On W3-algebra zero mode traces in the Potts Model

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The Virasoro algebra

$$[L_m, L_n] = (m - n)\, L_{m+n} + \delta_{m+n,0} \frac{c}{12} m \left(m^2 - 1\right)$$

The W_3 algebra

$$[L_m, L_n] = (m-n)\,L_{m+n} + \delta_{m+n,0}\frac{c}{12}m\left(m^2-1\right)$$

$$[L_m, W_n] = (2m-n)\,W_{m+n}$$

$$\begin{aligned}[W_m, W_n] = {} & (m-n)\left[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right]L_{m+n} \\ & + \beta\,(m-n)\,\Lambda_{m+n} + \delta_{m+n,0}\frac{c}{360}m\left(m^2-1\right)\left(m^2-4\right)\end{aligned}$$

The Λ operator

The Λ operator is defined as

$$\Lambda_n = \sum_{p\leq -2} L_p L_{n-p} + \sum_{p\geq -1} L_{n-p} L_p - \frac{3}{10}(n+2)(n+3)L_n$$

It has commutation relations

$$[L_m, \Lambda_n] = (3m-n)\Lambda_{m+n} + \frac{22+5c}{30} m\left(m^2-1\right) L_{m+n}$$

$$\begin{aligned}[W_m, \Lambda_n] = \frac{n+3}{5}&\left[3(m+n+3)(m+n+4) - 8(m+1)(m+2)\right] W_{m+n} \\ &+ \sum_{p\leq -2}(2m-4n+4p) L_p W_{m+n-p} \\ &+ \sum_{p\geq -1}(2m-4n+4p) W_{m+n-p} L_p\end{aligned}$$

The field-operator and state-field correspondances

These operators are related to fields via

$$L(z) = \sum_m L_m z^{-m-2}$$
$$W(z) = \sum_m W_m z^{-m-3}$$
$$\Lambda(z) = \sum_m \Lambda_m z^{-m-4}$$

Generally, $F(z) = \sum_m F_m z^{-m-h_F}$, where h_F is the weight of $F(z)$.

There is also a relation between fields and states (and thus operators and states), given by $|F\rangle = \lim_{z\to 0} F(z)\,|0\rangle$

Null fields and the zero-mode relation

If $N(z)$ is a null field, then $\mathrm{tr}\left(N_0 q^{L_0}\right) = 0$[1]

In this talk I am interested in the null fields

$$\Lambda = (LL) - \frac{3}{10} L''$$
$$N^{(7)} = -\frac{1}{12} W'''' + \frac{52}{121}\left(W''L\right) - \frac{47}{121}\left(W'L'\right) - \frac{27}{121}\left(W\Lambda\right) + \frac{141}{605}\left(WL''\right)$$
$$N^{(6)} = (WW) - \frac{85}{78}\left(L'L'\right) - \frac{95}{117}\left(L\Lambda\right) - \frac{25}{234} L'''' + \frac{29}{39}\left(LL''\right)$$

where the brackets $(\dots)$ denote normal ordering of the enclosed fields.

[1] Gaberdiel & Keller, "Modular differential equations and null vectors", hep-th/0804.0489

Zero modes

Λ field:

$$\Lambda_0 = \sum_{n\le -2} L_n L_{-n} + \sum_{n\ge -1} L_{-n} L_n - \frac{9}{5} L_0$$

Level seven field:

$$N_0^{(7)} = \frac{1}{78}\sum_{n\le -2}\left(143n^2 - 290n + 8\right) L_n L_{-n} + \frac{1}{78}\sum_{n\ge -1}\left(143n^2 - 290n + 8\right) L_{-n} L_n$$
$$- \frac{95}{117}\sum_{n\le -2} L_n \Lambda_{-n} - \frac{95}{117}\sum_{n\ge -1} \Lambda_{-n} L_n - \frac{500}{39} L_0 + \sum_{n\le -3} W_n W_{-n} + \sum_{n\ge -2} W_{-n} W_n$$

Level six field:

$$N_0^{(6)} = \frac{6}{605}\sum_{n\le -3}\left(106n^2 + 225n + 426\right) W_n L_{-n} - \frac{27}{121}\sum_{n\le -3} W_n \Lambda_{-n}$$
$$+ \frac{6}{605}\sum_{n\ge -2}\left(106n^2 + 225n + 426\right) L_{-n} W_n - \frac{27}{121}\sum_{n\ge -2} \Lambda_{-n} W_n - 30 W_0$$

Traces of zero modes

Λ field:

$$0 = \mathrm{tr}\left(\Lambda_0 q^{L_0}\right) = f_2 \mathrm{tr}\left(L_0^2 q^{L_0}\right) + f_1 \mathrm{tr}\left(L_0 q^{L_0}\right) + f_0 \mathrm{tr}\left(q^{L_0}\right)$$

Level seven field:

$$0 = g_1 \mathrm{tr}\left(L_0 W_0 q^{L_0}\right) + g_0 \mathrm{tr}\left(W_0 q^{L_0}\right)$$

Level six field:

$$\mathrm{tr}\left(W_0^2 q^{L_0}\right) = h_3 \mathrm{tr}\left(L_0^3 q^{L_0}\right) + h_2 \mathrm{tr}\left(L_0^2 q^{L_0}\right) + h_1 \mathrm{tr}\left(L_0 q^{L_0}\right) + h_0 \mathrm{tr}\left(q^{L_0}\right)$$

Differential operators acting on characters

We can use

$$L_0 q^{L_0} |h, w\rangle = L_0 q^h |h, w\rangle = h q^h |h, w\rangle = q \frac{\partial}{\partial q} q^h |h, w\rangle = q \frac{\partial}{\partial q} q^{L_0} |h, w\rangle$$

to identify $L_0 = q \frac{\partial}{\partial q}$ and rewrite these expressions as

Λ field:

$$0 = \mathrm{tr}\left(\Lambda_0 q^{L_0}\right) = \left[f_2 \left(q \frac{\partial}{\partial q}\right)^2 + f_1 \left(q \frac{\partial}{\partial q}\right) + f_0\right] \mathrm{tr}\left(q^{L_0}\right)$$

Level seven field:

$$0 = \left[g_1 \left(q \frac{\partial}{\partial q}\right) + g_0\right] \mathrm{tr}\left(W_0 q^{L_0}\right)$$

Level six field:

$$\mathrm{tr}\left(W_0^2 q^{L_0}\right) = \left[h_3 \left(q \frac{\partial}{\partial q}\right)^3 + h_2 \left(q \frac{\partial}{\partial q}\right)^2 + h_1 \left(q \frac{\partial}{\partial q}\right) + h_0\right] \mathrm{tr}\left(q^{L_0}\right)$$

Calculating $\mathrm{tr}\left(\Lambda_0 q^{L_0}\right)$

- $c = -22/5$: $h = 0$ or $h = -1/5$

$$\mathrm{tr}_0\left(q^{L_0}\right) = \prod_{k=0}^{\infty} \frac{1}{1-q^2 q^{5k}} \frac{1}{1-q^3 q^{5k}}$$

$$\mathrm{tr}_{-\frac{1}{5}}\left(q^{L_0}\right) = q^{-1/5} \prod_{k=0}^{\infty} \frac{1}{1-q q^{5k}} \frac{1}{1-q^4 q^{5k}}$$

- Coefficient functions

$$f_2 = 1$$

$$f_1 = \frac{1}{5} + 4 \sum_{n \geq 1} \frac{n q^n}{1-q^n}$$

$$f_0 = \frac{2}{15} \sum_{n \geq 1} \frac{\left(n^3 - n\right) q^n}{1-q^n}$$