Lattice field theory

Problem Sheet 2 - Fermionic theory

1. Show that the Wilson-Dirac operator is γ_5 hermitian:

$$D^\dagger = \gamma_5 D \gamma_5 \,,$$

and is invariant under C, P, and T:

$$\begin{aligned} P: \psi(x) &\mapsto \gamma_0 \psi(x_P)\,, \\ \bar\psi(x) &\mapsto \bar\psi(x_P)\gamma_0\,. \\ C: \psi(x) &\mapsto \gamma_0\gamma_2 \bar\psi(x)^T\,, \\ \bar\psi(x) &\mapsto -\psi(x)^T(\gamma_0\gamma_2)^{-1}\,. \\ T: \psi(x) &\mapsto \gamma_0\gamma_5\psi(x_T)\,, \\ \bar\psi(x) &\mapsto \bar\psi(x_T)\gamma_5\gamma_0\,. \end{aligned}$$

2. Using γ_5 hermiticity show that the eigenvectors of D satisfy:

$$v_\lambda^\dagger \gamma_5 v_\lambda = 0\,, \quad \text{for } \lambda^* \neq \lambda\,.$$

3. Construct the transfer matrix for free Wilson fermions, for the case $r = 1$.

4. *Timeslices of Wilson fermions.* A timeslice of the two-point function can be obtained by summing over the spatial coordinate at fixed Euclidean time. Compute the correlator of two timeslices separated by time t:

$$C(t) = \frac{1}{V}\sum_{\mathbf{x}} \langle \psi(\mathbf{x}, t)\bar\psi(0)\rangle\,,$$

for Wilson fermions with $|r| < 1$ in the limit where the time extension $T \to \infty$.

5. Two flavors of twisted-mass fermions are defined by the Wilson action with a *twisted* mass term:

$$S_m = \sum_x \bar\psi(x)\left[m + i\mu\gamma_5\tau^3\right]\psi(x)\,.$$

Using a chiral rotation of the fields, show that the twisted-mass action is equivalent to the standard one in the naive continuum limit $a \to 0$.

The twisted action has the same symmetries as the standard one. However their implementation in term of the fields ψ is different. As an example, write the parity transformation for the twisted fields.

6. Check that the overlap operator

$$D = 1 + \gamma_5 \epsilon(H)$$

is a solution of the GW equation.

7. Discuss the eigenvalue spectrum for the overlap, and compare it to the spectrum of the continuum Dirac operator for $a \mapsto 0$.

8. Show that the propagator in momentum space for overlap fermions is:

$$(aD)^{-1} = \frac{1}{2} - \frac{1}{2}B\,,$$

where

$$B = \frac{i\gamma_\mu \hat{p}_\mu}{F(p)}\,, \quad F(p) = \left[\hat{p}^2 + \left(1 - \frac{1}{2}\tilde{p}^2\right)^2\right]^{1/2} - \left(1 - \tilde{p}^2\right)\,.$$