A Refined Approach to Holonomic Multi-Summation

Mark Round

ESR LHCPhenonet Fellow

(in collaboration with C Schneider)

Research Institute for Symbolic Computation
Johannes Kepler Universität
Linz, Austria

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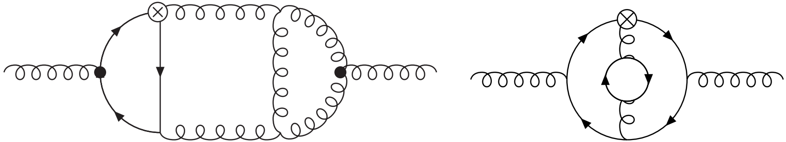
Talk Outline

1. A mathematical problem in particle physics
2. Existing tools
3. A refined holonomic approach
4. Illustrative result
5. Summary

- Consider perturbative QFT. Such as corrections to the high energy proton sub-structure from heavy quarks at 3-loops (and for $p^2 >> m^2_{\mathrm{HQ}}$). BIERENBAUM, BLÜMLEIN, KLEIN '09



- Such diagrams yield nasty integrals which are extremely hard to work with. A simpler form is needed. Our goal is to simplify the object.
- Physically speaking, the only restriction on the Feynman diagrams is that they contain atmost one mass.

- Momentum integrals are straightforward but the Feynman parameter integrals are not. Notice the operator insertion (the $\otimes$ in the diagram) will give us a Mellin parameter, n. Diagrams obey homogeneous difference equations in n. Blümlein, Kauers, Klein, Schneider '09
- A first step in trying to understand the remaining integrals is to convert them into definite nested hypergeometric and harmonic multi-sums. Blümlein & Kurth '98 and Vermaseren '98
- There is an algorithm to do this! Blümlein, Klein, Schneider & Stan '12

Statement of Problem

Any given Feynman diagram (with atmost one mass) can be expressed as a definite nested multi-sum,

$$F_n(\epsilon) = \sum_{i_1=\alpha_1}^{L_1(n)} \cdots \sum_{i_m=\alpha_m}^{L_m(n,i_1,\ldots i_m)} \sum_{j_1=\beta_1}^{\infty} \cdots \sum_{j_N=\beta_N}^{\infty} f(\epsilon, n, i_1, \ldots, i_m, j_1, \ldots, j_N).$$

In this talk $\epsilon = 0$.

Aim

Given a definite nested multi-sum find, in terms of indefinite nested multi-sums, another (simpler) representation.

$$\sum_{i=2}^{n+1}\sum_{j=2}^{n+2-i}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{\binom{n+2}{i}\binom{n+2-i}{j}(-1)^{i+j}B[i,j](j+i-1)}{(j+i+a+b)(i+j+a+b-1)(j+b)(i+b)}$$

$$\begin{aligned}
= \quad & \frac{2n^3+5n^2+4n-2}{n+1}S_2+\left[\frac{5}{2}S_2-\frac{n^2+2n+2}{n+1}\right]S_1^2-2nS_{2,1} \\
- \quad & 2n(n+1)\zeta_3 - S_1^3+\frac{1}{4}S_1^4-\frac{1}{4}S_2^2+2(n+1)S_3-\frac{1}{2}S_4-2S_{3,1} \\
+ \quad & [4\zeta_3+(2n-1)S_2+2S_3-2(n+1)-4S_{2,1}]\,S_1+2S_{2,1,1}
\end{aligned}$$

where $S_{a,\ldots,b} = S_{a,\ldots,b}(n)$ & $B[x,y] = \Gamma[x]\Gamma[y]/\Gamma[x+y]$ for convenience.

- We have gone from a tough to simple, if long, expression.

Existing Tools

Sigma

- To solve summation problems one needs a general toolkit to apply. Of the examples available, we will use a collection of algorithms found in Schneider '04, '05... related to Karr '81 & Chyzak '00.
- A Mathematica package, Sigma, is available that implements the algorithms.
- Sigma exploits the mathematical frame work of refined difference field theory.
- Note that Sigma computes linear recurrences with polynomial co-efficients.

$$F_n = \sum_{i_1=\alpha_1}^{L_1(n)} \cdots \sum_{i_m=\alpha_m}^{L_m(n,i_1,\ldots i_m)} \sum_{j_1=\beta_1}^{\infty} \cdots \sum_{j_N=\beta_N}^{\infty} f(n,\{i\},\{j\})$$

$\downarrow$

Apply MultiSum package (tough)

$\downarrow$

$$g_0(n)F_n + \ldots + g_r(n)F_{n+r} = G(n)$$

Developed and used for example by Wegschaider '08, Blümlein, Klein, Schneider & Stan '12

$$F_n = \sum_{i_1=\alpha_1}^{L_1(n)} \cdots \sum_{i_m=\alpha_m}^{L_m(n,i_1,\dots i_m)} \sum_{j_1=\beta_1}^{\infty} \cdots \sum_{j_N=\beta_N}^{\infty} f(n,\{i\},\{j\})$$

↓

Zoom to the innermost sum

Solve Recurrence and substitute in solution

$$\hat{F}_n = \sum_{j_N=\beta_N}^{\infty \text{ or } L_N} f(n,\{i\},\{j\})$$

↓

Find a recurrence for the inner sum,

$$g_0(n)\hat{F}_n + \ldots + g_r(n)\hat{F}_{n+r} \;=\; G(n)$$

Refined Holonomic Approach

$$F_n = \sum_{i_1=\alpha_1}^{L_1(n)} \cdots \sum_{i_m=\alpha_m}^{L_m(n,i_1,\dots i_m)} \sum_{j_1=\beta_1}^{\infty} \cdots \sum_{j_N=\beta_N}^{\infty} f(n, \{i\}, \{j\})$$

Zoom to the innermost sum

$$\hat{F}_n = \sum_{j_N=\beta_N}^{\infty \text{ or } L_N} f(n, \{i\}, \{j\})$$

Find a recurrence for the inner sum,

$$g_0(n)\hat{F}_n + \ldots + g_r(n)\hat{F}_{n+r} = G(n)$$

Replace innermost sum with the sequence and its recurrences

A Worked Example

Consider the following definite nested multi-sum,

$$F_n = \sum_{i_1=0}^{n-2} \sum_{i_2=0}^{n-i_1-2} \frac{4 i_1!(-1)^{i_2}(n-i_1-1)(1+i_2)!}{(1+i_1+i_2)^2(2+i_1+i_2)^2(i_1+i_2)!} \binom{n-i_1-2}{i_2}$$

To find a recurrence for F_n, first re-write the sum,

$$4 \sum_{i_1=0}^{n-2} i_1!(n-i_1-1) a_{i_1,n}$$

$$a_{i_1,n} = \sum_{i_2=0}^{n-i_1-2} \frac{(-1)^{i_2}(1+i_2)!}{(1+i_1+i_2)^2(2+i_1+i_2)^2(i_1+i_2)!} \binom{n-i_1-2}{i_2}$$

now apply a recurrence finding algorithm, e.g. Sigma.

A Worked Example

One finds that,

$$
\begin{aligned}
(2+i_1+n+i_1 n)a_{i_1,n} - (2+i_1-n)(1+i_1+n)a_{i_1+1,n} &= \frac{1}{(1+i_1^2)i_1!},\\
-(1+i_1-n)(1+2i_1+i_1 n)a_{i,n} - (1+n)(1+i_1+i_1 n)a_{i,n+1} &= -\frac{1}{(1+n)i_1!}.
\end{aligned}
$$

- The two recurrences plus initial conditions specify the summand — a **holonomic** sequence.
- The right-hand sides are non-zero, which is where the **'refined'** enters!

Combining them one finds, for example, a zero-order recurrence,

$$F_n = \frac{4n}{2+n} - \frac{8S(n)}{(1+n)(2+n)}, \quad S_1(n) = \sum_{i=1}^{n} \frac{1}{i}.$$

Illustrative Result

It is not correct to think of one method as quickest in general.
Take our example sum,

$$S = \sum_{i_1=0}^{n-2} \sum_{i_2=0}^{n-i_1-2} \frac{4 i_1!(-1)^{i_2}(n-i_1-1)(1+i_2)!}{(1+i_1+i_2)^2(2+i_1+i_2)^2(i_1+i_2)!} \binom{n-i_1-2}{i_2}$$

The refined approach is 4 times faster than the current ‘state-of-the-art’ algorithm and some examples can be as much as 10 times faster.

Summary

- A problem in the mathematics of summation was motivated from physics.
- Various approaches and the new refined approach to multi-summation were explained.
- An illustration of the potential improvement of the algorithm was given. Although one should not think about 'best and worst' but a toolkit of algorithms.

Definition of Summand

Definition

Let $\{f(n,k)\}_{n\geq 0, k\geq 0}$ be a multivariate (here bivariate) sequence over a field $\mathbb{F}$. f is hypergeometric in n and k if $\exists d \in \mathbb{Z}^+$ and $r_1(x,y), r_2(x,y) \in \mathbb{F}(x,y)$ such that $f(n+1,k)/f(k) = r_1(n,k)$ and $f(n,k+1)/f(n,k) = r_2(n,k)\,\forall n,k \geq d$.

Definition

Restrict to sequences where:
$\exists d \in \mathbb{Z}^+$, $a_i, b_i, m_i \in \mathbb{Z}$ and $c_i \in \mathbb{F}$ for $i \in [1,r]$ where $a_i n + b_i k + c_i \notin \mathbb{Z}^- \forall n,k \geq d$ and $\exists\, p(x,y), q(x,y) \in \mathbb{F}[x,y]$ where $q \neq 0$ factors linearly over $\mathbb{F}$:
$f(n,k) = p(n,k)/q(n,k) \prod_{i=1}^{r} \Gamma(a_i n + b_i k + c_i)^{m_i}\,\forall n,k \geq d$.

n.b. $\mathbb{Z}^{\pm}$ both include zero.

Statement of Problem

Summary

Allow the summand to be as defined, times harmonic numbers and linear combinations of such terms.

$$F_n(\epsilon) = \sum_{i_1=\alpha_1}^{L_1(n)} \cdots \sum_{i_m=\alpha_m}^{L_m(n,i_1,\dots i_m)} \sum_{j_1=\beta_1}^{\infty} \cdots \sum_{j_N=\beta_N}^{\infty} f(\epsilon, n, i_1, \dots, i_m, j_1, \dots, j_N)$$

where $n \in \mathbb{N}$ and $n \geq n_0 \in \mathbb{N}$, all the upper bounds, L_k, are integer-linear in n, $\{i\}$ and $\{j\}$. All the lower bounds are specified constants, $\{\alpha\}, \{\beta\} \in \mathbb{N}$. The summand, F, is hypergeometric (defined shortly) with respect to n, $\{i\}$ and $\{j\}$. In this talk $\epsilon = 0$

Aim

Find an expression for a definite nested multi-sum, using the outlined summand, in terms of indefinite nested multi-sums.

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