Symmetric Calorons

Alexander Cockburn

September 2012

Introduction to Solitons

- Stable, particle-like objects with a smooth structure.
- Solutions to non-linear field equations whose stability often follows from their topology.
- Ubiquitous: solitons appear in cosmology as cosmic strings, as instantons and monopoles in quantum field theory, as vortices in superconductors...
- Solitons have driven many developments in mathematics, especially the geometry of smooth 4-manifolds.

Magnetic Monopoles

- Dirac (1931) realised monopoles could explain the quantisation of electric charge.
- 't Hooft and Polyakov (1974) discovered that non-abelian gauge theories can have solitonic magnetic monopole solutions.
- Monopoles have never been seen in nature.
- But they provide a constraint on Grand Unified Theories and are of great mathematical interest in themselves.

Static BPS Monopoles

Static BPS monopoles are field configurations (A_i, Φ) minimising the energy functional:

$$E = -\int \frac{1}{4}\mathrm{Tr}(F_{ij}F_{ij}) + \frac{1}{2}\mathrm{Tr}(D_i\Phi D_i\Phi)\, d^3x \qquad (1)$$

This is an $SU(K)$ gauge theory. Here the Higgs field Φ is a Lie algebra-valued scalar in the adjoint, F_{ij} is the field strength for a Lie algebra-valued gauge potential A_i, and D_i is the covariant derivative.

We impose the boundary condition that $-\frac{1}{2}\mathrm{Tr}(\Phi^2) \to 1$ as $|x| \to \infty$. If the gauge group is $SU(2)$ this gives a map $\Phi : S^2_\infty \to S^2$ whose degree in appropriate units is the magnetic charge of the monopole.

The Bogomolny equation

The field equations for this energy functional are:

$$D_i D_i \Phi = 0 \tag{2}$$

$$D_i F_{ij} = [D_j \Phi, \Phi] \tag{3}$$

These are very difficult to solve. However, Bogomolny showed that if the topological degree of Φ is N, then the energy satisfies the 'Bogomolny bound'

$$E \geq 2\pi N \tag{4}$$

with equality iff

$$\frac{1}{2}\epsilon_{ijk} F_{jk} = -D_i \Phi \tag{5}$$

This is the Bogomolny equation.

Instantons

- In quantum field theory one encounters path integrals which must be analytically continued to become well-defined.
- A Wick rotation $t \longrightarrow it$ turns a path integral in Minkowskian $\mathbf{R}^{3,1}$ to one in Euclidean $\mathbf{R}^4$.
- The classical solutions to the field equations in $\mathbf{R}^4$ dominate the path integral. They can therefore be used to model non-perturbative quantum tunnelling effects.
- Solutions to pure Yang-Mills theory in $\mathbf{R}^4$ are called instantons.

Yang-Mills Instantons

Instantons are field configurations A_μ minimising the action for pure $SU(K)$ Yang-Mills on Euclidean $\mathbf{R}^4$:

$$S = -\frac{1}{8}\int F_{\mu\nu}F_{\mu\nu}\, d^4x \tag{6}$$

The field equations coming from this action are:

$$D_\mu F_{\mu\nu} = 0 \tag{7}$$

If $K = 2$, one can again apply a Bogomolny argument to show that:

$$S \geq \pi^2 N \tag{8}$$

where N is another topological quantity, with equality iff A_μ satisfies the (anti)-self-dual Yang-Mills equation:

$$F_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{9}$$

Monopoles and Instantons

Monopoles and Instantons are closely connected. In fact the Bogomolny equation is a time-independent version of the self-dual Yang-Mills equation. One can write the self-dual Yang-Mills equation as:

$$F_{4i} = \frac{1}{2}\epsilon_{ijk}F_{jk} \tag{10}$$

If A_μ is independent of Euclidean time x_4, then this simplifies to:

$$-\partial_i A_4 - [A_i, A_4] = \frac{1}{2}\epsilon_{ijk}F_{jk} \tag{11}$$

Identifying A_4 with the Higgs field Φ gives the Bogomolny equation $\frac{1}{2}\epsilon_{ijk}F_{jk} = -D_i\Phi$

Calorons

- Calorons are solutions to the self-dual Yang-Mills which are periodic in Euclidean time.
- They describe non-perturbative quantum effects in finite-temperature Yang-Mills theory.
- Calorons form an intermediate case between monopoles and instantons. If the caloron period tends to infinity, we obtain an instanton. If the period tends to zero, we obtain a monopole.
- Calorons, instantons, and monopoles are also related to a solitonic model of nuclear physics called the Skyrme model.

Symmetric Calorons

- The Bogomolny and self-dual Yang-Mills equations are very difficult to solve.
- One can obtain solutions by imposing symmetries on the monopole/instanton.
- Do all symmetric monopoles extend to a symmetric monopole-caloron-instanton family?
- Do all symmetric instantons extend to a family of symmetric calorons?
- What is the role of harmonic maps, which are related to symmetric monopoles?