Generalizations of Harmonic Sums and Polylogarithms and the Package HarmonicSums.

Jakob Ablinger
LHCPhenonet Fellow
DESY-Zeuthen, Germany
RISC, J. Kepler University Linz, Austria

joint work with
J. Blümlein (DESY) and C. Schneider (RISC)

LHCPhenoNet Annual Meeting March 2012

"Background"

- The following examples arise in the context of 2– and 3–loop massive single scale Feynman diagrams with operator insertion.
- These are related to the QCD anomalous dimensions and massive operator matrix elements.
- At 2-loop order all respective calculations are finished:

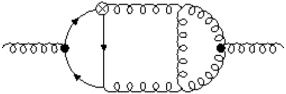
  M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;
  I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;
  I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008) 1;

  and lead to representations in terms of harmonic sums.

Example 1: All N-Results for 3–Loop Ladder Graphs

Joint work with J. Blümlein (DESY), C. Schneider (RISC)
A. Hasselhuhn (DESY), S. Klein (RWTH)

Consider, e.g., the diagram



(containing three massive fermion propagators)

$$\Downarrow$$

Around 1000 sums have to be calculated

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma!S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$
$$+\dots$$

where, e.g.,

$$S_{-2,1,-2}(N)=\sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{(-1)^k}{k^2}}{j}}{i^2}$$

Vermaseren 98;Blümlein/Kurth 98

A typical sum

$$\sum_{j=0}^{N-2}\sum_{s=1}^{j+1}\sum_{r=0}^{N+s-j-2}\sum_{\sigma=0}^{\infty}\frac{-2(-1)^{s+r}\binom{j+1}{s}\binom{-j+N+s-2}{r}(N-j)!(s-1)!\sigma! S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$=\frac{\left(2N^2+6N+5\right)S_{-2}(N)^2}{2(N+1)(N+2)}+S_{-2,-1,2}(N)+S_{-2,1,-2}(N)$$

$$+\cdots-S_{2,1,1,1}(-1,2,\frac{1}{2},-1;N)+S_{2,1,1,1}(1,\frac{1}{2},1,2;N)$$

$$+\ldots$$

where, e.g.,

145 S–sums occur

$$S_{2,1,1,1}(1,\frac{1}{2},1,2;N)=\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\left(\frac{1}{2}\right)^{j}\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02

S-Sums

For $a_i \in \mathbb{N}$ and $x_i \in \mathbb{R}^*$ we define

$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n) = \sum_{n \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{a_1}} \cdots \frac{x_k^{i_k}}{i_k^{a_k}}.$$

S-Sums

For $a_i \in \mathbb{N}$ and $x_i \in \mathbb{R}^*$ we define

$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n) = \sum_{n \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{a_1}} \cdots \frac{x_k^{i_k}}{i_k^{a_k}}.$$

Example

$$S_{2,3,1}\left(2,\frac{1}{2},-1;n\right) = \sum_{k=1}^{n} \frac{2^k \sum_{j+1}^{k} \frac{\left(\frac{1}{2}\right)^j \sum_{i=1}^{j} \frac{(-1)^i}{i}}{j^3}}{k^2}$$

$$S_{2,3,1}(-1,1,-1;n) = \sum_{k=1}^{n} \frac{(-1)^k \sum_{j+1}^{k} \frac{\sum_{i=1}^{j} \frac{(-1)^i}{i}}{j^3}}{k^2} = S_{-2,3,-1}(n)$$

Algebraic Relations

$$\mathrm{S}_{a_1,a_1,a_2}(x_1,x_2,x_3;n) =$$
$$\mathrm{S}_{a_2}(x_3;n)\,\mathrm{S}_{a_1,a_1}(x_1,x_2;n) + \mathrm{S}_{a_1,a_1+a_2}(x_1,x_2x_3;n) - \mathrm{S}_{a_1}(x_1;n)$$
$$\mathrm{S}_{a_2,a_1}(x_3,x_2;n) - \mathrm{S}_{a_2,2a_1}(x_3,x_1x_2;n) + \mathrm{S}_{a_2,a_1,a_1}(x_3,x_2,x_1;n),$$

$$\mathrm{S}_{a_1,a_1,a_2}(x_1,x_3,x_2;n) =$$
$$\mathrm{S}_{a_2}(x_2;n)\,\mathrm{S}_{a_1,a_1}(x_1,x_3;n) + \mathrm{S}_{a_1,a_1+a_2}(x_1,x_2x_3;n) - \mathrm{S}_{a_1}(x_1;n)\,\mathrm{S}_{a_2,a_1}(x_2,x_3;n)$$
$$-\mathrm{S}_{a_2,2a_1}(x_2,x_1x_3;n) + \mathrm{S}_{a_2,a_1,a_1}(x_2,x_3,x_1;n),$$

$$\mathrm{S}_{a_1,a_1,a_2}(x_2,x_1,x_3;n) =$$
$$-\mathrm{S}_{a_2}(x_3;n)\,\mathrm{S}_{a_1,a_1}(x_1,x_2;n) + \mathrm{S}_{a_1}(x_2;n)\,\mathrm{S}_{a_1,a_2}(x_1,x_3;n) + \mathrm{S}_{2a_1,a_2}(x_1x_2,x_3;n)$$
$$-\mathrm{S}_{a_1+a_2,a_1}(x_1x_3,x_2;n) + \mathrm{S}_{a_2,a_1,a_1}(x_3,x_1,x_2;n)$$

Differential Relations

- integral representation

$$S_{1,2,1}\left(2,\frac{1}{2},1;n\right) = \int_0^1 \frac{1}{x_1-1}\int_1^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3-2}\int_2^{x_3}\frac{x_4^n-1}{x_4-1}dx_4dx_3dx_2dx_1$$

Differential Relations

- integral representation

$$S_{1,2,1}\left(2,\frac{1}{2},1;n\right) = \int_0^1 \frac{1}{x_1-1}\int_1^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3-2}\int_2^{x_3}\frac{x_4^n-1}{x_4-1}dx_4dx_3dx_2dx_1$$

- differentiation

$$\frac{\partial S_{1,2,1}\left(2,\frac{1}{2},1;n\right)}{\partial n} = \int_0^1 \frac{1}{x_1-1}\int_1^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3-2}\int_2^{x_3}\frac{x_4^n\log(x_4)}{x_4-1}dx_4dx_3dx_2dx_1$$

Differential Relations

- integral representation

$$S_{1,2,1}\left(2,\frac{1}{2},1;n\right) = \int_0^1 \frac{1}{x_1-1}\int_1^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3-2}\int_2^{x_3}\frac{x_4^n-1}{x_4-1}dx_4dx_3dx_2dx_1$$

- differentiation

$$\frac{\partial S_{1,2,1}\left(2,\frac{1}{2},1;n\right)}{\partial n} = \int_0^1 \frac{1}{x_1-1}\int_1^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3-2}\int_2^{x_3}\frac{x_4^n\log(x_4)}{x_4-1}dx_4dx_3dx_2dx_1$$

- integration of right hand side

$$\begin{aligned}\frac{\partial S_{1,2,1}\left(2,\frac{1}{2},1;n\right)}{\partial n} =& -2H_{0,0,1,2,1}(1)-4H_{0,0,2,1,1}(1)-H_{0,1,0,2,1}(1)-H_{0,1,2,0,1}(1)-2H_{0,2,0,1,1}(1)\\ &-H_{0,2,1,0,1}(1)+(H_2(1)(H_{0,1,2}(1)+H_{0,2,1}(1))+2H_{0,0,1,2}(1)+2H_{0,0,2,1}(1)\\ &+H_{0,1,0,2}(1))S_1(2;n)+S_2(\infty)\left(\frac{3(H_{0,1,2}(1)+H_{0,2,1}(1))}{2}+S_{1,2}\left(2,\frac{1}{2};n\right)\right)\\ &-H_2(1)S_{1,2,1}\left(2,\frac{1}{2},1;n\right)-S_{-1}(\infty)S_{1,2,1}\left(2,\frac{1}{2},1;n\right)-S_{1,2,2}\left(2,\frac{1}{2},1;n\right)\\ &-2S_{1,3,1}\left(2,\frac{1}{2},1;n\right)-S_{2,2,1}\left(2,\frac{1}{2},1;n\right)\end{aligned}$$

Multiple Polylogarithms (M-Logs)

Let $a \in \mathbb{R}$ and

$$q = \begin{cases} a, & \text{if } a > 0 \\ \infty, & \text{otherwise} \end{cases}$$

We define f as follows:

$$f_a : (0, q) \mapsto \mathbb{R}$$
$$f_a(x) = \begin{cases} \frac{1}{x}, & \text{if } a = 0 \\ \frac{1}{|a| - \text{sign}(a)\, x}, & \text{otherwise.} \end{cases}$$

Multiple Polylogarithms (M-Logs)

Let $a \in \mathbb{R}$ and

$$q = \begin{cases} a, & \text{if } a > 0 \\ \infty, & \text{otherwise} \end{cases}$$

We define f as follows:

$$f_a : (0, q) \mapsto \mathbb{R}$$
$$f_a(x) = \begin{cases} \frac{1}{x}, & \text{if } a = 0 \\ \frac{1}{|a| - \text{sign}(a)\, x}, & \text{otherwise.} \end{cases}$$

Multiple polylogarithms: Let $m_i \in \mathbb{R}$ and let $q = \min_{m_i > 0} m_i$, we define for $x \in (0, q)$:

$$\begin{aligned} \mathrm{H}(x) &= 1, \\ \mathrm{H}_{m_1, m_2, \ldots, m_k}(x) &= \begin{cases} \frac{1}{k!}(\log x)^k, & \text{if } (m_1, \ldots, m_k) = \mathbf{0} \\ \int_0^x f_{m_1}(y) \mathrm{H}_{m_2, \ldots, m_k}(y) dy, & \text{otherwise.} \end{cases} \end{aligned}$$

Number of Basic S-Sums

We consider

$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n)$$

with $x_i \in \{1,-1,1/2,-1/2,2,-2\}$.

We consider

$$S_{a_1,\dots,a_k}(x_1,\dots,x_k;n)$$

with $x_i \in \{1,-1,1/2,-1/2,2,-2\}$.

Example (Harmonic Sums ($x_i \in \{1,-1\}$))

w	Number of							
	All	N_A	N_D	N_H	N_{AD}	N_{AH}	N_{DH}	N_{ADH}
1	2	2	2	1	2	1	1	1
2	6	3	4	4	1	2	3	1
3	18	8	12	14	5	6	10	4
4	54	18	36	46	10	15	32	9
5	162	48	108	146	30	42	100	27
6	486	116	324	454	68	107	308	65
7	1458	312	972	1394	196	294	940	187
8	4374	810	2916	4246	498	780	2852	486

Number of Basic S-Sums

We consider

$$S_{a_1,\ldots,a_k}(x_1,\ldots,x_k;n)$$

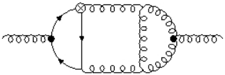
with $x_i \in \{1,-1,1/2,-1/2,2,-2\}$.

Example

Each of the indices $\{1/2,-1/2,2,-2\}$ is allowed to appear just once in each sum.

Weight	Number of			
	All	N_A	N_D	N_{AD}
1	6	6	6	6
2	38	23	32	17
3	222	120	184	97
4	1206	654	984	543
5	6150	3536	4944	2882
6	29718	18280	23568	14744

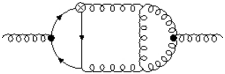
Example 1: continued



$\Downarrow$ C. Schneider's Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums

Example 1: continued



$\Downarrow$ C. Schneider's Sigma.m

Around 1000 sums are calculated containing in total 533 S–sums

$\Downarrow$ HarmonicSums.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_{1}(N), S_{2}(N), S_{3}(N), S_{4}(N), S_{-3,1}(N),$$
$$S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

We say that the function $f : \mathbb{R} \to \mathbb{R}$ is expanded in an asymptotic series

$$f(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k}, \; x \to \infty,$$

where a_k are constants, if for all $K \geq 0$

$$R_K(x) = f(x) - \sum_{k=0}^{K} \frac{a_k}{x^k} = o\left(\frac{1}{x^K}\right), \; x \to \infty.$$

Why do we need these expansions of harmonic sums?

E.g.,

- for limits of the form

$$\lim_{n\to\infty} n\left(S_2(n) - \zeta_2 - S_{2,2}(n) + \frac{7\,\zeta_2^2}{10}\right)$$

- for the approximation of the values of analytic continued harmonic sums at the complex plane

$$S_{2,-3}(-20+10i)$$

Basic Idea

$$S_{-1,3}(n) = (-1)^n \int_0^1 x^n \frac{\mathrm{H}_{1,0,0}(x)}{1+x} dx + const$$

Basic Idea

$$S_{-1,3}(n) = (-1)^n \int_0^1 x^n \overbrace{\frac{\mathrm{H}_{1,0,0}(x)}{1+x}}^{\varphi(x)} dx + const$$

Basic Idea

$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

$$S_{-1,3}(n) = (-1)^n \int_0^1 x^n \frac{\overbrace{\mathrm{H}_{1,0,0}(x)}}{1+x} dx + const$$

Basic Idea

$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

$$S_{-1,3}(n) = \underbrace{(-1)^n \int_0^1 x^n \frac{\overbrace{\mathrm{H}_{1,0,0}(x)}}{1+x} dx}_{\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1} k!}{n(n+1)\dots(n+k)}} + const$$

Basic Idea

$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

$$S_{-1,3}(n) = \underbrace{(-1)^n \int_0^1 x^n \frac{\overbrace{\mathrm{H}_{1,0,0}(x)}}{1+x} dx}_{} + const$$

$$\sum_{k=0}^{\infty} \frac{a_{k+1} k!}{n(n+1)\dots(n+k)} = \sum_{k=1}^{\infty} \frac{b_k}{n^k}$$

Basic Idea

$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

$$S_{-1,3}(n) = \underbrace{(-1)^n \int_0^1 x^n \frac{\overbrace{\mathrm{H}_{1,0,0}(x)}}{1+x} dx}\, + const$$

$$\sum_{k=0}^{\infty} \frac{a_{k+1} k!}{n(n+1)\dots(n+k)} = \sum_{k=1}^{\infty} \frac{b_k}{n^k}$$

$$b_1 = a_1$$

$$b_k = \sum_{l=0}^{k-2} (-1)^l S_{k-l}^l a_{k-l} (k-l)!$$

Basic Idea

$$\varphi(x) \longrightarrow \varphi(1-x) = \sum_{k=0}^{\infty} a_k x^k$$

$$S_{-1,3}(n) = (-1)^n \underbrace{\int_0^1 x^n \overbrace{\frac{\mathrm{H}_{1,0,0}(x)}{1+x}}\, dx}_{} + const$$

$$\sum_{k=0}^{\infty} \frac{a_{k+1} k!}{n(n+1)\dots(n+k)} = \sum_{k=1}^{\infty} \frac{b_k}{n^k}$$

$$b_1 = a_1$$

$$b_k = \sum_{l=0}^{k-2} (-1)^l S_{k-l}^l a_{k-l} (k-l)!$$

$$\begin{aligned} S_{-1,3}(n) \sim\ & (-1)^n \left(-\frac{1}{4n^3} + \frac{5}{8n^4} - \frac{5}{8n^5} - \frac{5}{16n^6} \right) + \frac{3\ \log(2)\ \zeta_3}{4} \\ & + (-1)^n \left(\frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{8n^4} - \frac{1}{4n^6} \right) \zeta_3 - \frac{19\ \zeta_2^2}{40} \end{aligned}$$

Example:

In[1]:= **SExpansion[S[-1, 3, n], n, 10]**

Out[1]=

$$(-1)^n\left(-\frac{1}{4n^3}+\frac{5}{8n^4}-\frac{5}{8n^5}-\frac{5}{16n^6}+\frac{31}{24n^7}+\frac{133}{96n^8}-\frac{169}{24n^9}-\frac{163}{16n^{10}}\right)+\frac{3\,\ln2\ z3}{4}+(-1)^n\left(\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{8n^4}-\frac{1}{4n^6}+\frac{17}{16n^8}-\frac{31}{4n^{10}}\right)z3-\frac{19\ z2^2}{40}$$

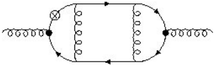
In[2]:= **GetApproximation[S[-1,3,n], {-2.5, 2}]**

Out[2]= $-0.795096 - 0.105476\ i$

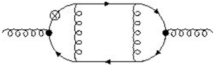
In[3]:= **HLimit[n*(S[2, n] - z2 - S[2, 2, n] + 7*z2^2/10), n]**

Out[3]= $-1+z2$

We consider another diagram:



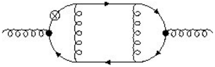
We consider another diagram:



$$\downarrow \quad \text{Sigma} \ldots$$

$$
\begin{aligned}
I_3 = & \frac{C_3}{(N+1)(N+2)(N+3)}\Bigg\{\frac{1}{6}S_1^3+\frac{N^2+12N+16}{2(N+1)(N+2)}S_1^2+\frac{4(2N+3)}{(N+1)^2(N+2)}S_1 \\
& +\frac{8(2N+3)}{(N+1)^3(N+2)}+2\left[-2^{N+3}+3-(-1)^N\right]\zeta_3-(-1)^N S_{-3}+\left[\frac{3N^2+40N+56}{2(N+1)(N+2)}-\frac{1}{2}S_1\right]S_2 \\
& -\frac{3N+17}{3}S_3-2(-1)^N S_{-2,1}-(N+3)S_{2,1}+2^{N+4}S_{1,2}\left(\frac{1}{2},1;N\right) \\
& +2^{N+3}S_{1,1,1}\left(\frac{1}{2},1,1;N\right)\Bigg\}+O(\epsilon)
\end{aligned}
$$

We consider another diagram:



Sigma ...

$$I_3 = \frac{C_3}{(N+1)(N+2)(N+3)}\Bigg\{\frac{1}{6}S_1^3 + \frac{N^2+12N+16}{2(N+1)(N+2)}S_1^2 + \frac{4(2N+3)}{(N+1)^2(N+2)}S_1$$
$$+\frac{8(2N+3)}{(N+1)^3(N+2)} + 2\left[-2^{N+3}+3-(-1)^N\right]\zeta_3 - (-1)^N S_{-3} + \left[\frac{3N^2+40N+56}{2(N+1)(N+2)} - \frac{1}{2}S_1\right]S_2$$
$$-\frac{3N+17}{3}S_3 - 2(-1)^N S_{-2,1} - (N+3)S_{2,1} + 2^{N+4}S_{1,2}\left(\frac{1}{2},1;N\right)$$
$$+2^{N+3}S_{1,1,1}\left(\frac{1}{2},1,1;N\right)\Bigg\} + O(\epsilon)$$

remaining sums:

$$\mathrm{S}_1(n), \mathrm{S}_2(n), \mathrm{S}_3(n), \mathrm{S}_{-3}(n), \mathrm{S}_{2,1}(n), \mathrm{S}_{-2,1}(n), \mathrm{S}_{1,2}\left(\frac{1}{2},1;n\right), \mathrm{S}_{1,1,1}\left(\frac{1}{2},1,1;n\right)$$

Asymptotic Expansion: Basic Idea

- again we start from the integral representation

$$S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) = \int_0^{\frac{1}{6}} \frac{1}{x_1-\frac{1}{6}} \int_{\frac{1}{6}}^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3-\frac{1}{3}} \int_{\frac{1}{3}}^{x_3} \frac{x_4^n-1}{x_4-1} dx_4 dx_3 dx_2 dx_1$$

Asymptotic Expansion: Basic Idea

- again we start from the integral representation

$$S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) = \int_0^{\frac{1}{6}} \frac{1}{x_1-\frac{1}{6}} \int_{\frac{1}{6}}^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3-\frac{1}{3}} \int_{\frac{1}{3}}^{x_3} \frac{x_4^n-1}{x_4-1} dx_4 dx_3 dx_2 dx_1$$

- integration by parts leads to integrals of depth one:

$$\begin{aligned} S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) = & -\left(H_{\frac{1}{3},0,\frac{1}{6}}(\frac{1}{6}) - H_{\frac{1}{3}}(\frac{1}{6})H_{0,\frac{1}{6}}(\frac{1}{6})\right) \int_{\frac{1}{6}}^{\frac{1}{3}} \frac{x^n-1}{x-1} dx + H_{0,\frac{1}{6}}(\frac{1}{6}) \int_0^{\frac{1}{6}} \frac{x^n-1}{x-1} H_{\frac{1}{3}}(x) dx \\ & - \int_0^{\frac{1}{6}} \frac{x^n-1}{x-1} H_{\frac{1}{3},0,\frac{1}{6}}(x) dx \end{aligned}$$

Asymptotic Expansion: Basic Idea

- again we start from the integral representation

$$S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) = \int_0^{\frac{1}{6}} \frac{1}{x_1-\frac{1}{6}} \int_{\frac{1}{6}}^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{x_3-\frac{1}{3}} \int_{\frac{1}{3}}^{x_3} \frac{x_4^n-1}{x_4-1} dx_4 dx_3 dx_2 dx_1$$

- integration by parts leads to integrals of depth one:

$$\begin{aligned} S_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right) &= -\left(H_{\frac{1}{3},0,\frac{1}{6}}(\frac{1}{6}) - H_{\frac{1}{3}}(\frac{1}{6})H_{0,\frac{1}{6}}(\frac{1}{6})\right)\int_{\frac{1}{6}}^{\frac{1}{3}} \frac{x^n-1}{x-1}dx + H_{0,\frac{1}{6}}(\frac{1}{6})\int_0^{\frac{1}{6}} \frac{x^n-1}{x-1}H_{\frac{1}{3}}(x)dx \\ &\quad - \int_0^{\frac{1}{6}} \frac{x^n-1}{x-1}H_{\frac{1}{3},0,\frac{1}{6}}(x)dx \end{aligned}$$

- as before for harmonic sums we can expand these integrals and get

$$\begin{aligned} \int_{\frac{1}{6}}^{\frac{1}{3}} \frac{x^n-1}{x-1}dx &\sim -H_1(\frac{1}{6}) + H_1(\frac{1}{3}) + 3^{-n}\left(-\frac{15}{n^5} + \frac{33}{8n^4} - \frac{3}{2n^3} + \frac{3}{4n^2} - 2^{-n}\left(-\frac{4074}{3125n^5} + \frac{366}{625n^4} \right.\right. \\ &\quad \left.\left. - \frac{42}{125n^3} + \frac{6}{25n^2} - \frac{1}{5n}\right) - \frac{1}{2n}\right) \\ \int_0^{\frac{1}{6}} \frac{x^n-1}{x-1}H_{\frac{1}{3}}(x)dx &\sim H_{1,\frac{1}{3}}(\frac{1}{6}) + 6^{-n}\left(\log(2)\left(-\frac{4074}{3125n^5} + \frac{366}{625n^4} - \frac{42}{125n^3} + \frac{6}{25n^2} - \frac{1}{5n}\right) - \frac{10834}{625n^5} \right. \\ &\quad \left. + \frac{456}{125n^4} - \frac{22}{25n^3} + \frac{1}{5n^2}\right) \end{aligned}$$

$$
\begin{aligned}
&\mathrm{S}_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;n\right)\sim\\
&\mathrm{S}_{1,2,1}\left(\frac{1}{3},\frac{1}{2},1;\infty\right)+3^{-n}\Bigg(2^{-n}\Bigg(L(n)\Bigg(\frac{42\mathrm{H}_{0,2}(1)}{125n^3}-\frac{6\mathrm{H}_{0,2}(1)}{25n^2}+\frac{\mathrm{H}_{0,2}(1)}{5n}+\log(2)^2\left(\frac{21}{125n^3}-\frac{3}{25n^2}+\frac{1}{10n}\right)\\
&-\frac{21\zeta_2}{125n^3}+\frac{1}{5n^3}+\frac{3\zeta_2}{25n^2}-\frac{\zeta_2}{10n}\Bigg)+\log(2)\left(-\frac{42\mathrm{H}_{0,2}(1)}{125n^3}+\frac{6\mathrm{H}_{0,2}(1)}{25n^2}-\frac{\mathrm{H}_{0,2}(1)}{5n}\right)+\frac{42\mathrm{H}_2(1)\mathrm{H}_{0,2}(1)}{125n^3}\\
&-\frac{2461\mathrm{H}_{0,2}(1)}{1500n^3}+\frac{126\mathrm{H}_{0,0,2}(1)}{125n^3}-\frac{42(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,2,2}(1))}{125n^3}-\frac{42(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,6,2}(1))}{125n^3}-\frac{42(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{2,0,2}(1))}{125n^3}\\
&-\frac{42(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{6,0,2}(1))}{125n^3}-\frac{42\mathrm{H}_{6,2,0}(1)}{125n^3}-\frac{6\mathrm{H}_2(1)\mathrm{H}_{0,2}(1)}{25n^2}+\frac{27\mathrm{H}_{0,2}(1)}{50n^2}-\frac{18\mathrm{H}_{0,0,2}(1)}{25n^2}+\frac{6(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,2,2}(1))}{25n^2}\\
&+\frac{6(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,6,2}(1))}{25n^2}+\frac{6(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{2,0,2}(1))}{25n^2}+\frac{6(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{6,0,2}(1))}{25n^2}+\frac{6\mathrm{H}_{6,2,0}(1)}{25n^2}\\
&+\frac{\mathrm{H}_2(1)\mathrm{H}_{0,2}(1)}{5n}+\frac{3\mathrm{H}_{0,0,2}(1)}{5n}-\frac{\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,2,2}(1)}{5n}-\frac{\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,6,2}(1)}{5n}-\frac{\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{2,0,2}(1)}{5n}\\
&-\frac{\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{6,0,2}(1)}{5n}-\frac{\mathrm{H}_{6,2,0}(1)}{5n}+\log(2)^3\left(-\frac{7}{125n^3}+\frac{1}{25n^2}-\frac{1}{30n}\right)+\log(2)^2\left(\frac{27}{100n^2}-\frac{2461}{3000n^3}\right)\\
&+\frac{2461\zeta_2}{3000n^3}+\frac{63\zeta_3}{250n^3}-\frac{27\zeta_2}{100n^2}-\frac{9\zeta_3}{50n^2}+\frac{3\zeta_3}{20n}\Bigg)-\frac{3\mathrm{H}_2(1)\mathrm{H}_{0,2}(1)}{2n^3}-\frac{9\mathrm{H}_{0,0,2}(1)}{2n^3}+\frac{3(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,2,2}(1))}{2n^3}\\
&+\frac{3(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{2,0,2}(1))}{2n^3}+\frac{3\mathrm{H}_2(1)\mathrm{H}_{0,2}(1)}{4n^2}+\frac{9\mathrm{H}_{0,0,2}(1)}{4n^2}-\frac{3(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,2,2}(1))}{4n^2}-\frac{3(\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{2,0,2}(1))}{4n^2}\\
&-\frac{\mathrm{H}_2(1)\mathrm{H}_{0,2}(1)}{2n}-\frac{3\mathrm{H}_{0,0,2}(1)}{2n}+\frac{\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{0,2,2}(1)}{2n}+\frac{\mathrm{H}_{0,0,2}(1)+\mathrm{H}_{2,0,2}(1)}{2n}\Bigg)
\end{aligned}
$$

Cyclotomic Harmonic Sums

$$S_{(a_1,b_1,c_1),\dots,(a_l,b_l,c_l)}(s_1,\dots,s_l;n) = \sum_{k_1=1}^{n} \frac{s_1^k}{(a_1k_1+b_1)^{c_1}} S_{(a_2,b_2,c_2),\dots,(a_l,b_l,c_l)}(s_2,\dots,s_l;k_1),$$

with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}, b_i \in \mathbb{N}_0, \quad s_i = \pm 1, a_i > b_i$.

Cyclotomic Harmonic Sums

$$S_{(a_1,b_1,c_1),\ldots,(a_l,b_l,c_l)}(s_1,\ldots,s_l;n) = \sum_{k_1=1}^{n} \frac{s_1^k}{(a_1k_1+b_1)^{c_1}} S_{(a_2,b_2,c_2),\ldots,(a_l,b_l,c_l)}(s_2,\ldots,s_l;k_1),$$

with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}, b_i \in \mathbb{N}_0, \quad s_i = \pm 1, a_i > b_i.$

- algebraic relations

Cyclotomic Harmonic Sums

$$S_{(a_1,b_1,c_1),\ldots,(a_l,b_l,c_l)}(s_1,\ldots,s_l;n) = \sum_{k_1=1}^{n} \frac{s_1^k}{(a_1k_1+b_1)^{c_1}} S_{(a_2,b_2,c_2),\ldots,(a_l,b_l,c_l)}(s_2,\ldots,s_l;k_1),$$

with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}, b_i \in \mathbb{N}_0, \quad s_i = \pm 1, a_i > b_i$.

- algebraic relations
- differential relations

Cyclotomic Harmonic Sums

$$S_{(a_1,b_1,c_1),\ldots,(a_l,b_l,c_l)}(s_1,\ldots,s_l;n) = \sum_{k_1=1}^{n} \frac{s_1^k}{(a_1k_1+b_1)^{c_1}} S_{(a_2,b_2,c_2),\ldots,(a_l,b_l,c_l)}(s_2,\ldots,s_l;k_1),$$

with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}, b_i \in \mathbb{N}_0, \quad s_i = \pm 1, a_i > b_i$.

- algebraic relations
- differential relations
- multiple argument relations

Cyclotomic Harmonic Sums

$$S_{(a_1,b_1,c_1),\ldots,(a_l,b_l,c_l)}(s_1,\ldots,s_l;n) = \sum_{k_1=1}^{n} \frac{s_1^k}{(a_1 k_1 + b_1)^{c_1}} S_{(a_2,b_2,c_2),\ldots,(a_l,b_l,c_l)}(s_2,\ldots,s_l;k_1),$$

with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}, b_i \in \mathbb{N}_0, \quad s_i = \pm 1, a_i > b_i$.

- algebraic relations
- differential relations
- multiple argument relations
- asymptotic expansion

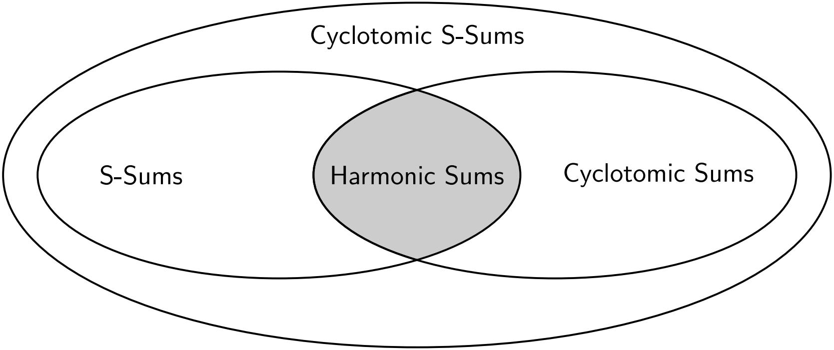
Cyclotomic Harmonic Sums

$$S_{(a_1,b_1,c_1),\ldots,(a_l,b_l,c_l)}(s_1,\ldots,s_l;n) = \sum_{k_1=1}^{n} \frac{s_1^k}{(a_1k_1+b_1)^{c_1}} S_{(a_2,b_2,c_2),\ldots,(a_l,b_l,c_l)}(s_2,\ldots,s_l;k_1),$$

with $S_\emptyset = 1$ and $N, a_i, c_i \in \mathbb{N}, b_i \in \mathbb{N}_0, \quad s_i = \pm 1, a_i > b_i$.

- algebraic relations
- differential relations
- multiple argument relations
- asymptotic expansion
- allowing $s_i \in \mathbb{R}$ leads to cyclotomic S-sums.

Harmonic Sums and their Generalizations



Cyclotomic Harmonic Polylogarithms (C-Logs)

We now define the alphabet

$$\mathfrak{A} := \left\{ \frac{1}{x} \right\} \cup \left\{ \frac{x^l}{\Phi_k(x)} \middle| k \in \mathbb{N}, 0 \leq l < \varphi(k) \right\},$$

where $\Phi_k(x)$ denotes the kth cyclotomic polynomial and φ is Euler's totient function.

Cyclotomic Harmonic Polylogarithms (C-Logs)

We now define the alphabet

$$\mathfrak{A} := \left\{ \frac{1}{x} \right\} \cup \left\{ \frac{x^l}{\Phi_k(x)} \middle| k \in \mathbb{N}, 0 \leq l < \varphi(k) \right\},$$

where $\Phi_k(x)$ denotes the kth cyclotomic polynomial and φ is Euler's totient function.

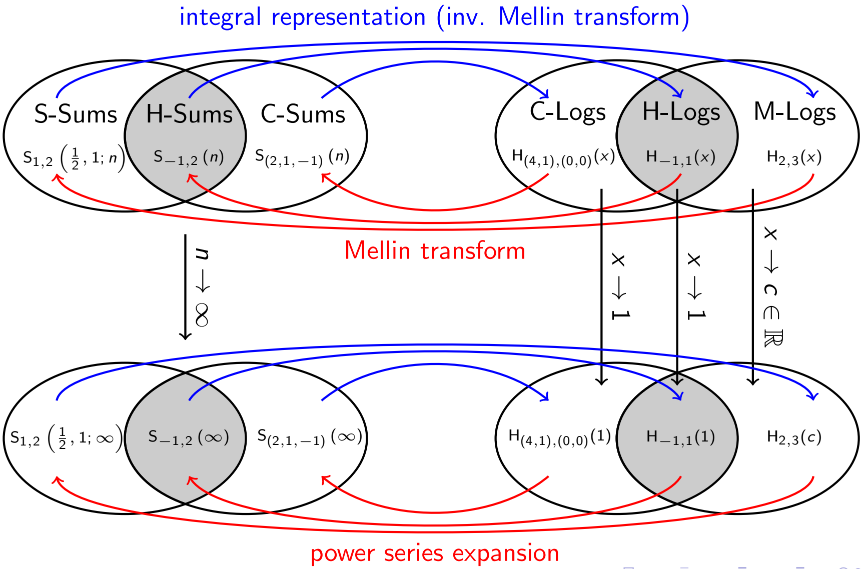
Definition (Cyclotomic Harmonic Polylogarithms)

Let $m_i \in \mathfrak{A}$ we define for $x \in (0, 1)$:

$$\begin{aligned} \mathrm{H}(x) &= 1, \\ \mathrm{H}_{m_1,m_2,\ldots,m_k}(x) &= \begin{cases} \frac{1}{k!}(\log x)^k, & \text{if } (m_1,\ldots,m_k) \\ & \quad = (\frac{1}{x},\ldots,\frac{1}{x}) \\ \int_0^x m_1 \mathrm{H}_{m_2,\ldots,m_k}(y)dy, & \text{otherwise.} \end{cases} \end{aligned}$$

k is called the depth of $\mathrm{H}_{\mathbf{m}}(x)$.

Connection between these structures



The Package HarmonicSums

The package HarmonicSums offers functions to

- find algebraic and structural relations of harmonic sums and their generalizations

The package HarmonicSums offers functions to

- find algebraic and structural relations of harmonic sums and their generalizations
- compute the inverse Mellin transform of harmonic sums and their generalizations, this leads to harmonic polylogarithms and their generalizations

The Package HarmonicSums

The package `HarmonicSums` offers functions to

- find algebraic and structural relations of harmonic sums and their generalizations
- compute the inverse Mellin transform of harmonic sums and their generalizations, this leads to harmonic polylogarithms and their generalizations
- find algebraic and structural relations of harmonic polylogarithms and their generalizations

The Package `HarmonicSums`

The package `HarmonicSums` offers functions to

- find algebraic and structural relations of harmonic sums and their generalizations
- compute the inverse Mellin transform of harmonic sums and their generalizations, this leads to harmonic polylogarithms and their generalizations
- find algebraic and structural relations of harmonic polylogarithms and their generalizations
- apply algebraic and structural relations to harmonic sums

The Package HarmonicSums

The package `HarmonicSums` offers functions to

- find algebraic and structural relations of harmonic sums and their generalizations
- compute the inverse Mellin transform of harmonic sums and their generalizations, this leads to harmonic polylogarithms and their generalizations
- find algebraic and structural relations of harmonic polylogarithms and their generalizations
- apply algebraic and structural relations to harmonic sums
- calculate the asymptotic expansion of harmonic sums and their generalizations

The Package `HarmonicSums`

The package `HarmonicSums` offers functions to

- find algebraic and structural relations of harmonic sums and their generalizations
- compute the inverse Mellin transform of harmonic sums and their generalizations, this leads to harmonic polylogarithms and their generalizations
- find algebraic and structural relations of harmonic polylogarithms and their generalizations
- apply algebraic and structural relations to harmonic sums
- calculate the asymptotic expansion of harmonic sums and their generalizations
- perform several other tasks not mentioned in this talk (see, e.g., my PhD thesis, April 2012)