Recent Results on Three Loop Corrections to Heavy Quark Contributions to DIS Structure Functions

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- 3-Loop Gluonic OMEs $O(n_f T_F^2 C_{F,A})$ and FVNS
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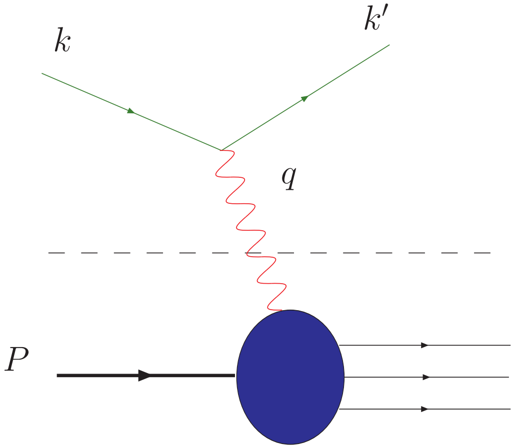
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle$$

$$= \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x,Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x,Q^2) \ .$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x)\otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x-yz) f(y) g(z)\ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) := \int_0^1 dx\, x^{N-1} f(x)\ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right) = \langle j \mid O_i \mid j \rangle .$$

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten, 1976 Nucl.Phys.B; Babcock, Sivers, 1978 Phys.Rev.D; Shifman, Vainshtein, Zakharov, 1978 Nucl.Phys.B; Leveille, Weiler, 1979 Nucl.Phys.B; Glück, Reya, 1979 Phys.Lett.B; Glück, Hoffmann, Reya, 1982 Z.Phys.C.]

Next-to-Leading Order : [Laenen, van Neerven, Riemersma, Smith, 1993 Nucl. Phys. B]
[Large Q^2/m^2: Buza, Matiounine, Smith, Migneron, van Neerven, 1996 Nucl.Phys.B] IBP
[Bierenbaum, Blümlein, Klein, 2007 Nucl.Phys.B] via ${}_pF_q$'s, more compact results
[Bierenbaum, Blümlein, Klein 2008 Nucl.Phys.B, 2009 Phys.Lett.B]:$O(\alpha_s^2\varepsilon)$ contributions (all N)

NNLO:[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] Moments for F_2: $N = 2...10(14)$
[Blümlein, Klein, Tödtli 2009 Phys. Rev. D] contrib. to transversity: $N = 1...13$
[Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl.Phys.B] contrib. $\propto n_f$ to F_2 (all N):

At 3-loop order known:

- $A_{qq,Q}^{\rm PS}$, $A_{qg,Q}$: complete
- A_{Qg}, $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS}A_{qq,Q}^{\rm NS,TR}$: all terms of $O(n_f T_F^2 C_{A/F})$
- $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS}A_{qq,Q}^{\rm NS,TR}$: all terms of $O(T_F^2 C_{A/F})$
- $A_{gq,Q}$, $A_{gg,Q}$: see this talk $\longrightarrow$ all terms of $O(n_f T_F^2 C_{A/F})$

VFNS Relations for PDFs

The matching conditions for the the VFNS:

[Buza, Matiounine, Smith, van Neerven 1998 Eur.Phys.J.C]→ NLO

[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] → NNLO

$$\begin{aligned}
&f_k(N,n_f+1,\mu^2,m^2)+f_{\bar{k}}(N,n_f+1,\mu^2,m^2)\\
&\qquad = A^{\mathrm{NS}}_{qq,Q}\left(N,n_f,\frac{\mu^2}{m^2}\right)\otimes\left[f_k(N,n_f,\mu^2,m^2)+f_{\bar{k}}(N,n_f,\mu^2,m^2)\right]\\
&\qquad +\frac{1}{n_f}A^{\mathrm{PS}}_{qq,Q}\left(N,n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(N,n_f,\mu^2,x)+\frac{1}{n_f}A_{qg,Q}\left(N,n_f,\frac{\mu^2}{m^2}\right)\otimes G(N,n_f,\mu^2,x)
\end{aligned}$$

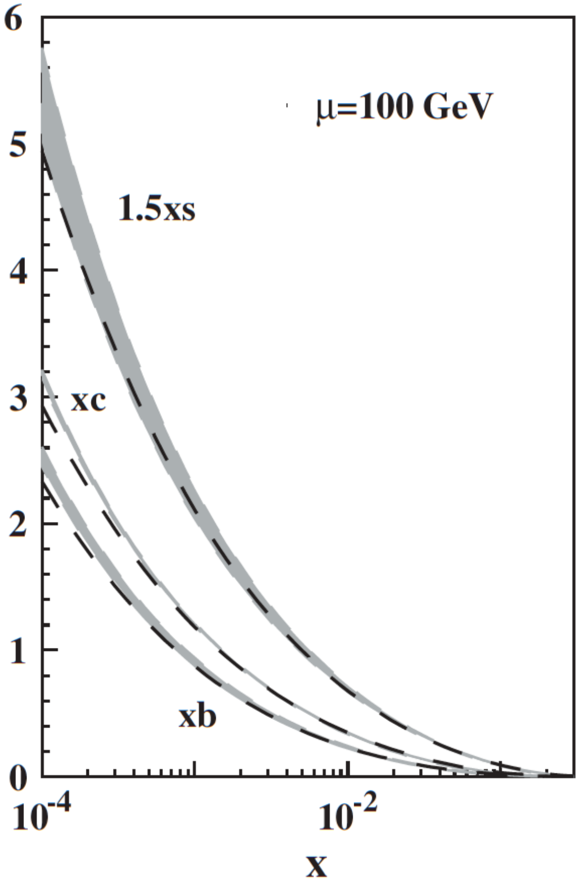
$$\begin{aligned}
&f_Q(N,n_f+1,\mu^2,m^2)+f_{\bar{Q}}(N,n_f+1,\mu^2,m^2)\\
&\qquad = A^{\mathrm{PS}}_{Qq}\left(N,n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(N,n_f,\mu^2,m^2)+A_{Qg}\left(N,n_f,\frac{\mu 2}{m^2}\right)\otimes G(N,n_f,\mu^2,m^2)
\end{aligned}$$

$$\begin{aligned}
&G(N,n_f+1,\mu^2,m^2)\\
&\qquad = A_{gq,Q}\left(N,n_f,\frac{\mu^2}{m^2}\right)\otimes\Sigma(N,n_f,\mu^2,m^2)+A_{gg,Q}\left(N,n_f,\frac{\mu^2}{m^2}\right)\otimes G(N,n_f,\mu^2,m^2)
\end{aligned}$$

where: $\left(\Sigma(N,n_f,...)=\sum_{k=1}^{n_f}(f_k+f_{\bar{k}}),\quad n_f=3\right)$

The ABKM09 heavy flavor PDFs: (grey band) compared to MSTW2008 NNLO (dashed line):

[Alekhin, Blümlein, Klein, Moch 2010 Phys.Rev.D] [Martin, Stirling, Thorne, Watt 2010 Eur.Phys.J.C]



Influence on W^+ production at the LHC (e.g. $\sqrt{s} = 7$ TeV):

ABM11($n_f = 5$): $\sigma_{W^+} = 59.53\,^{+0.38}_{-0.23}\,^{+0.88}_{-0.88}$

ABM11($n_f = 4$): $\sigma_{W^+} = 59.08\,^{+0.30}_{-0.14}\,^{+0.87}_{-0.87}$

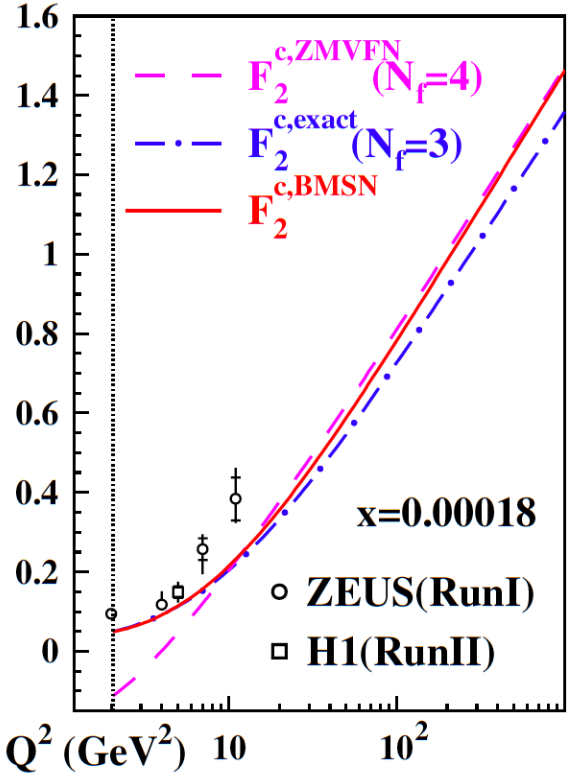
[Alekhin, Blümlein, Moch 2012]

BMSN-scheme for the heavy quark part F_2^h in F_2:

[Buza, Matiounine, Smith, van Neerven 1998 Eur.Phys.J.C], [Alekhin, Blümlein, Klein, Moch 2010 Phys.Rev.D]

$$F_2^{h,\mathrm{BMSN}}(n_f+1,x,Q^2) = F_2^{h,\mathrm{exact}}(n_f,x,Q^2) + F_2^{h,\mathrm{ZMVFN}}(n_f+1,x,Q^2) - F_2^{h,\mathrm{asymp}}(n_f,x,Q^2)$$

- $F_2^{h,\mathrm{exact}}$ = exact heavy quark contributions to F_2 in the n_f-flavor scheme
- $F_2^{h,\mathrm{ZMVFN}}$ = similar to a (n_f+1)-flavor scheme with PDFs defined from the matching
- $F_2^{h,\mathrm{asymp}} \approx F_2^{h,\mathrm{exact}}$ for $Q^2 \gg m_h^2$ without power corrections



The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gg,Q}$

calculation of the 1PI part of $A_{gg,Q}^{(3),n_f T_F^2}$

- generation of Diagrams with `QGRAF` [Nogueira 1993 J. Comput. Phys] → 76 Diagrams
- momentum integrals (regularized in $4+\varepsilon$ dimensions) → Feynman parameterization → finite sums and hypergeometric functions
- All-ε representation: maximum nestedness 2, hypergeometric functions ${}_3F_2$

Moments were tested using earlier calculations based on `MATAD` by [M. Steinhauser,2000 CPC].
Then the package `Sigma` [C. Schneider, 2005–] is used for:

- reducing the sums to a small number of key sums
- expanding the summands in ε
- simplifying by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–]
- harmonic sums are algebraically reduced using the package HarmonicSums (Ablinger) [Ablinger, Blümlein, Schneider 2011]

⟶ single harmonic sums and ζ-values of max. weight 3 $\quad S_i \equiv \sum_{j=1}^{N} \frac{1}{j^i}$

$$
\begin{aligned}
A_{ggQ}^{n_f T_f^2 1\mathrm{PI}} = {} & S_\varepsilon^3 a_s^3 n_f T_F^2 \frac{1+(-1)^N}{2} \left(\frac{m^2}{\mu^2}\right)^{\frac{3}{2}\varepsilon} \Biggl\{ \frac{1}{\boldsymbol{\varepsilon}^3} \Biggl(\mathbf{C_A} \left[\frac{512}{27} S_1 - \frac{64\left(3N^4+6N^3+13N^2+10N+16\right)}{27(N-1)N(N+1)(N+2)} \right] \\
& - \mathbf{C_F} \frac{512\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)} \Biggr) + \frac{1}{\boldsymbol{\varepsilon}^2} \Biggl(\mathbf{C_A} \left[\frac{1280}{81} S_1 - \frac{16P_1}{81(N-1)N^2(N+1)^2(N+2)} \right] \\
& + \mathbf{C_F} \frac{1}{(N-1)(N+2)} \left[\frac{128\left(N^2+N+2\right)^2}{9N^2(N+1)^2} S_1 - \frac{128P_2}{27N^3(N+1)^3} \right] \Biggr) \\
& + \frac{1}{\boldsymbol{\varepsilon}} \Biggl(\mathbf{C_A} \frac{1}{(N-1)(N+2)} \Biggl[-\frac{4P_8}{81N^3(N+1)^3} - \frac{8\left(3N^4+6N^3+13N^2+10N+16\right)}{9N(N+1)} \zeta_2 \\
& + \frac{32P_9}{27N^2(N+1)^2} S_1 + \frac{64}{9}(N-1)(N+2)\zeta_2 S_1 \Biggr] + \mathbf{C_F} \frac{1}{(N-1)(N+2)} \Biggl[-\frac{160\left(N^2+N+2\right)^2}{9N^2(N+1)^2} S_1^2 \\
& - \frac{64\left(N^2+N+2\right)^2}{3N^2(N+1)^2} \zeta_2 + \frac{32\left(N^2+N+2\right)^2}{3N^2(N+1)^2} S_2 - \frac{64P_{10}}{81N^4(N+1)^4} + \frac{64P_{11}}{27N^3(N+1)^3} S_1 \Biggr] \Biggr) \\
& + \mathbf{C_A} \frac{1}{(N-1)(N+2)} \Biggl[\frac{4P_3}{27N^2(N+1)^2} S_1^2 + \frac{8P_4}{729N^3(N+1)^3} S_1 + \frac{160}{27}(N-1)(N+2)\zeta_2 S_1 \\
& - \frac{448}{27}(N-1)(N+2)\zeta_3 S_1 + \frac{P_5}{729N^4(N+1)^4} - \frac{2P_6}{27N^2(N+1)^2}\zeta_2 - \frac{4P_7}{27N^2(N+1)^2} S_2 \\
& + \frac{56\left(3N^4+6N^3+13N^2+10N+16\right)}{27N(N+1)} \zeta_3 \Biggr] + \mathbf{C_F} \frac{1}{(N-1)(N+2)} \Biggl[\frac{112\left(N^2+N+2\right)^2}{27N^2(N+1)^2} S_1^3 \\
& - \frac{16P_{12}}{27N^3(N+1)^3} S_1^2 + \frac{32P_{13}}{81N^4(N+1)^4} S_1 + \frac{16\left(N^2+N+2\right)^2}{3N^2(N+1)^2} \zeta_2 S_1 + \frac{16\left(N^2+N+2\right)^2}{3N^2(N+1)^2} S_2 S_1 \\
& - \frac{32P_{14}}{243N^5(N+1)^5} - \frac{16P_2}{9N^3(N+1)^3}\zeta_2 + \frac{448\left(N^2+N+2\right)^2}{9N^2(N+1)^2}\zeta_3 + \frac{16P_{15}}{9N^3(N+1)^3} S_2 - \frac{160\left(N^2+N+2\right)^2}{27N^2(N+1)^2} S_3 \Biggr] \Biggr\}
\end{aligned}
$$

Renormalization of the OME:

[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

1. include contributions from reducible diagrams
2. perform on-shell mass renormalization
3. renormalize the coupling in a MOM-scheme, using the background field method
4. remove remaining UV singularities defining operator Z-factors
5. remove collinear singularities via coll. factorization
6. transform coupling constant to $\overline{\mathsf{MS}}$
7. choice: m on-shell or $m_{\overline{\mathsf{MS}}}$

Combining the 1PI part with the gluon self energy contribution:

$$\begin{aligned} \hat{\Pi}^{ab}_{\mu\nu}(p^2, \hat{m}^2, \mu^2, \hat{a}_s^2) &= i\delta^{ab}\left[-g_{\mu\nu}p^2 + p_\mu p_\nu\right] \sum_{k-1}^{\infty} \hat{a}_s^k \hat{\Pi}^{(k)}(p^2, \hat{m}^2, \mu^2) \\ \hat{\Pi}^{(k)} &\equiv \hat{\Pi}^{(k)}(0, \hat{m}^2, \mu^2) \end{aligned}$$

giving:

$$\begin{aligned} \hat{\hat{A}}^{(3)}_{gg,Q} &= \hat{\hat{A}}^{(3),1\mathrm{PI}}_{gg,Q} - \hat{\Pi}^{(3)} - \hat{\hat{A}}^{(2),1\mathrm{PI}}_{gg,Q}\hat{\Pi}^{(1)} - 2\hat{\hat{A}}^{(1)}_{gg,Q}\hat{\Pi}^{(2)} + \hat{\hat{A}}^{(1)}_{gg,Q}\hat{\Pi}^{(1)}\hat{\Pi}^{(1)} \\ &\equiv \frac{a^{(3,0)}_{gg,Q}}{\varepsilon^3} + \frac{a^{(3,1)}_{gg,Q}}{\varepsilon^2} + \frac{a^{(3,2)}_{gg,Q}}{\varepsilon} + a^{(3)}_{gg,Q} \end{aligned}$$

The renormalization determines the coefficients of the ε-poles in terms of coefficients from the mass renormalization: $\delta m_1^{(-1)}, \delta m_1^{(0)}, \delta m_1^{(1)}, \delta m_2^{(-2)}, \delta m_2^{(-1)}, \delta m_2^{(0)}$

contributions β_i to the $\overline{\mathsf{MS}}$-β-function and

$\beta_{i,Q}$ to the MOM-beta-function in QCD,

splitting functions: $\gamma_{ij} \equiv \sum_{l=1}^{\infty} a_s^{\overline{\mathsf{MS}}\,l} \gamma_{ij}^{(l)}$ with $\hat{\gamma}_{ij}^{(k)} \equiv \gamma_{ij}^{(k)}(n_f+1) - \gamma_{ij}^{(k)}(n_f)$ and

lower order contributions to the unrenormalized OMEs:

$$\hat{\hat{A}}_{gg,Q}^{(2)} \equiv \frac{a_{gg,Q}^{(2,0)}}{\varepsilon^2} + \frac{a_{gg,Q}^{(2,1)}}{\varepsilon} + a_{gg,Q}^{(2)} + \varepsilon \bar{a}_{gg,Q}^{(2)} + O(\varepsilon^2)$$

$$\hat{\hat{A}}_{gq,Q}^{(2)} \equiv \frac{a_{gq,Q}^{(2,0)}}{\varepsilon^2} + \frac{a_{gq,Q}^{(2,1)}}{\varepsilon} + a_{gq,Q}^{(2)} + \varepsilon \bar{a}_{gq,Q}^{(2)} + O(\varepsilon^2)$$

$$\hat{\hat{A}}_{qg,Q}^{(2)} \equiv \frac{a_{Qg}^{(2,0)}}{\varepsilon^2} + \frac{a_{Qg}^{(2,1)}}{\varepsilon} + a_{Qg}^{(2)} + \varepsilon \bar{a}_{Qg}^{(2)} + O(\varepsilon^2)$$

[Bierenbaum, Blümlein, Klein 2007,2008 Nucl.Phys.B, 2009 Phys.Lett.B]

The structure of the unrenormalized OME: [Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

$$
\begin{aligned}
\hat{\hat{A}}_{gg,Q}^{(3)} = {} & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2} \Biggl[\frac{1}{\varepsilon^3} \Biggl(-\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{6} \Bigl[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 4n_f\beta_{0,Q} + 10\beta_{0,Q}\Bigr] - \frac{2\gamma_{gg}^{(0)}\beta_{0,Q}}{3}\Bigl[2\beta_0 + 7\beta_{0,Q}\Bigr] \\
& -\frac{4\beta_{0,Q}}{3}\Bigl[2\beta_0^2 + 7\beta_{0,Q}\beta_0 + 6\beta_{0,Q}^2\Bigr]\Biggr) + \frac{1}{\varepsilon^2}\Biggl(\frac{\hat{\gamma}_{qg}^{(0)}}{6}\Bigl[\gamma_{gq}^{(1)} - (2n_f - 1)\hat{\gamma}_{gq}^{(1)}\Bigr] + \frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}}{3} - \frac{\hat{\gamma}_{gg}^{(1)}}{3}\Bigl[4\beta_0 + 7\beta_{0,Q}\Bigr] \\
& +\frac{2\beta_{0,Q}}{3}\Bigl[\gamma_{gg}^{(1)} + \beta_1 + \beta_{1,Q}\Bigr] + \frac{2\gamma_{gg}^{(0)}\beta_{1,Q}}{3} + \delta m_1^{(-1)}\Bigl[-\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 2\beta_{0,Q}\gamma_{gg}^{(0)} - 10\beta_{0,Q}^2 - 6\beta_{0,Q}\beta_0\Bigr]\Biggr) \\
& +\frac{1}{\varepsilon}\Biggl(\frac{\hat{\gamma}_{gg}^{(2)}}{3} - 2(2\beta_0 + 3\beta_{0,Q})\mathbf{a_{gg,Q}^{(2)}} - n_f\hat{\gamma}_{qg}^{(0)}\mathbf{a_{gq,Q}^{(2)}} + \gamma_{gq}^{(0)}\mathbf{a_{Qg}^{(2)}} + \beta_{1,Q}^{(1)}\gamma_{gg}^{(0)} + \frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} \\
& +2(2n_f + 1)\beta_{0,Q} + 6\beta_0\Bigr] + \frac{\beta_{0,Q}\zeta_2}{4}\Bigl[\gamma_{gg}^{(0)}\{2\beta_0 - \beta_{0,Q}\} + 4\beta_0^2 - 2\beta_{0,Q}\beta_0 - 12\beta_{0,Q}^2\Bigr] \\
& +\delta m_1^{(-1)}\Bigl[-3\delta m_1^{(-1)}\beta_{0,Q} - 2\delta m_1^{(0)}\beta_{0,Q} - \hat{\gamma}_{gg}^{(1)}\Bigr] + \delta m_1^{(0)}\Bigl[-\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 2\gamma_{gg}^{(0)}\beta_{0,Q} - 4\beta_{0,Q}\beta_0 - 8\beta_{0,Q}^2\Bigr] \\
& +2\delta m_2^{(-1)}\beta_{0,Q}\Biggr) + \boxed{a_{gg,Q}^{(3)}}\Biggr] .
\end{aligned}
$$

⟶ use for checking the ε singular parts

We confirm the $n_f T_F^2$ part of the 3-Loop anomalous dimension:

[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$
\begin{aligned}
\hat{\gamma}_{gg}^{(2)} \; = \; & n_f T_F^2 \mathbf{C_A} \Bigg[-\frac{32\left(8N^6+24N^5-19N^4-78N^3-253N^2-210N-96\right)}{27(N-1)N^2(N+1)^2(N+2)} S_1 \\
& -\frac{8\left(87N^8+348N^7+848N^6+1326N^5+2609N^4+3414N^3+2632N^2+1088N+192\right)}{27(N-1)N^3(N+1)^3(N+2)} \Bigg] \\
& + n_f T_F^2 \mathbf{C_F} \Bigg[\frac{64\left(N^2+N+2\right)^2}{3(N-1)N^2(N+1)^2(N+2)} \left(S_1^2 - 3S_2\right) - \frac{16P_1}{27(N-1)N^4(N+1)^4(N+2)} \\
& + \frac{128\left(4N^6+3N^5-50N^4-129N^3-100N^2-56N-24\right)}{9(N-1)N^3(N+1)^3(N+2)} S_1 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
P_1 \; = \; & 33N^{10}+165N^9+256N^8-542N^7-3287N^6-8783N^5-11074N^4-9624N^3 \\
& -5960N^2-2112N-288
\end{aligned}
$$

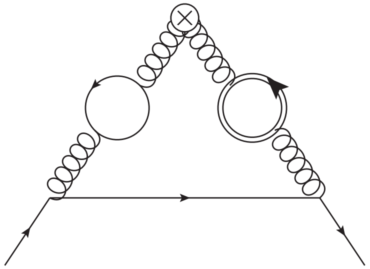
First diagrammatic recalculation

$$
\begin{aligned}
A_{gg,Q}^{(3),\overline{\mathsf{MS}}} = {} & \frac{1}{48}\Biggl\{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\Bigl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-6\beta_0-4n_f\beta_{0,Q}-10\beta_{0,Q}\Bigr)-4\Bigl(\gamma_{gg}^{(0)}\Bigl[2\beta_0+7\beta_{0,Q}\Bigr]+4\beta_0^2+14\beta_{0,Q}\beta_0 \\
& +12\beta_{0,Q}^2\Bigr)\beta_{0,Q}\Biggr\}\ln^{\mathbf{3}}\left(\frac{\mathbf{m^2}}{\boldsymbol{\mu^2}}\right)+\frac{1}{8}\Biggl\{\hat{\gamma}_{qg}^{(0)}\Bigl(\gamma_{gq}^{(1)}+(1-n_f)\hat{\gamma}_{gq}^{(1)}\Bigr)+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}-4\hat{\gamma}_{gg}^{(1)}[\beta_0+2\beta_{0,Q}] \\
& +4\gamma_{gg}^{(1)}\beta_{0,Q}+4[\beta_1+\beta_{1,Q}]\beta_{0,Q}+2\gamma_{gg}^{(0)}\beta_{1,Q}\Biggr\}\ln^{\mathbf{2}}\left(\frac{\mathbf{m^2}}{\boldsymbol{\mu^2}}\right)+\frac{1}{16}\Biggl\{8\hat{\gamma}_{gg}^{(2)}-8n_f a_{gq,Q}^{(2)}\hat{\gamma}_{qg}^{(0)}+8\gamma_{gq}^{(0)}a_{Qg}^{(2)} \\
& -16a_{gg,Q}^{(2)}(2\beta_0+3\beta_{0,Q})+8\gamma_{gg}^{(0)}\beta_{1,Q}^{(1)}+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_2\Bigl(\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+6\beta_0+4n_f\beta_{0,Q}+6\beta_{0,Q}\Bigr) \\
& +4\beta_{0,Q}\zeta_2\Bigl(\gamma_{gg}^{(0)}+2\beta_0\Bigr)\Bigl(2\beta_0+3\beta_{0,Q}\Bigr)\Biggr\}\ln\left(\frac{\mathbf{m^2}}{\boldsymbol{\mu^2}}\right)+2(2\beta_0+3\beta_{0,Q})\overline{\mathbf{a}}_{\mathbf{gg,Q}}^{\mathbf{(2)}}+n_f\hat{\gamma}_{qg}^{(0)}\overline{\mathbf{a}}_{\mathbf{gq,Q}}^{\mathbf{(2)}}-\gamma_{gq}^{(0)}\overline{\mathbf{a}}_{\mathbf{Qg}}^{\mathbf{(2)}} \\
& -\beta_{1,Q}^{(2)}\gamma_{gg}^{(0)}+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_3}{48}\Bigl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-2[2n_f+1]\beta_{0,Q}-6\beta_0\Bigr)+\frac{\beta_{0,Q}\zeta_3}{12}\Bigl([\beta_{0,Q}-2\beta_0]\gamma_{gg}^{(0)} \\
& +2[\beta_0+6\beta_{0,Q}]\beta_{0,Q}-4\beta_0^2\Bigr)-\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\Bigl(\gamma_{gq}^{(1)}+\hat{\gamma}_{gq}^{(1)}\Bigr)+\frac{\beta_{0,Q}\zeta_2}{8}\Bigl(\hat{\gamma}_{gg}^{(1)}-2\gamma_{gg}^{(1)}-2\beta_1-2\beta_{1,Q}\Bigr) \\
& +\frac{\delta m_1^{(-1)}}{4}\Bigl(8a_{gg,Q}^{(2)}+24\delta m_1^{(0)}\beta_{0,Q}+8\delta m_1^{(1)}\beta_{0,Q}+\zeta_2\beta_{0,Q}\beta_0+9\zeta_2\beta_{0,Q}^2\Bigr)+\delta m_1^{(0)}\Bigl(\beta_{0,Q}\delta m_1^{(0)}+\hat{\gamma}_{gg}^{(1)}\Bigr) \\
& +\delta m_1^{(1)}\Bigl(\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}+2\beta_{0,Q}\gamma_{gg}^{(0)}+4\beta_{0,Q}\beta_0+8\beta_{0,Q}^2\Bigr)-2\delta m_2^{(0)}\beta_{0,Q}+a_{gg,Q}^{(3)}
\end{aligned}
$$

The final renormalized contribution with the $\overline{\mathsf{MS}}$-mass $\bar{m}$:

$$
\begin{aligned}
A_{gg,Q}^{(3),n_f T_F^2,\overline{\mathsf{MS}}} = n_f T_F^2 \Bigg\{ & \Bigg(\mathbf{C_F} \frac{64\left(N^2+N+2\right)^2}{9(N-1)N^2(N+1)^2(N+2)} + \mathbf{C_A} \left[\frac{128\left(N^2+N+1\right)}{27(N-1)N(N+1)(N+2)} - \frac{64}{27} S_1 \right] \Bigg) \boldsymbol{\ln^3\left(\frac{\bar{m}^2}{\mu^2}\right)} \\
& - \mathbf{C_F} \frac{16}{3} \boldsymbol{\ln^2\left(\frac{\bar{m}^2}{\mu^2}\right)} + \Bigg(\mathbf{C_A} \frac{1}{(N-1)(N+2)} \left[- \frac{4P_1}{81N^3(N+1)^3} - \frac{16P_2}{81N^2(N+1)^2} S_1 \right] \\
& + \mathbf{C_F} \frac{1}{(N-1)(N+2)} \left[\frac{16\left(N^2+N+2\right)^2}{N^2(N+1)^2} \left(S_1^2 - \frac{5}{3} S_2 \right) - \frac{4P_3}{9N^4(N+1)^4} - \frac{32P_4}{3N^3(N+1)^3} S_1 \right] \Bigg) \boldsymbol{\ln\left(\frac{\bar{m}^2}{\mu^2}\right)} \\
& + \mathbf{C_A} \frac{1}{(N-1)(N+2)} \Bigg[- \frac{4P_5}{27N^2(N+1)^2} S_1^2 - \frac{8P_6}{729N^3(N+1)^3} S_1 + \frac{512}{27}(N-1)(N+2)\zeta_3 S_1 \\
& - \frac{2P_7}{729N^4(N+1)^4} - \frac{1024\left(N^2+N+1\right)}{27N(N+1)} \zeta_3 + \frac{4P_8}{27N^2(N+1)^2} S_2 \Bigg] \\
& + \mathbf{C_F} \frac{1}{(N-1)(N+2)} \Bigg[\frac{64\left(N^2+N+2\right)^2}{9N^2(N+1)^2} \left(-\frac{1}{3} S_1^3 - 8\zeta_3 + \frac{4}{3} S_3 \right) + \frac{32P_9}{27N^3(N+1)^3} S_1^2 \\
& - \frac{64P_{10}}{81N^4(N+1)^4} S_1 - \frac{32P_{11}}{243N^5(N+1)^5} - \frac{32P_{12}}{3N^3(N+1)^3} S_2 \Bigg] \Bigg\}
\end{aligned}
$$

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gq,Q}$



The all-ε result constituting the color factor $T_F^2 n_f C_F$

$$\begin{aligned}\hat{\hat{A}}^{(3)}_{gq,T_F^2 n_f} = {}& -96 a_s^3 T_F^2 n_f C_F \left(\frac{m^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}} S_\varepsilon^3 \frac{1+(-1)^N}{2} e^{-\frac{3\varepsilon}{2}\gamma} \frac{(\varepsilon-1)^2(\varepsilon+2)\left(\varepsilon+N^2+N+2\right)}{\varepsilon(\varepsilon+1)(\varepsilon+3)} \\ & \times\Gamma(1-\varepsilon)^2\Gamma\left(-\frac{\varepsilon}{2}-4\right)\Gamma\left(\frac{\varepsilon}{2}+2\right)\frac{\Gamma\left(\frac{\varepsilon}{2}+5\right)\Gamma\left(-\frac{3\varepsilon}{2}\right)\Gamma(N-1)}{\Gamma(4-2\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+N+2\right)}\end{aligned}$$

yields the renormalized contribution

$$\begin{aligned}A_{gq,Q}^{(3),n_f T_F^2,\overline{\mathsf{MS}}} = {}& n_f T_F^2 \frac{1+(-1)^N}{2}\Biggl\{\mathbf{C_F}\frac{32\left(N^2+N+2\right)}{9(N-1)N(N+1)}\mathbf{\ln^3}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right)+\mathbf{C_F}\Biggl[-\frac{16\left(N^2+N+2\right)}{3(N-1)N(N+1)}\left(S_1^2+S_2\right) \\ & +\frac{32\left(8N^3+13N^2+27N+16\right)}{9(N-1)N(N+1)^2}S_1+\frac{32\left(19N^4+81N^3+86N^2+80N+38\right)}{27(N-1)N(N+1)^3}\Biggr]\mathbf{\ln}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right) \\ & +\mathbf{C_F}\Biggl[\frac{32\left(N^2+N+2\right)}{27(N-1)N(N+1)}\left(S_1^3+3S_2S_1+2S_3-24\zeta_3\right)-\frac{32\left(8N^3+13N^2+27N+16\right)}{27(N-1)N(N+1)^2}\left(S_1^2+S_2\right) \\ & +\frac{64\left(4N^4+4N^3+23N^2+25N+8\right)}{27(N-1)N(N+1)^3}S_1+\frac{64\left(197N^5+824N^4+1540N^3+1961N^2+1388N+394\right)}{243(N-1)N(N+1)^4}\Biggr]\Biggr\}\end{aligned}$$

Here we confirm the n_f contribution to the anomalous dimension:

[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$\begin{aligned}\hat{\gamma}_{gq}^{(2),n_f} \quad = \quad & n_f T_F^2 C_F \Bigg(\frac{64\,(N^2+N+2)}{3(N-1)N(N+1)} - \left(S_1^2 + S_2\right) + \frac{128\,(8N^3+13N^2+27N+16)}{9(N-1)N(N+1)^2} S_1 \\ & - \frac{128\,(4N^4+4N^3+23N^2+25N+8)}{9(N-1)N(N+1)^3} \Bigg)\end{aligned}$$

in an independent calculation.

Furthermore we are able to check a result for the combination

$$\tilde{\gamma}_{gg}^{(2)} + \frac{\tilde{\gamma}_{gq}^{(2)} \gamma_{qg}^{(0)}}{\tilde{\gamma}_{gg}^{(0)} n_f}$$

of 3-loop anomalous dimensions, derived from the large n_f expansion in QCD by [Bennett, Gracey 1997]; where we denote with $\tilde{\gamma}_{ij}^{(k)}$ the leading n_f coefficient of $\gamma_{ij}^{(k)}$.

Graphs with m_c and m_b

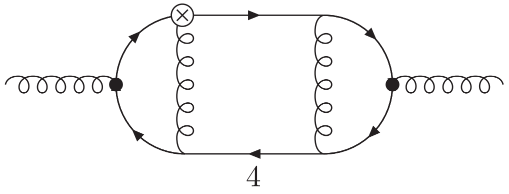
$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) \quad = \quad & T_F^2 C_A \Biggl\{ \frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\Bigr] + \ln^3\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\frac{324148}{19845} + \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{156992}{6615} \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\Bigr] + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{68332}{6615} \\
& + \ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\Bigr] + \ln^3\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{412808}{19845} \Biggr\} \\
& + T_F^2 C_F \Biggl\{ - \frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{1980566069882672}{2467763508585375}x^3 \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\Bigr] - \ln^3\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\frac{111848}{19845} - \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{223696}{46305} \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\Bigr] + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{223696}{46305} \\
& + \ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\Bigr] - \ln^3\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{1230328}{138915} \Biggr\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

These moments have been calculated referring `qexp` by Steinhauser et al. with operator insertions. Despite being universal, these contribution do not belong to the charm or bottom PDF. This is then the end of the VFNF.

Calculation of Convergent Massive 3-Loop Graphs

- Aim:
  - Compute fixed Mellin moments of convergent 3-loop diagrams
  - Find general N representations for all convergent 3-loop topologies
- We use the α-representation to solve the momentum integrals and obtain integrals
- The corresponding graph polynomials of a graph G are given by
  - $U = \sum_T \prod_{l \notin T} \alpha_l$, where T denotes the spanning trees of G
  - $V = \sum_{l \in massive} \alpha_l$
  - different Dodgson polynomials, which can be derived from the corresponding tadpole diagram, for the operator insertions

Calculation of Convergent Massive 3-Loop Graphs



$$\begin{aligned}
I_4(N) &= \int \cdots \int d\alpha_1\ d\alpha_2\ d\alpha_3\ d\alpha_4\ d\alpha_5\ d\alpha_6\ d\alpha_7\ d\alpha_8\ \frac{\sum_{j=0}^{N} T_{4\alpha}^{N-j} T_{4b}^{j}}{U^2 V^2} \\
T_{4\alpha} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad +\alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad +\alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad +\alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad +\alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad +\alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
V &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
\end{aligned}$$

- The integral above is a projective integral, one α-parameter may be set 1
- The operators sit on on-shell diagrams which obey specific symmetries. These are generally not obeyed by the operator insertion.
- For example above: After applying symmetry transformations $\alpha_1 \to x_1 - \alpha_2$, $\alpha_3 \to x_2 - \alpha_4$, $\alpha_5 \to x_5 - \alpha_6$ $\alpha_2, \alpha_4, \alpha_6$ are only contained in the operator polynomials and may be integrated out at this stage.

Calculation of Convergent Massive 3-Loop Graphs

- Feynman parameter integrals are performed in terms of Hyperlogarithms, [Brown 2008 Comm. Math. Phys.]

  $L(\overrightarrow{w}, z) : \mathbb{C}\backslash\Sigma \to \mathbb{C}$, where

  - $\Sigma = \{\sigma_0, \sigma_1, ..., \sigma_N\}$ are distinct points in $\mathbb{C}$ which may contain variables
  - $\overrightarrow{w}$ is a word over the alphabet $\mathfrak{A} = \{a_0, a_1, ..., a_N\}$ where each letter a_i corresponds to a point σ_i

- $L\left(\overrightarrow{w}, z\right)$ is uniquely defined by the following properties

  1. $L\left(\{\}, z\right) = 1$, and $L\left(0^n, z, \right) = \frac{1}{n!}\log^n(z)$ for $n \geq 1$
  2. $\frac{\partial}{\partial z} L\left(\{a_i \overrightarrow{w}\}, z\right) = \frac{1}{z-\sigma_i} L\left(\overrightarrow{w}, z\right)$ for $z \in \mathbb{C}\backslash\Sigma$
  3. If $\overrightarrow{w}$ is not of the form $w = (0, 0, \cdots, 0)$, then $\lim_{z\to 0} L\left(\overrightarrow{w}, z\right) = 0$.

- e.g. $L\left(\{a_i\}, z\right) = \log(z - \sigma_i) - \log(\sigma_i)$
- The weight of $L\left(\overrightarrow{w}, z\right)$ is given by the number of letters in $\overrightarrow{w}$

- The hyperlogarithms satisfy shuffle relations $L\left(\overrightarrow{w_1}, z\right) L\left(\overrightarrow{w_2}, z\right) = L\left(\overrightarrow{w_1} \sqcup\!\sqcup \overrightarrow{w_2}, z\right)$, e.g.: $L\left(\{a_1, a_2\}, z\right) L\left(\{a_3\}, z\right) = L\left(\{a_3, a_1, a_2\}, z\right) + L\left(\{a_1, a_3, a_2\}, z\right) + L\left(\{a_1, a_2, a_3\}, z\right)$
- using these properties after partial fractioning and integration by parts, one can express any primitive for expressions consisting of rational and hyperlogarithmic functions in terms of different hyperlogarthmic functions
- These primitives have to be evaluated at the respective integration limits
  - Limit at $z \to 0$ is trivially obtained by computing the regularized Taylor series for the hyperlogarithmic functions
  - Limit at $z \to \infty$ is more sophisticated. General idea:
    1. Choose the integration order
    2. Compute the derivative with respect to the next integration variable x, (this lowers the weight by one)
    3. Perform the series expansion of the derivative.
    4. Perform the indefinite integration with respect to x
    5. Determine the respective integration constant

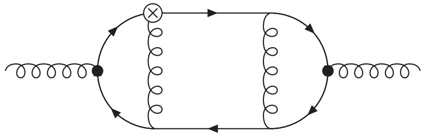
- For example: $lim_{y\to\infty}L\left(\{-x-1,0\},y\right)$
- Compute $\frac{\partial}{\partial x}L\left(\{-x-1,0\},y\right)$:
  - $\frac{\partial}{\partial x}\frac{\partial}{\partial y}L\left(\{-x-1,0\},y\right)=-\frac{L(\{0\},y)}{(y+1+x)^2}$

$$\begin{aligned}\frac{\partial}{\partial x}L\left(\{-x-1,0\},y\right) &= \int dy\frac{\partial}{\partial x}\frac{\partial}{\partial y}L\left(\{-x-1,0\},y\right)+Const\\ &= \left(\frac{1}{y+x+1}-\frac{1}{x+1}\right)L\left(\{0\},y\right)+\frac{L\left(\{-x-1\},y\right)}{x+1}+Const\end{aligned}$$

  - Fix $Const$ such that $\lim_{y\to 0}\frac{\partial}{\partial x}L\left(\{-x-1,0\},y\right)=0$, in this case $Const=0$
- The asymptotic limit of the derivative is then given by $-\frac{L(\{-1\},x1)}{x1+1}$
- $\int\lim_{y\to\infty}\frac{\partial}{\partial x}L\left(\{-x-1,0\},y\right)=L\left(\{-1,-1\},x1\right)$
- We have to add the respective integration constant, which is given by $\lim_{y\to\infty}L\left(\{-1,0\},y\right)=L\left(\{0,0\},y\right)-\zeta_2$
- Thus $lim_{y\to\infty}L\left(\{-x-1,0\},y\right)=L\left(\{-1,-1\},x1\right)+L\left(\{0,0\},y\right)-\zeta_2$.

Fixed Mellin Moments

- Using this method we have computed a number of fixed Mellin-Moments from $N = 0..19$ e.g.:



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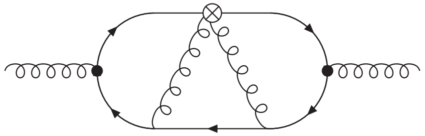

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N	Diag 4	Diag 5_a	Diag 5_b
0	$2 - 2\zeta_3$	$2\zeta_3$	$2\zeta_3$
1	$-2 + 2\zeta_3$	$-\frac{5}{2} - \zeta_3$	$-2 - 2\zeta_3$
2			
3			
4			
$\ldots$	$\ldots$	$\ldots$	$\ldots$
19	$-\frac{5825158236879253094413489658569181}{2503562235895708381108915200000}$ $-\frac{10489980717474386425 3}{541923759913536 00}\zeta_3$	$-\frac{1280902668906280290626432157835 49}{133523319247771113659142144000}$ $+\frac{2383887939492174 97}{301068755507520}\zeta_3$	$-\frac{2541169035757973854110502577 69}{25288507433289983647564800000 0}$ $-\frac{1968329}{635040}\zeta_3$

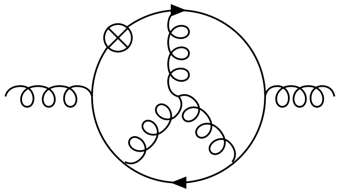
General Values of N

- Due to the operator-insertions leading to power-type functions, the integrals do not fit directly into the framework of the algorithm for general values of N
- In order to use the algorithm also on integrals with general values of N, a generating function is constructed e.g. by the mapping

$$p\left(\alpha_1, \cdots, \alpha_n\right)^N \to \frac{1}{1 - x\; p\left(\alpha_1, \cdots, \alpha_n\right)} \,.$$

- Performing the Feynman-parameter integrations then leads to an expression which contains Hyperlogarithms $L_w(x)$ in the variable x
- Finally the Nth coefficient of this expression in x has to be extracted analytically. This has been done with the package `HarmonicSums` by J.Ablinger

General Values of N

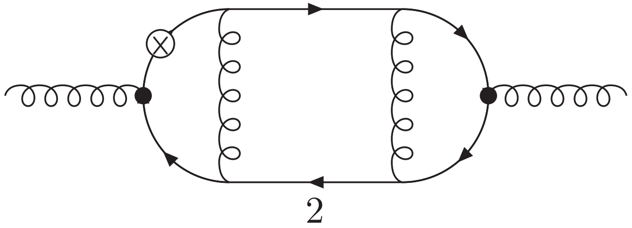


$$
\begin{aligned}
I(x) \quad = \quad & \frac{1}{(1+N)(2+N)x}\Bigg\{\zeta_3\Big[2L\left(\{-1\},x\right)-2(-1+2x)L\left(\{1\},x\right)-4L\left(\{1,1\},x\right)\Big]-3L\left(\{-1,0,0,1\},x\right)\\
& +2L\left(\{-1,0,1,1\},x\right)-2xL\left(\{0,0,1,1\},x\right)+3xL\left(\{0,1,0,1\},x\right)-xL\left(\{0,1,1,1\},x\right)\\
& +(-3+2x)L\left(\{1,0,0,1\},x\right)+2xL\left(\{1,0,1,1\},x\right)-(-1+5x)L\left(\{1,1,0,1\},x\right)+xL\left(\{1,1,1,1\},x\right)\\
& -2L\left(\{1,0,0,1,1\},x\right)+3L\left(\{1,0,1,0,1\},x\right)-L\left(\{1,0,1,1,1\},x\right)+2L\left(\{1,1,0,0,1\},x\right)\\
& +2L\left(\{1,1,0,1,1\},x\right)-5L\left(\{1,1,1,0,1\},x\right)+L\left(\{1,1,1,1,1\},x\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
I(N) \quad = \quad & \frac{1}{(N+1)(N+2)(N+3)}\Bigg\{\frac{648+1512N+1458N^2+744N^3+212N^4+32N^5+2N^6}{(1+N)^3(2+N)^3(3+N)^3}\\
& -\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3-(-1)^N S_{-3}-\frac{N}{6(1+N)}S_1^3+\frac{1}{24}S_1^4\\
& -\frac{\left(7+22N+10N^2\right)}{2(1+N)^2(2+N)}S_2-\frac{19}{8}S_2^2-\frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2+\frac{9}{4}S_2-\frac{(-9+4N)}{3(1+N)}S_3\\
& -\frac{1}{4}S_4-2(-1)^N S_{-2,1}+\frac{(-1+6N)}{(1+N)}S_{2,1}+\frac{54+207N+246N^2+130N^3+32N^4+3N^5}{(1+N)^3(2+N)^2(3+N)^2}S_1\\
& +4\zeta_3 S_1-\frac{(-2+7N)}{2(1+N)}S_2S_1+\frac{13}{3}S_3S_1-7S_{2,1}S_1-7S_{3,1}+10S_{2,1,1}\Bigg\}
\end{aligned}
$$

General Values of N

The method has been applied also to other graphs, e.g.:



$$\begin{aligned}
I_{2a} \quad = \quad & \frac{1}{(N+1)(N+2)(N+3)}\left\{\frac{8(3+2N)}{(1+N)(2+N)} - 2\left(-3+(-1)^N+2^{3+N}\right)\zeta_3\right. \\
& -(-1)^N S_{-3} + \frac{\left(16+12N+N^2\right)S_1^2}{2(1+N)(2+N)} + \frac{1}{6}S_1^3 + \frac{\left(56+40N+3N^2\right)S_2}{2(1+N)(2+N)} \\
& +\frac{4(3+2N)}{(1+N)(2+N)}S_1 - \frac{1}{2}S_1 S_2 - (17+3N)S_3 - 2(-1)^N S_{-2,1} \\
& \left. -(N+3)S_{2,1} + 2^{4+N}S_{1,2}\left(\frac{1}{2}\right) + 2^{3+N}S_{1,1,1}\left(\frac{1}{2}\right)\right\}
\end{aligned}$$

Conclusions

- A series of moments for the transition matrix elements A_{ij} at 3-loop order were given in [Bierenbaum, Blümlein, Klein 2009 Nucl. Phys. B].
- The corresponding quarkonic 3-loop contributions of $O(n_f T_F^2 C_{A,F})$ to A_{qq} and A_{qg} were calculated in [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl. Phys. B]. Now also $A_{gg,Q}$ and $A_{gq,Q}$ have been obtained for these color coefficients at general N.
- A series of OMEs were fully calculated A_{qq} and A_{qg} in $O(T_F^2 C_{A,F})$
- Ladder topologies, including poles, are currently calculated using `Sigma`.
- The moments N = 2,4,6 have been calculated for graphs depending on both m_c and m_b. Starting with 3-loops, graphs exist which conflict with the ideology of the VFNS. This was expected, since the heavy quarks are produced in the final state!
- With the help of Hyperlogarithms non-divergent 3-Loop graphs can be calculated, if moments are considered. For general values of N first analytic results have been obtained, including Benz-topologies, performing the calculation automatically.