prod_{1 \le i < j \le 4+\ell} x_{ij}^2}$$

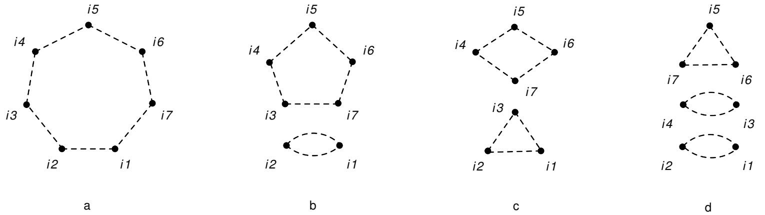
$P^{(\ell)}$ is invariant under interchange of any of the $4+\ell$ points

$$P^{(\ell)}(\ldots, x_i, \ldots, x_j, \ldots) = P^{(\ell)}(\ldots, x_j, \ldots, x_i, \ldots)\,.$$

$$\begin{aligned}
P^{(1)}(x_1, \ldots, x_5) &= 1\,, \\
P^{(2)}(x_1, \ldots, x_6) &= \frac{1}{48} \sum_{\sigma \in S_6} x^2_{\sigma(1)\sigma(2)} x^2_{\sigma(3)\sigma(4)} x^2_{\sigma(5)\sigma(6)} \\
P^{(3)}(x_1, \ldots, x_7) &= \text{symmetric polynomial, weight 2 at each point}
\end{aligned}$$

3 loops

- Analyse possible numerators $P^{(3)}$ at 3 loops:
- 7 point multigraphs of degree 2



Only 4 possibilities:

(a) heptagon: $x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 \; + \; S_7$ perms

(b) 2-gon $\times$ pentagon: $(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \; + \; S_7$ perms

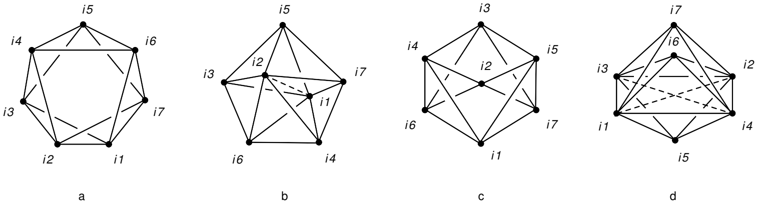
(c) triangle $\times$ square: $(x_{12}^2 x_{23}^2 x_{31}^2)(x_{45}^2 x_{56}^2 x_{67}^2 x_{74}^2) \; + \; S_7$ perms

(d) 2-gon $\times$ 2-gon $\times$ triangle: $(x_{12}^4)(x_{34}^4)(x_{56}^2 x_{67}^2 x_{75}^2) \; + \; S_7$ perms

Integrand graphs

From numerator graphs to f-graphs (graph complement)

$$f^{(3)}(x_1, \dots, x_7) = \frac{P^{(3)}(x_1, \dots, x_7)}{\prod_{1 \le i < j \le 7} x_{ij}^2} \,,$$



- (S_7 symmetric: graph theory, non-isomorphic graphs)
- Solid lines: propagators $1/x_{ij}^2$
- Dashed lines: remaining terms in numerator x_{ij}^2

Determining the coefficients

3 loop correlator

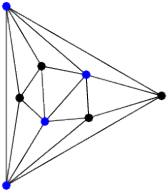
Above simple arguments determine the 3 loop correlator (for any gauge group / any N) as a linear combination of just four terms, with to-be-determined coefficients.

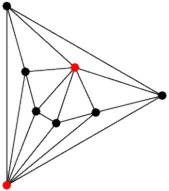
- How to determine the coefficients?
- **Method 1:** (In the planar theory only) we can use the correlator/amplitude duality. Very straightforward.
- **Method 2:** Even better: by analysing divergences at the integrand level we can fix it without direct reference to amplitudes (planar and non-planar)!

We find: only topology b contributes (no non-planar corrections at 3 loops)

Four-loops too (and five- and six-)

- Amplitude duality with four loop amplitude result [Bern Czakon Dixon Kosower Smirnov 2006] gives the following three terms:


1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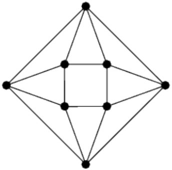

3

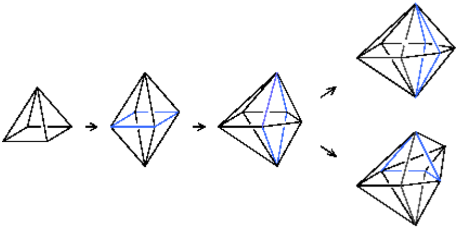
- Counting numerator multigraphs, there were 32 possibilities: these are the only planar *f*-topologies.
- Topologies 1 and 2 are generated by gluing lower loop graphs together. 3 is a new one (and comes with coefficient -1).

Planar limit ⇒ planar f-graphs

- It seems that in the large N limit all the f-graphs (except at one-loop) are planar graphs (indeed they can be thought of as edges and vertices of polyhedra.)

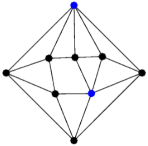
- This is true at 4-, 5- and 6-loops and seems to be true in general.

- gluing tetrahedra together to obtain higher loop f-graphs = rung rule. Derived using amplitude/correlator duality



Five-loop planar correlator

All planar graphs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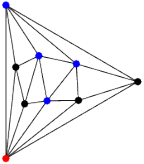
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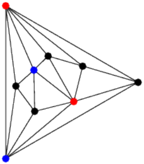
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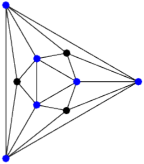
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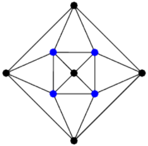
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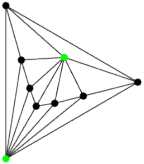
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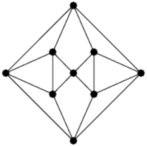
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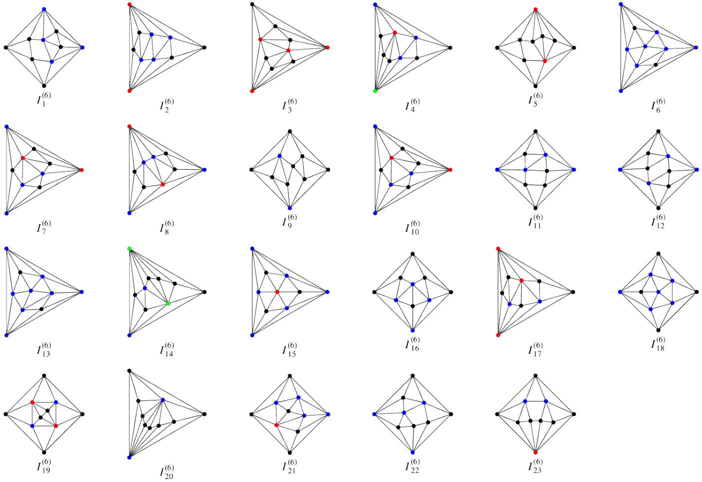
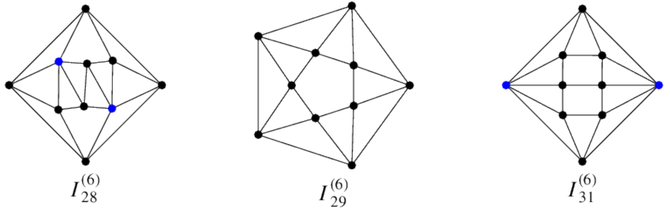
Six-loop correlator

Induced by rung rule



Six-loop graphs continued

+ 3 extra



- Graph 29 has coefficient 2 (all the rest have coefficient ± 1)
- Simple rule (topology of planar graph) which gives the coefficient?
- Check: in the light-like limit this agrees with the amplitude found independently (unpublished) by
  [Bourjailly, Dike, Shaikh, Spradlin, Volovich, Carrasco Johannson]
- $\sim$ 230 amplitude graphs, $>$1000 correlator graphs

More on four-point correlators: integrability

$\langle \mathcal{O}\bar{\mathcal{O}}\mathcal{O}\bar{\mathcal{O}} \rangle$ contains all info about the OPE

$$\mathcal{O}(x_1)\bar{\mathcal{O}}(x_2) \sim c(x_{12})\hat{\mathcal{O}}(x_2)$$

- eg $\hat{O}$= Konishi operator $Tr(\phi_{AB}\phi^{AB})$ (dual to first massive "string state")
- cusp anomalous dimension solved for all "a" [Beisert Eden Staudacher]
- conjectures for anomalous dimensions of Konishi and other finite spin operators
- 5 loop Konishi interesting since there are new aspects for integrability "dressing factor"
- Correlators contain the amplitude + much more!

Exciting by-products

- four-point correlator $\rightarrow$ anomalous dimensions etc. (integrability)
- compute integrals in limit $x_1 \rightarrow x_2,\ x_3 \rightarrow x_4$, we obtain 3-, 4-, 5-loop Konishi anomalous dimensions

  (also first two 3 loop higher twist ops using conformal partial wave techniques developed by [Dolan Osborn] and 4 loop non-planar Konishi (up to an integer constant))

First derivation of 5 loop Konishi anomalous dimension obtained using integrability + string theory by [Bajnok Hegedus Janik Lukowski] (dressing factor)

$$\left(\frac{2528}{5} + \frac{1152}{5}\zeta(3) - \frac{864}{5}[\zeta(3)]^2 - 288\zeta(5) + 1008\zeta(7)\right) \times \frac{1}{4^5}$$

Conclusion:

- Fun, beautiful, but with practical offshoots
- Lots achieved recently for high loop amplitudes and correlators, both analytically and at the integrand level.
  - Obtaining high loop results without Feynman diagrams
  - Apply symbol in QCD?
  - Correlation functions as regulated amplitudes?
  - symmetry beyond 4-points?
- Much still to explore....

Method 1: determining the coeffs using the duality

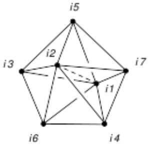
Recall the duality:

$$\lim_{x_{i,i+1}^2 \to 0} (G_4 / G_4^{(0)}) = \mathcal{M}_4^2 \,.$$

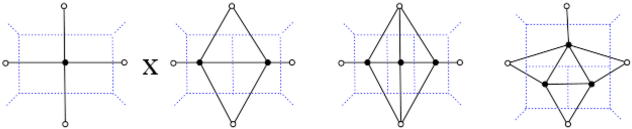
Expanding out to 3rd order:

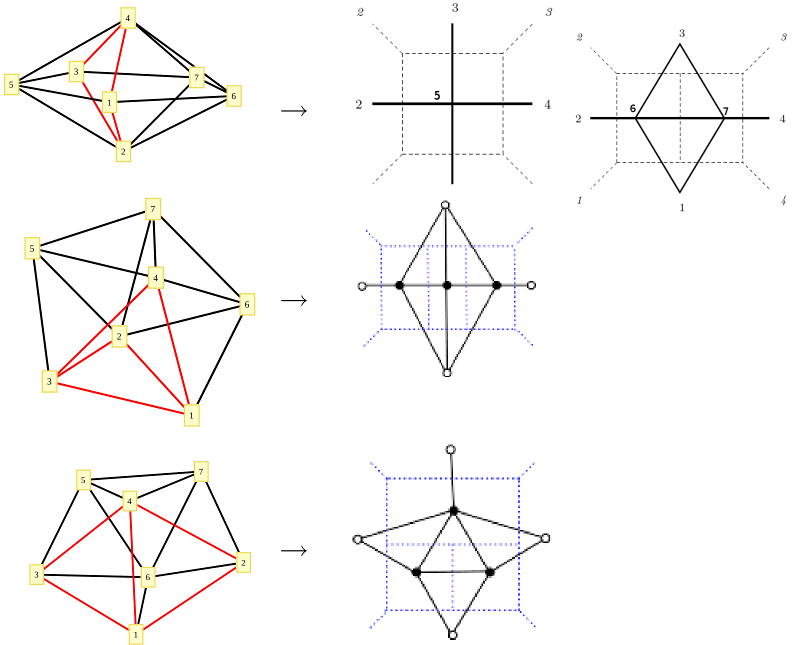
$$\lim_{x_{i,i+1}^2 \to 0} (G_4^{(3)}(x) / G_4^{(0)}(x)) = 2\mathcal{M}_4^{(3)} + 2\mathcal{M}_4^{(1)}\mathcal{M}_4^{(2)} \,.$$

Remarkably: 1 topology, b, $\rightarrow$ amplitude graphs [Bern Dixon Smirnov 2005]:



$\rightarrow$

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- Going backwards. If one knows the amplitude integrand graphs it is straightforward to obtain the f-graph. Simply add all "external lines". You will clearly get the same f-graph more than once.

- At 3 loops anyhow, this fixes the planar correlator to be given by the single term (topology b) with the precise coefficient.

Crucial insight: hidden symmetry

Eg. 2- loop correlator integrand [Eden Schubert Sokatchev]

Rewrite over common denominator:

$$\Big[\frac{1}{2}(x_{12}^2x_{34}^2 + x_{13}^2x_{24}^2 + x_{14}^2x_{23}^2)(g(1,2,3,4))^2$$

$$+ x_{12}^2 h(1,2,3;1,2,4) + x_{23}^2 h(1,2,3;2,3,4) + x_{34}^2 h(1,3,4;2,3,4)$$

$$+ x_{41}^2 h(1,2,4;1,3,4) + x_{13}^2 h(1,2,3;1,3,4) + x_{24}^2 h(1,2,4;2,3,4)\Big]$$

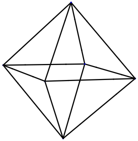
$$= x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^2 \left[\frac{\frac{1}{48}\sum_{\sigma\in S_6} x_{\sigma_1\sigma_2}^2 x_{\sigma_3\sigma_4}^2 x_{\sigma_5\sigma_6}^2}{\prod_{1\le i<j\le 6} x_{ij}^2}\right]$$

- $g(1,2,3,4)^2$ = (1 loop box)^2
- $h(1,2,3;1,2,4)$= 2 loop ladder
- Bit in brackets is S_6 invariant, mixing external variables $\{x_1, x_2, x_3, x_4\}$ with integration variables $\{x_5, x_6\}$

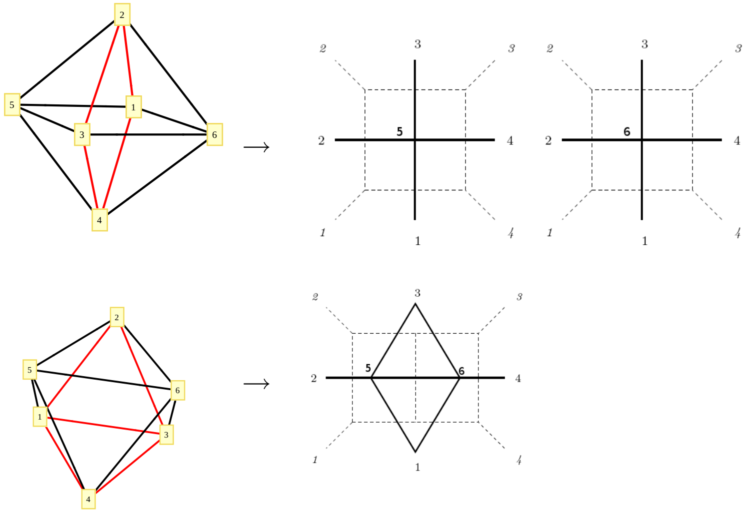
2-loops graphically

We can represent the symmetric integrand via a graph (f-graph)

$$\left[\frac{\frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_5\sigma_6}}{\prod_{1\leq i<j\leq 6} x^2_{ij}}\right] \rightarrow \sum_{\text{vertex labellings}}$$



- We can exctract the different integrals from this single object:
- Choose 4-points as external points $x_1, \ldots x_4$
- Remove all external propagators (ie remove the square.)



First line is $g(1,2,3,4)^2$ secons line is $h(1,2,3;1,3,4)$. Summing over all different inequivalent choices of vertex gives the result.

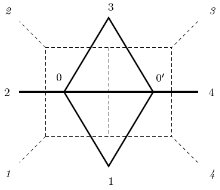
- integrand defined using Lagrangian insertion procedure
- $\Rightarrow \ell$ loop integrand = tree level $4 + \ell$-point correlator (with ℓ Lagrangian operators).
- In superspace this is proportional to a unique nilpotent superconformal invariant...
- We find a version of this invariant which is $S_{4+\ell}$ invariant
- Together with crossing symmetry this give the afore-mentioned $S_{4+\ell}$ symmetry

Example 2: 4-points 2 loop [Eden Schubert Sokatchev]

$$G_4^{(2)}(x_1,x_2,x_3,x_4) \sim \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2}(1-u-v)\Big[\frac{1}{2}(x_{12}^2x_{34}^2 + x_{13}^2x_{24}^2 + x_{14}^2x_{23}^2)(g(1,2,3,4))^2$$
$$+ x_{12}^2 h(1,2,3;1,2,4) + x_{23}^2 h(1,2,3;2,3,4) + x_{34}^2 h(1,3,4;2,3,4)$$
$$+ x_{41}^2 h(1,2,4;1,3,4) + x_{13}^2 h(1,2,3;1,3,4) + x_{24}^2 h(1,2,4;2,3,4)\Big]$$

$h(1,2,3;1,3,4) =$ 2 loop ladder =

$$\int d^4x_0 \int d^4x_0' \, \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{00'}^2 x_{10'}^2 x_{30'}^2 x_{40'}^2} =$$



so in the light-like limit

$$\lim_{x^2_{i,i+1}\to 0} \frac{G_4^{(2)}}{G_{4;0}^{\text{tree}}} =$$

$$2x_{13}^2x_{24}^2\left(\frac{1}{2}x_{13}^2x_{24}^2 g(1,2,3,4)^2 + x_{13}^2 h(1,2,3;1,3,4) + x_{24}^2 h(1,2,4;2,3,4)\right)$$

Compare with the amplitude [Bern Rozowsky Yan]

$$\frac{A_4^{(2)}}{A_4^{(0)}} = x_{13}^2x_{24}^2\left(x_{13}^2 h(1,2,3;1,3,4) + x_{24}^2 h(1,2,4;2,3,4)\right)$$

- $g(1,2,3,4)^2$ term comes from taking the square of the amplitude
- $(1 + aM^1 + a^2M^{(2)} + \ldots)^2 = 1 + 2aM^{(1)} + a^2(2M^{(2)} + (M^{(1)})^2) + \ldots$

Integrands = correlators with Lagrangian insertions

- Loop corrections $\Rightarrow$ Lagrangian insertions.

1 loop correlator

$$G_{n;0}^{(1)} = \int d^4x_0 \, \langle \mathcal{L}(x_0)\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle^{(0)}$$

- so the *Born-level* $(n+1)$-point correlator with the (chiral part of the on-shell) Lagrangian inserted at new point x_0 defines the 1 loop integrand
- ℓ-loops $\Rightarrow$ ℓ Lagrangian insertions

Beyond MHV: superduality

[Eden Korchemsky Sokatchev PH]

- So far, specific amplitude (MHV) $\leftrightarrow$ specific correlation function
- How general is the duality?
- amplitude $\rightarrow$ superamplitude
- MHV $\rightarrow$ N^{k}MHV
- Correlation functions $\rightarrow$ supercorrelation functions

Superspaces: superamplitudes

- Use Nair's $\mathcal{N}$=4 on-shell superspace, all particles → superparticle

super-particle

$$\Phi(p,\eta) = G^{+}(p) + \eta\psi + \eta^{2}\phi(p) + \eta^{3}\bar{\psi}(p) + \eta^{4}G^{-}(p)$$

- All amplitudes → superamplitudes

$$A(x_i) \rightarrow \mathcal{A}(x_i,\eta_i)$$

super-amplitude structure

$$\begin{aligned}\mathcal{A}(x_i,\eta_i) &= [\eta^{8}]A_{MHV} + [\eta^{12}]A_{NMHV} + [\eta^{16}]A_{NNMHV} + \ldots \\ &= A^{\text{tree}}_{MHV}\left(\hat{A}_{MHV} + [\eta^{4}]\hat{A}_{NMHV} + [\eta^{8}]\hat{A}_{NNMHV} + \ldots\right)\end{aligned}$$

Superspace: correlation functions

Similar expansion for correlation functions:

energy momentum supermultiplet

$$\mathcal{T}(x,\theta,\bar{\theta}) = \mathcal{O} + \ldots + \theta^4 \mathcal{L} + \ldots + \bar{\theta}^4 \bar{\mathcal{L}} + \ldots + (\theta\sigma^{\mu}\bar{\theta})(\theta\sigma^{\nu}\bar{\theta}) T_{\mu\nu} + \ldots\,,$$

- correlation function of $\mathcal{T}$s: θ-expansion organised in powers of $\theta^m\bar{\theta}^n$ with $m - n = 4k$ (non-chiral)
- How can we compare with the amplitude?

- Simply set $\bar{\theta} = 0$. Somewhat unnatural, but...

Similar superspace expansion to the superamplitude

$$\begin{aligned} G_n|_{\bar{\theta}=0} &:= \langle \mathcal{T}(1)\mathcal{T}(2)\dots\mathcal{T}(n)\rangle \\ &= G_{n;0} + \left[\theta^4\right] G_{n;1} + \left[\theta^8\right] G_{n;2} + \dots \end{aligned}$$

Dual superconformal symmetry

- Correlation functions have full superconformal symmetry, but $\bar{Q}, S$ is explicitly broken by setting $\bar{\theta} = 0$
- All amplitudes have this same symmetry "dual superconformal symmetry" [Drummond Henn Korchemsky Sokatchev]
  - proven: full symmetry at tree-level; partial symmetry at one-loop [Brandhuber Travaglini PH]
  - but $\bar{Q}, S$ is broken at loop level: no contradiction
- Predicts unexpected recovery of the full symmetry at tree-level for the correlation functions with $\bar{\theta} = 0$

Superamplitude/ supercorrelation function duality

[Eden Korchemsky Sokatchev PH]

We conjecture that (a='tHooft coupling)

Superduality

$$\lim_{x^2_{i\,i+1}\to 0} \frac{\langle \mathcal{T}(1)\ldots\mathcal{T}(n)\rangle}{\langle \mathcal{T}(1)\ldots\mathcal{T}(n)\rangle^{\text{tree}}_{n;0}}(x, a^{-1/4}\theta, \bar\theta = 0) = \left(\frac{\mathcal{A}_n}{\mathcal{A}^{\text{tree}}_{n;\text{MHV}}}(x,\eta)\right)^2$$

- duality considered at the level of the integrand...
- technical difficulty in identifying the superspaces on the two sides solved

Lagrangian insertions are also T-correlators!

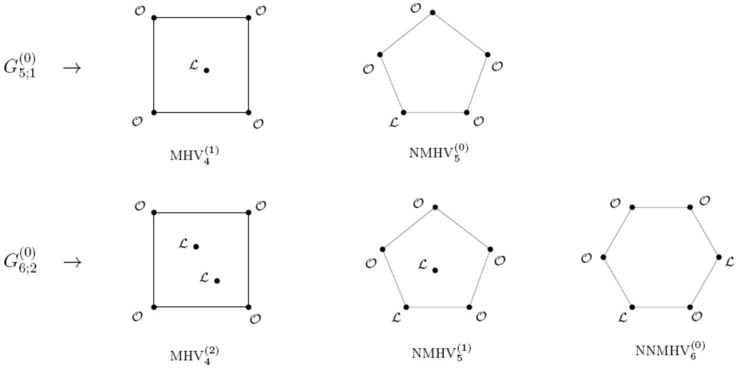
- Defined integrands of loop level correlators via insertion of $\mathcal{L}$
- But we saw that $\mathcal{L}$ is part of $\mathcal{T}$ (at $O(\theta^4)$)!
- Therefore integrands of loop level correlators of $\mathcal{T}$'s are in fact Born level correlators of $\mathcal{T}$'s

$$
\begin{aligned}
\langle \mathcal{T}(1) \dots \mathcal{T}(n) \rangle^{(l)} &= \int d^4x_{0_1} \dots d^4x_{0_l} \langle \mathcal{L}(0_1) \dots \mathcal{L}(0_l) \mathcal{T}(1) \dots \mathcal{T}(n) \rangle^{(0)} \\
&= \int d\mu_{0_1} \dots d\mu_{0_l} \langle \mathcal{T}(0_1) \dots \mathcal{T}(0_l) \mathcal{T}(1) \dots \mathcal{T}(n) \rangle^{(0)} ,
\end{aligned}
$$

- $d\mu := d^4x d^4\theta$

All loop amplitude integrands from Born level correlators

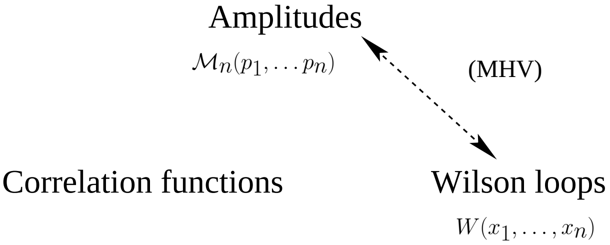
- Putting all these facts together we predict that *all* loop amplitude integrands can be obtained from Born level correlators
- Same correlator gives rise to different amplitudes



Further checks beyond MHV

- tree level: all *n*-point NMHV and 6 point NNMHV
- 1 loop: 5,6 points NMHV
- 6 point 1-loop NMHV integrand in particular has a highly non-trivial parity odd sector [Arkani-Hamed Bourjaily Cachazo Trnka + Caron-Huot] which is reproduced exactly (shown numerically)

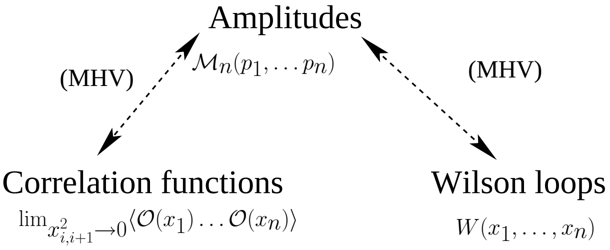
Triality



Discovered first...more later

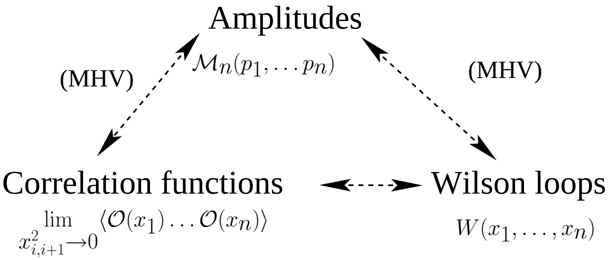
[Alday Maldacena 2007, Drummond Henn Korchemsky Sokatchev, Brandhuber Travaglini PH]

Triality



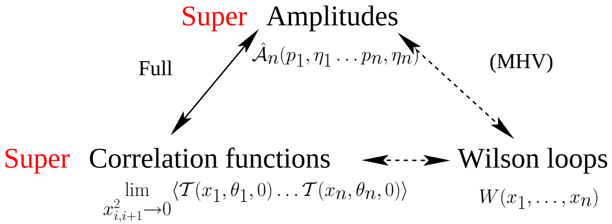
Last summer, [Eden Korchemsky Sokatchev]. Tests, later integrand identity. Dual conformal symmetry → conformal symmetry of correlators

Triality



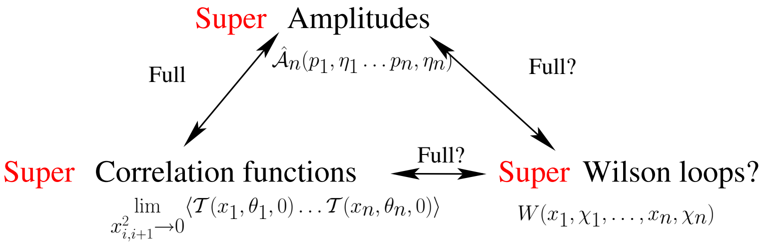
The same day, [Alday Eden Korchemsky Maldacena Sokatchev]. Regularisation: correlators as regularised Wilson loops/amplitudes?

Triality



(super) correlation function/ (super) amplitude duality

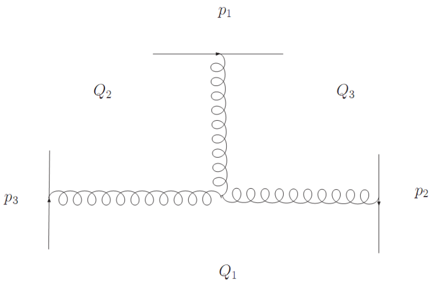
[Eden Korchemsky Sokatchev PH]



- Super Wilson loop on twistor space: [Adamo Bullimore Mason Skinner]
- Super Wilson loop on Minkowski space: [Caron-Huot]
- Correlation functions on twistor space $\leftrightarrow$ super Wilson loop on twistor space (finite N_c adjoint rep.) [Adamo Bullimore Mason Skinner]
- However at the moment the super Wilson loop is formal, not regularised (even integrands are divergent).
- Only attempt so far to regularise (dim reg) lead to an anomaly [Belitsky Korchemsky Sokatchev]

Other Integrals

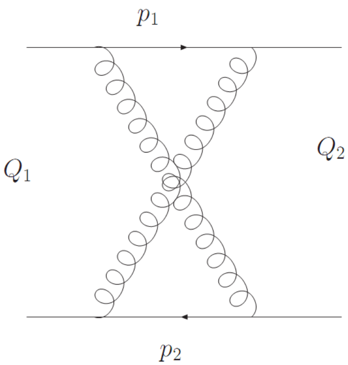
Hard integral



$$\frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 d\tau_1 d\tau_2 d\tau_3 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3)\times(\alpha_1\alpha_2\alpha_3)^{\epsilon}\frac{\mathcal{N}}{\mathcal{D}^{2+2\epsilon}}$$

Other Integrals

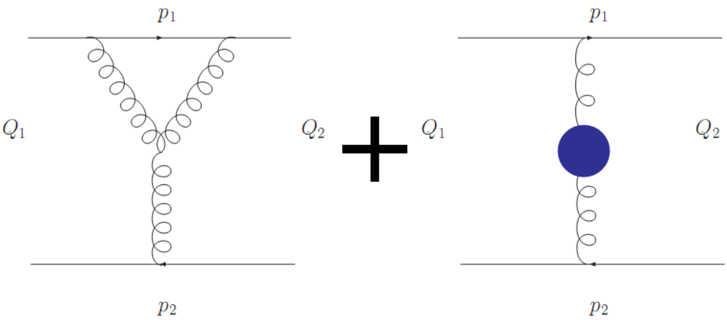
The cross integral



$$-\frac{1}{2}\int_0^1 d\sigma_1 d\tau_2 \int_0^{\sigma_1} d\tau_1 \int_0^{\tau_2} d\sigma_2 \frac{(p_1 p_2)}{\left(-2(p_1 p_2)\sigma_1\sigma_2 - 2p_1 Q_2\sigma_1 - 2p_2 Q_2\sigma_2 - Q_2^2\right)^{1+\epsilon}}$$
$$\frac{(p_1 p_2)}{\left(-2(p_1 p_2)\tau_1\tau_2 - 2p_1 Q_2\tau_1 - 2p_2 Q_2\tau_2 - Q_2^2\right)^{1+\epsilon}}$$

Other Integrals

The Y diagram + self-energy diagram



$$-\frac{p_1 \cdot p_2}{8} \frac{1}{\epsilon} \frac{\Gamma(1+2\epsilon)}{\Gamma^2(1+\epsilon)}$$

$$\times \int_0^1 d\sigma \int_0^1 d\tau_1 d\tau_2 \Bigg[-\frac{\sigma^\epsilon(1-\sigma)^\epsilon}{(-Q_1^2 - 2(Q_1 p_2)\tau_2 - 2(Q_1 p_1)\sigma\tau_1 - 2(p_1 p_2)\sigma\tau_1\tau_2)^{1+2\epsilon}} - \frac{\sigma^\epsilon(1-\sigma)^\epsilon}{(-Q_2^2 - 2(Q_2 p_2)\tau_2 - 2(Q_2 p_1)\sigma\tau_1 - 2(p_1 p_2)\sigma\tau_1\tau_2)^{1+2\epsilon}} \Bigg]$$

(Multi-)collinear limits: the general case

- As k in-coming momenta become collinear we have

Collinear factorisation

$$\mathcal{M}_n(\epsilon) \to r_k(\epsilon)\mathcal{M}_{n-k+1}(\epsilon)$$

- Putting this with the modified BDS formula

$$\log \mathcal{M}_n(\epsilon) = \sum_{l=1}^{\infty} f^{(l)}(\epsilon)\mathcal{M}_n^{(l)}(l\epsilon) + C + \mathcal{R}_n$$

$$\log \mathcal{M}_n(\epsilon) \to \log r_k(\epsilon) + \sum_{l=1}^{\infty} f^{(l)}(\epsilon)\mathcal{M}_{n-k+1}^{(l)}(l\epsilon) + C + \mathcal{R}_{n-k+1}$$

$$\downarrow$$

$$\sum_{l=1}^{\infty} f^{(l)}(\epsilon)\big(\mathcal{M}_{n-k+1}^{(l)}(l\epsilon) + r_k^{(l)}(\epsilon)\big) + C + \lim \mathcal{R}_n$$

- ie $\log r_k(\epsilon) = \sum_{l=1}^{\infty} f^{(l)}(\epsilon) r_k^{(1)}(l\epsilon) + \lim \mathcal{R}_n - \mathcal{R}_{n-k+1}$
- putting in $n = k + 3$ gives $\log r_k(\epsilon) = \sum_{l=1}^{\infty} f^{(l)}(\epsilon) r_k^{(1)}(l\epsilon) + \lim \mathcal{R}_{k+3}$
- putting this back in the first equation then gives

$$\lim \mathcal{R}_n = \mathcal{R}_{n-k+1} + \lim \mathcal{R}_{k+3}$$

n-point 1+1 kinematics

- number of independent cross-ratios reduces:

$$
\begin{aligned}
u_{i\,,i+\mathrm{odd}} &= 1 \\
u_{2i+1\,,2j+1} &= u^+_{ij} \\
u_{2i\,,2j} &= u^-_{ij}
\end{aligned}
$$

- vertices have zig-zag light-cone representation:

$$
x_{2i} = (x^+_i, x^-_i)\,, \qquad x_{2i+1} = (x^+_i, x^-_{i+1})\,, \qquad i = 1, \ldots, n
$$

$$
\Rightarrow \qquad x^2_{2i,2j} = x^+_{ij}\, x^-_{ij} \;\; etc.
$$

- light-cone cross-ratios $u^{\pm}_{ij}$:

$$
u^+_{ij} := \frac{x^+_{ij+1}\, x^+_{i+1j}}{x^+_{ij}\, x^+_{i+1j+1}}\,, \qquad u^-_{ij} := \frac{x^-_{ij+1}\, x^-_{i+1j}}{x^-_{ij}\, x^-_{i+1j+1}}\,,
$$

Aside: Relation with Y-system of [Alday Maldacena Sever Vieira]

We have the simple identity

$$(1 - 1/u^{\pm}_{i\,j})(1 - 1/u^{\pm}_{i+1\,j+1}) = (1 - u^{\pm}_{i\,j+1})(1 - u^{\pm}_{i+1\,j})$$

$$u_{i,i+1} = u_{i+1,i} = 0 \qquad u_{i,i} = \infty$$

Geometrically

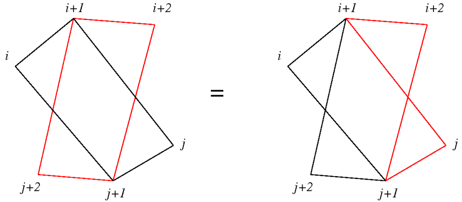


Figure: LHS: black = $(1 - 1/u^{\pm}_{i\,j})$ and red = $(1 - 1/u^{\pm}_{i+1\,j+1})$. RHS black = $1 - u_{ij+1}$ and red = $1 - u^{\pm}_{i+1j}$.

aside: interpreted at strong coupling [Alday Maldacena Sever Vieira] as a Y-system, amplitude is the free energy of the TBA integrable system

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- Eden Heslop Korchemsky Sokatchev + Smirnov
- Arkani-Hamed Bourjaily Cachazo Trnka
- Gaiotto Maldacena, Amit Sever, Pedro Vieira
- Witten
- Cachazo Svrcek Witten
- Mansfieldc
- Boels
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- Bartels Lipatov Prygarin
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- Fiamberti Santambrogio Sieg Zanon
- Duhr Gangl
- Kotikov Lipatov Onishchenko
- Bern Rozowsky Yan
- Caron-Huot Song He
- Bullimore Skinner
- Bajnok Hegedus Janik Lukowski
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- ...