Reduction of one and two-loop amplitudes at the Integrand level

Who? Ioannis Malamos

From? Institute de Fisica Corpuscular, Valencia

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Introduction

- Phenomenology at new colliders requires NLO (and NNLO) calculations
- One of the parts of these calculations is the virtual part (loop diagrams)
- Reduction methods are used due to the complexity and the large number of Feynman integrals
- One-loop case completely solved,towards a two-loop reduction method

Historical Background

- Already attempts in the 60's (D.B.Melrose (1965),G.Källèn and J.Toll (1965))
- Passarino-Veltman reduction (general applicability, major achievements but not designed at amplitude level)
- Unitarity based methods- Bern,Dixon,Dunbar,Kosower (major advantage: designed to work at amplitude level, limited applications)
- Quadruple and triple cuts- Britto, Cachazo, Feng (major simplifications)
- Reduction at the integrand level (Ossola,Papadopoulos,Pittau)

Definitions

- A general scalar Feynman integral of order n is given

$$\int d^4q \frac{1}{D_1 D_2 ... D_n} \tag{1}$$

with $D_i = (q + p_i)^2 - m_i^2 = q^2 + 2p_i \cdot q + \mu_i$ the inverse propagators

- p_i's are the momenta entering the propagators, related to the external momenta
- we don't assume momentum conservation,we rather deal with integrand-Graphs or i-Graphs given by

$$\frac{1}{D_1 D_2 ... D_n}$$

instead of Feynman Graphs

- We deal with scalar integrals without loss of generality

Reduction at one loop

- Write the number ‘one’ in terms of denominators

$$1 = \sum_i^n T_i(q) D_i \tag{2}$$

If we find such T_i’s then reduction is achieved

$$\frac{1}{D_1 \cdots D_n} = \frac{T_1(q)}{D_2 D_3 \cdots D_n} + \cdots + \frac{T_n(q)}{D_1 D_2 \cdots D_{n-1}} \, , \tag{3}$$

- Equation polynomial in q

Reduction with trivial coefficients

- We start by assuming that the T_i's are constants in the loop momentum

$$T_j(q) = x_j$$

- For any value of q

$$q^2 \sum_{j=1}^{n} x_j + 2q_\mu \sum_{j=1}^{n} x_j {p_j}^\mu + \sum_{j+1}^{n} x_j \mu_j = 1 \qquad (4)$$

- For the $d+2$ equations we need $d+2$ coefficients
- In 4 dimensions a hexagon is decomposed to pentagons with trivial coefficients

Reduction with coefficients linear in q

- We assume now

$$T_j(q) = x_j + \sum_{k=1}^{4} x_{j,k}(q \cdot t_k) \tag{5}$$

- t_k are d linearly independent arbitrary vectors, forming a base in d dimensions
- We start with $(d+1) \times n$ coefficients,not all being independent.
- The tensor structures we have to construct are denoted by

$$1 \ , \ q^\mu \ , \ q^\mu q^\nu \ , \ q^2 q^\mu \ . \tag{6}$$

- Tensor structures are given by $\frac{d^2+5d}{2} + 1$ (19 in $d = 4$)
- Number of independent coefficients numerically (with rounding errors)

n	$d = 6$	$d = 5$	$d = 4$	$d = 3$	$d = 2$	$d = 1$
2	14-0	12-0	10-0	8-0	6-0	4-0
3	21-1	18-1	15-1	12-1	9-1	6-2
4	28 -3	24-3	20-3	16-3	12-4	8-4
5	35-6	30-6	25-6	20-7	15-7	10-6
6	42-10	36-10	30-11	24-11	18-10	12-8
7	49-15	42-16	35-16	28-15	21-13	14-10
8	56-22	48-22	40-21	32-19	24-16	16-12

The rank of M for various d and n,
given as the difference $\xi - q$.

Decomposability with linear terms

- Every integral of order $d+1$ is decomposed to an integral of order d with coefficients linear in the loop momentum
- In the most interesting case of $d=4$ a pentagon is decomposed to boxes
- Situation does not improve with coefficients of higher order
- Uniqueness of the decomposition

Reduction at two loops

- A generic two loop diagram has three kind of propagators

$$D(l_1+p_1), D(l_1+p_2), ..., D(l_1+p_n),$$
$$D(l_2+q_1), D(l_2+q_2), ..., D(l_2+q_k),$$
$$D(l_1+l_2+r_1), D(l_1+l_2+r_2), ..., D(l_1+l_2+r_s), \quad (7)$$

- We denote this iGraph by (n,s,k) and we define it to be of order $n+k+s$.
- Due to these mixed propagators the problem is not just a double copy of a one-loop case

Reduction at two loops

- At one-loop the base of Master Integrals is known a priori.At two loop not
- Cannot be just scalar integrals (i.e. $l_1.q_1$ is not reducible,notice the difference with one loop)
- Reduction seems to depend on what kind of iGraphs one wants to reduce to
- Our motivation is to find out what is the highest number of denominators every iGraph can be decomposed to (i.e. in one-loop it is d)
- For unitarity reasons we expect this to be 2d

Reduction at two loops

- 'Counting to one' again

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_1 + l_2 + p_j) = 1 \tag{8}$$

- Notice that $n_{1,2,3} \leq d$, otherwise the problem can be solved as a one loop problem

Reduction at two loops

- Imagine the iGraph (5,1,1) in 4 dimensions. We can put all coefficients of l_2 to zero and solve the pentagon to boxes problem for l_1 only (as we saw before with linear terms)
- Restricted number of iGraphs to analyse-highest being the (d,d,d)
- An iGraph of order $2d + 4$ is decomposed to an iGraph of order $2d + 3$ with trivial coefficients x_j

Reduction at two loops with linear terms

- We try now the linear terms of the following type

$$x_j = \sum_i (a_j + b_{ij}(l_1 \cdot t_i) + c_{ij}(l_2 \cdot t_i)) \qquad (9)$$

- We again find the number of independent coefficients (with rounding errors) and compare it with the number of tensor structures. We give the table with our findings, using again a horizontal line for the reducible cases. T(d) is the number of tensor structures

Reduction at two loops with linear terms

$$L = 2, \text{ linear}$$
$$T(d) = (4d^2 + 18d + 2)/2$$

n	$d = 6$	$d = 5$	$d = 4$	$d = 3$	$d = 2$
3	39-0	33-0	27-0	21-0	15-0
4	52-0	44-0	36-0	28-0	20-0
5	65-1	55-1	45-1	35-1	25-1
6	78-3	66-3	54-3	42-3	30-3
7	91-6	77-6	63-6	49-6	35-8
8	104-10	88-10	72-10	56-10	40-10
9	111-15	99-15	81-15	63-17	45-18
10	130-21	110-21	90-21	70-24	50-23
11	143-28	121-28	99-30	77-31	55-28
12	156-36	132-36	108-39	84-36	60-33
13	169-45	143-47	117-48	91-45	65-38
14	182-55	154-58	126-57	98-52	70-43
$T(d)$	127	96	69	46	27

Reduction at two loops with linear terms

- In d dimensions, every $2d+2$ iGraph is decomposed to a $2d+1$ with linear terms.
- In order to go one step further we have to consider $2d+1$ iGraphs with coefficients of quadratic,cubic,quartic dependence in the loop momenta
- The set of iGraphs of interest is even more restricted. In $d=2$ we have to consider the (2,1,2) only, in $d=3$ the (3,1,3),(3,2,2) and in $d=4$ the (4,1,4),(4,3,2) and (3,3,3)

Reduction at two loops with higher order terms

- We repeat the proceedure using quadratic, and cubic terms terms. The reducibility of diagrams with higher terms depends on the number of dimensions
- We find that the (2,1,2) is reducible in 2 dimensions with quadratic terms, while the similar $2d+1$ iGraphs in 3 and 4 dimensions are not.
- We find solutions for the iGraphs of order 7 in 3 dimensions with cubic terms
- We find solutions for the iGraphs of order 9 in 4 dimensions with cubic terms

Reduction at two loops with higher order terms

- For all the cases above we solve numerically the $1 = 1$ equation.
- We construct a base for two loop iGraphs with at most 2d denominators.We proved it in 2,3 and 4 dimensions.
- We are interested in a base for the moment (not necessarily the minimal base)
- Once again, when considering iGraphs with 2d denominators, they must have a maximal cut. For example, a (6,1,1) in 4 dimensions does not belong in this category as we saw and as expected from Unitarity.

Not the end of the story yet!

- Consider the following Feynman integral in 2 dimensions

$$\int d^2l_1 d^2l_2 \frac{1}{l_1^2(l_1+p)^2(l_1+l_2)^2 l_2^2(l_2-p)^2} \qquad (10)$$

- It should be decomposable. We write:

$$1 = a_1 l_1^2 + a_2(l_1+p)^2 + \cdots + a_5(l_2-p)^2 \qquad (11)$$

- There should be coefficients that this holds for any l_1 and l_2

Not the end of the story yet!

- Find l_1 and l_2 such that

$$l_1^2 = (l_1 + p)^2 = l_2^2 = (l_2 - p)^2 = 0 \qquad (12)$$

- However,for such l_1 and l_2 also

$$(l_1 + l_2)^2 = 0$$

It means that there are l_1 and l_2 for which $1 = 0$! This diagram is NOT decomposable!

Solution

- The diagram above has a problemtaic maximal cut. However, there is a category of terms that one can add to the $1 = 1$ equation that vanish upon integration and can make the equation hold for every l_1 and l_2.
- Any total derivative in dimensional regularisation vanishes (I.B.P) (Chetyrkin and Tkachov)
- An example of an I.B.P. identity is the following

$$\int \frac{\partial}{\partial l_1^{\mu}} \left(\frac{(l_1 + p_1)^{\mu}}{D(l_1 + p_1) D(l_2 + p_2) D(l_1 + l_2 + p_3)} \right) = 0 \tag{13}$$

Solution

- One can now add at the Integrand level all these total derivatives (together with the previous terms we had) and ask for coefficients
- The problem is now solved!
- For the particular 2-dimensional problem, after integrating we get the following solution (which agrees with the literature)

$$\int \frac{1}{l_1^2(l_1+p)^2(l_1+l_2)^2 l_2^2(l_2-p)^2} = \frac{-1}{p^2}\int \frac{1}{l_1^2(l_1+p)^2 l_2^2(l_2-p)^2} + \frac{4}{(p^2)^2}\int \frac{1}{l_1^2(l_1+l_2)^2(l_2-p)^2} \tag{14}$$

I.B.P. at the Integrand level

- We tested the I.B.P.'S equations for other known examples and they work
- As a by-product, we can use the I.B.P.'s at the integrand level at one loop as well and generalise the OPP method with denominators of higher powers

Conclusions

- We presented Reductions at the Integrand level for one and two loop amplitudes
- Although the one loop case is already known we could rediscover things with a slightly different method. Important because the coefficients of the reduction depend on the base one asks to reduce to
- At two loops we showed why we expect unitarity to work, and how by simple counting one can decompose two-loop iGraphs
- For special cases one needs to use Integration By Parts identities. We use them at the integrand level for the first time

Conclusions

- In principle both counting and I.B.P's should be used
- Our method is very general (does not depend on masses or planar or non-planar Graphs) and is very simple as well.

BACKUP SLIDES

Finding the independent coefficients

- Choose random values for external momenta and masses
- Choose as many random values for q as the number of coefficients and substitute in equation 2
- For these ξ values we get a set of ξ linear equations for the ξ unknowns x:

$$\sum_{j=1}^{\xi} M^i{}_j x^j = 1 \quad , \quad j = 1, \ldots, \xi \quad . \tag{15}$$

- In case the number of independent tensor structures we can form are less than ξ the determinant of M vanishes. In a numerical computation there will be of course rounding errors

Finding the independent coefficients

- For q zero eigenvalues this Matrix will have a determinant of the order 10^{-pq} where p is the number of significant digits
- Running for different precisions we get the value q and the rank of the Matrix from $\xi - q$
- If the rank of the Matrix is equal to the number of tensor structures the integral is decomposable
- For different dimensions and number of propagators we give the results of the number of independent coefficients
- We denote the limit of decomposability with horizontal lines

Decomposability with spurious terms

- In the conventional approach, a pentagon is decomposed is decomposed to boxes with spurious terms
$$1 = \sum_{i=1}^{5} (a_i + \tilde{a}_i S_i(q)) D_i \qquad (16)$$
- The S_i's vanish after integration, by construction. Their form (q-dependence) is known
- Triangles in our case vanish giving the same decomposition
- The resulting base for one loop integrals is scalar integrals up to 4 denominators