## Lecture 2

# Fermion fields

## 2.1 Introduction

Let us start from the four-dimensional Euclidean action for a free fermion field in the continuum:

$$S_E[\psi,\bar{\psi}] = \int d^4x \, \left[\bar{\psi}(x)\gamma_\mu\partial_\mu\psi(x) + m_0\bar{\psi}(x)\psi(x)\right] \, . \tag{2.1}$$

We have introduced the Euclidean gamma matrices:

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix} \, , \quad \begin{cases} e_4 = \mathbb{1} \, , \\ e_k = \sigma_k \, , \quad k=1,2,3 \end{cases} \tag{2.2}$$

They satisfy the usual relations:

$$\gamma_\mu^\dagger = \gamma_\mu \, , \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \, . \tag{2.3}$$

Let us also define the Euclidean $\gamma_5$, which will be used below:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \, , \quad \gamma_5^\dagger = \gamma_5 \, , \quad \gamma_5^2 = \mathbb{1} \, . \tag{2.4}$$

**KL representation** As for the scalar theory, the particle content of the fermionic theory can be read from the KL representation of the two-point field correlator:

$$\langle\psi(x)\bar{\psi}(0)\rangle\big|_{x_4>0} = \sum_n \int \frac{d^3p}{(2\pi)^3 2E_n(\mathbf{p})} \, e^{-E_n(\mathbf{p})x_4 + i\mathbf{p}\cdot\mathbf{x}} \, (i\not{p} + m_n)|Z_n|^2 \, , \tag{2.5}$$

where

$$p_4 = i\sqrt{\mathbf{p}^2 + m_n^2} = iE_n(\mathbf{p}) \, . \tag{2.6}$$

## 2.2 Naive fermions

The continuum action in Eq. (2.1) is discretized by simply replacing the partial derivatives with finite differences:

$$S = \sum_x \left[\bar{\psi}(x)\frac{1}{2}\left(\nabla_\mu + \nabla_\mu^*\right)\gamma_\mu\psi(x) + m_0\bar{\psi}(x)\psi(x)\right] \, . \tag{2.7}$$

Let us find the KL representation for the propagator and discuss the spectrum of the theory defined by the naive discretization of the Dirac equation. The propagator is diagonal in momentum space:

$$K(p,q)_{\alpha\beta} = \left[\sum_\mu i\tilde{p}_\mu\gamma_\mu + m_0\right]_{\alpha\beta} (2\pi)^4\delta(p+q) \, , \tag{2.8}$$

where:

$$\tilde{p}_\mu = \frac{1}{a}\sin(ap_\mu)\,. \tag{2.9}$$

The exponential decay of the two-point function in position space is determined by the poles of $\Delta(p) = K(p)^{-1}$ in the complex $p_4$ plane such that $|\mathrm{Re}\ p_4| \leq \pi/a$ and $\mathrm{Im}\ p_4 \geq 0$. Note that in this case there are *two* poles:

$$e^{ip_4 a} = \pm e^{-\omega(\mathbf{p})a} = \pm\left(\sqrt{1+\hat{M}_p^2} - \hat{M}_p\right)\,, \tag{2.10}$$

$$\hat{M}_p^2 = m_0^2 a^2 + \sum_{k=1}^{3} \sin^2(p_k a)\,. \tag{2.11}$$

Therefore:

$$\begin{aligned}\langle \psi_\alpha(x)\bar{\psi}_\beta(0)\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot\mathbf{x}-\omega(\mathbf{p})x_4}}{\sinh\left(2\omega(\mathbf{p})a\right)}\Bigg\{\left(\gamma_4\sinh\omega(\mathbf{p})a - i\sum_k \gamma_k \sin p_k a + m_0 a\right) + \\ + (-)^{x_4/a}\left(-\gamma_4\sinh\omega(\mathbf{p})a - i\sum_k \gamma_k \sin p_k a + m_0 a\right)\Bigg\}_{\alpha\beta}\,.\end{aligned} \tag{2.12}$$

There are two terms in the sum, corresponding to the same energy, but with different residues. The energy function $\omega(\mathbf{p})$ has $2^3$ minima inside the Brillouin zone. They correspond to $\bar{p}_k = n_k\frac{\pi}{a}$, and $n_k = 0, 1$. In the naive continuum limit:

$$\lim_{a\to 0}\omega(\mathbf{p})|_{\bar{p}} = m\,. \tag{2.13}$$

The integral in Eq. (2.12) is dominated by the modes near the minima of $\omega$. We shall therefore rewrite the integral as the sum of integrals in the neighbourhood of the mimima by setting:

$$p_j = \bar{p}_j^{(\alpha)} + k_j, \quad j = 1, 2, 3\,, \tag{2.14}$$

while the index $\alpha$ runs over the $2^3$ minima of $\omega$. Expanding the integrand for small values of $ak_j$ yields:

$$\langle \psi(x)\bar{\psi}(0)\rangle = \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}\cdot x}\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}-\omega x_4}}{2k_4}\left[\gamma_4\cos(\bar{p}_4^{(\alpha)}a)k_4 - i\sum_j \gamma_j\cos(\bar{p}_j^{(\alpha)}a)k_j + m_0\right]\,, \tag{2.15}$$

where we have introduced a fourth component to the vectors $\bar{p}$, in order to rewrite in a single form the two terms that were added together in Eq. (2.12). The sixteen contributions to the sum in Eq. (2.15) correspond to the values:

$$\bar{p}_\mu = n_\mu\frac{\pi}{a}, \quad n_\mu = 0, 1\,. \tag{2.16}$$

It is possible to define unitary operators $S_\alpha$ such that:

$$S_\alpha \gamma_\mu S_\alpha^\dagger = \gamma_\mu \cos\left(\bar{p}_\mu^{(\alpha)} a\right) . \tag{2.17}$$

so that we finally obtain:

$$\langle \psi(x)\bar{\psi}(0)\rangle = \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}\cdot x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}-\omega x_4}}{2k_4} S_\alpha \left[\gamma_4 k_4 - i\sum_j \gamma_j k_j + m_0\right] S_\alpha^\dagger . \tag{2.18}$$

We recognize in this last expression the contribution of sixteen equivalent free relativistic fermions. The $S_\alpha$ simply perform a change of basis in the space of spin states, *i.e.* they correspond to a redefinition of the $\gamma_\mu$ matrices.

In general in $D$ dimensions, the naive discretization of the Dirac operator describes $2^D$ relativistic fermions. This is the so-called *fermion doubling* problem.

## 2.3 Nielsen-Ninomiya theorem

The appearance of doublers in the previous section has a very subtle origin related to chiral symmetry. A Dirac field is made of two Weyl spinors, obtained by acting on the original field with a right-handed (resp. left-handed) projector:

$$\psi_R = \frac{1+\gamma_5}{2}\psi, \quad \psi_L = \frac{1-\gamma_5}{2}\psi . \tag{2.19}$$

In the continuum massless theory the LH and RH components are not coupled. The Dirac field is really made of two independent Weyl spinors.

Repeating the lattice computation leading to Eq. (2.18) above, but starting from a LL correlator in the massless case yields:

$$\begin{aligned}\langle \psi_L(x)\bar{\psi}_L(0)\rangle &= \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}\cdot x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}-\omega x_4}}{2k_4} S_\alpha \left[\gamma_4 k_4 - i\sum_j \gamma_j k_j\right] S_\alpha^\dagger \frac{1-\gamma_5}{2} \\ &= \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}\cdot x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}-\omega x_4}}{2k_4} S_\alpha \left[\left(\gamma_4 k_4 - i\sum_j \gamma_j k_j\right)\left(\frac{1-\eta^\alpha\gamma_5}{2}\right)\right] S_\alpha^\dagger ,\end{aligned} \tag{2.20}$$

where $\eta^\alpha = (-)^{\sum_\mu n_\mu^{(\alpha)}}$. The latest equation shows explicitly that half the doublers are RH states and half of them are LH, even though we started from the lattice correlator of two LH fields!

This result can be generalized to other lattice actions as well. Consider a massless lattice action:

$$S = \sum_{x,y} \bar{\psi}(x) D(x-y) \psi(y) \,. \tag{2.21}$$

The Nielsen & Ninomiya theorem states that the properties:

1. $D(x)$ is local;
2. $D(p) = i\gamma_\mu p_\mu + O(ap^2)$;
3. $D(p)$ is invertible for $p \neq 0$;
4. $\gamma_5 D + D\gamma_5 = 0$

cannot hold simultaneously. As we will see later, this is also related to the existence of the axial anomaly in non-Abelian gauge theories. In momentum space locality implies that the propagator is a periodic, analytic function of $p_\mu$. Violating this condition leads to discontinuities in the derivatives of the propagator, and therefore to singular terms in Ward identities. The second and third conditions correspond to having a single Dirac flavor in the continuum limit. The doubling of the spectrum that we have seen for naive fermions corresponds to a violation of the third condition.

Any attempt to solve the doubling problem will entail some violation of the four conditions listed above.

## 2.4 Wilson fermions

The simplest solution to the doubling problem is to break chiral symmetry explicitly, thereby evading the Nielsen-Ninomiya theorem. Wilson's idea was to break chiral symmetry by an irrelevant operator:

$$\Delta S = -\frac{ra}{2} \sum_x \bar{\psi}(x) \nabla^*_\mu \nabla_\mu \psi(x) \,, \tag{2.22}$$

where $r$ is a constant. We will choose here to work with $r = 1$. Note that $\Delta S$ is a bilinear in the fermion field and has the same spin structure as a mass term, thereby coupling the LH and RH components of the Dirac field, *i.e.* breaking chiral symmetry.

The Wilson term induces a momentum-dependent mass term in the propagator:

$$\tilde{M}(p) = m_0 + \frac{1}{a} \sum_\mu \left(1 - \cos p_\mu a\right) \,. \tag{2.23}$$

As a consequence, the poles of the integrand in the complex $p_4$ plane are changed. They correspond to the solutions of the equation:

$$\tilde{p}^2 + \tilde{M}(p)^2 = 0 \implies p_4 = i\omega(\mathbf{p}) , \tag{2.24}$$

where now:

$$\cosh\left(\omega(\mathbf{p})a\right) = \frac{1 + \sum_k \sin^2(p_k a) + \left[am_0 + 1 + \sum_k (1 - \cos p_k a)\right]^2}{2\left[am_0 + 1 + \sum_k (1 - \cos p_k a)\right]} . \tag{2.25}$$

It is important to note that there is now only one solution to the equation within the Brillouin zone; shifting $p_4$ by $\pi/a$ does not correspond to a new pole. This is different from what we observed for the naive discretization in the previous section.

Once again, the integral is dominated by the regions around the minima of $\omega$:

$$p_k = \bar{p}_k^{(\alpha)} = n_k^{(\alpha)} \frac{\pi}{a} , \quad n_k^{(\alpha)} = 0, 1 . \tag{2.26}$$

The corresponding value for the energy is:

$$\omega^{(\alpha)} a = \log \left[ am_0 + 1 + 2 \sum_k n_k^{(\alpha)} \right] . \tag{2.27}$$

As we approach the continuum limit, $am_0 \to 0$, we find:

$$\lim_{a\to 0} a\omega^{(\alpha)} = 0 , \qquad \text{if } n_k^{(\alpha)} = 0 , \forall k ; \tag{2.28}$$

$$\lim_{a\to 0} a\omega^{(\alpha)} = \text{finite} , \ \text{if } n_k^{(\alpha)} \neq 0 . \tag{2.29}$$

All the doublers with $n_k^{(\alpha)} \neq 0$ have an energy $\omega \sim a^{-1}$, *i.e.* they are states with energies at the cutoff scale and therefore decoupled from the low-energy dynamics that we are interested in. We are left with just one state with mass $\ll a^{-1}$, *i.e.* we have removed all the doublers. In doing so we have explicitly broken chiral symmetry. In particular chiral symmetry no longer protects quantities from quantum corrections, and therefore the renormalization of Wilson fermions has some additional complications. For instance the fermion mass acquires an additive renormalization and the chiral limit can only be reached by fine-tuning the bare fermion mass. Similarly the renormalization of composite operators requires greater care.

## 2.5 Symmetries of Wilson fermions

As discussed above, Wilson fermions break chiral symmetry. We shall summarize in this section the symmetries of the Wilson-Dirac operator. The proof of these properties is left as an exercise to the reader.

### 2.5.1 $\gamma_5$ hermiticity

The Wilson-Dirac operator satisfies:

$$D^\dagger = \gamma_5 D \gamma_5 \,. \tag{2.30}$$

### 2.5.2 Discrete symmetries

The discrete symmetries $C, P, T$ are implemented on the lattice as:

$$\begin{aligned} P :& \psi(x) \mapsto \gamma_4 \psi(x_P) \\ & \bar\psi(x) \mapsto \bar\psi(x_P)\gamma_4 \\ \\ T :& \psi(x) \mapsto \gamma_4\gamma_5 \psi(x_T) \\ & \bar\psi(x) \mapsto \bar\psi(x_T)\gamma_5\gamma_4 \\ \\ C :& \psi(x) \mapsto C\bar\psi(x)^T \\ & \bar\psi(x) \mapsto \psi(x)^T C^{-1} \,, \end{aligned}$$

where $x_P = (-\mathbf{x}, x_4)$, $x_T = (\mathbf{x}, -x_4)$, and $C = \gamma_4\gamma_2$, so that $C\gamma_\mu C = -\gamma_\mu^*$.

### 2.5.3 Multiple flavors

If the theory has $n_f$ copies of identical fermions, then the Wilson-Dirac operator is symmetric under vectorial $\mathrm{U}(n_f)$ rotations of the fields. Let us denote by $\lambda^a$ the generators of $\mathrm{U}(n_f)$, we have:

$$\psi \mapsto (1 + i\omega^a\lambda^a)\psi \,, \tag{2.31}$$

$$\bar\psi \mapsto \bar\psi(1 - i\omega^a\lambda^a) \,, \tag{2.32}$$

$$[D, \lambda^a] = 0 \,. \tag{2.33}$$

## 2.6 Staggered fermions

A different way to tackle the problem of doublers was suggested by Kogut and Susskind in the early days of lattice field theory. The basic idea is rather ingenious: using some of the 16 doublers to construct the 4 spinor components of a Dirac fermion.

Let us see in detail how this is achieved. First of all the naive lattice action needs to be diagonalized in spinor space, *i.e.* we need to find a transformation $S(x)$ such that:

$$S(x)\gamma_\mu S(x + a\mu)^\dagger = \rho(x, \mu)\mathbb{1} \,. \tag{2.34}$$

You can verify that this is achived by choosing:

$$S(x) = \gamma_1^{n_1} \dots \gamma_4^{n_4}\,, \quad x = a(n_1, n_2, n_3, n_4)\,; \tag{2.35}$$

and therefore obtaining:

$$\rho(x,\mu) = (-)^{\sum_{\nu<\mu} n_\nu}\,. \tag{2.36}$$

Transforming the spinor fields using $S(x)$, we obtain for the action:

$$S_{\rm stag} = \sum_{x,\alpha} \left[ \sum_\mu \rho(x,\mu)\bar{\chi}^\alpha(x) \frac{1}{2}(\nabla_\mu + \nabla^*_\mu)\chi^\alpha(x) + m_0 \bar{\chi}^\alpha(x)\chi^\alpha(x) \right]\,. \tag{2.37}$$

We see that the action $S_{\rm stag}$ is completely diagonal in the spinor index $\alpha$. We can therefore remove the summation over $\alpha$, and consider a single field $\chi$. In this way we have reduced the number of doublers by a factor 4, and therefore the number of physical degrees of freedom associated to the staggered action is 4 rather than 16. The four components of the Dirac field are reconstructed from the values of the field $\chi$ at different space-time points separated at most by one lattice spacing. Both the Dirac and the flavor structure are obtained from the spatial dependence of the field $\chi$. Consider an hypercube with origin at $z_\mu = 2N_\mu a$, where $N_\mu$ are integers. The other points within the hypercube have coordinates:

$$r_\mu = z_\mu + \rho_\mu\,, \tag{2.38}$$

where $\rho_\mu = 0, 1$. The field can then be relabeled:

$$\chi_\rho(z_\mu) = \chi(z_\mu + a\rho)\,. \tag{2.39}$$

The four Dirac flavors are defined on a lattice with lattice spacing $2a$ as:

$$\psi^f_\alpha(z_\mu) = \sum_\rho (T_\rho)_{\alpha f}\, \chi_\rho(z_\mu)\,, \tag{2.40}$$

where:

$$T_\rho = \gamma_1^{\rho_1} \dots \gamma_4^{\rho_4}\,. \tag{2.41}$$

Staggered fermions have been used extensively for simulations of lattice QCD. They are fast to simulate, and have an exact U(1) chiral symmetry, which is sufficient to protect their mass from aditive renormalization. Unfortunatley staggered fermions do not have a positive-definite transfer matrix at finite lattice spacing, and do not allow a definition of a theory with less than 4 flavors. In order to simulate a theory with less than 4 flavors, the square root of the matrix has been often considered. There are hints that unitarity would be recovered in the continuum limit.

## 2.7 Ginsparg-Wilson relation

Ginsparg and Wilson studied the problem of chiral symmetry on the lattice starting from a continuum theory, and defining the lattice theory by some blocking procedure. Given a continuum field $\psi(z)$, they introduce a lattice field $\psi_{\mathrm{lat}}(an)$ by averaging over space-time cells centered around the lattice sites $x = an_\mu$:

$$\psi_{\mathrm{lat}}(an) = \int d^4z \psi(z) f(z - an) \,, \tag{2.42}$$

where the density function $f$ vanishes rapidly for $|z| > a$. We have denoted by $z$ the continuum coordinate, while we use the letter $x$ to indicate the dimensionful discrete quantity $x_\mu = an_\mu$. A lattice theory can then be defined by a path integral:

$$Z = \int \prod_x d\chi(x) d\bar{\chi}(x) \exp\left(-S_{\mathrm{lat}}[\chi, \bar{\chi}]\right) \,, \tag{2.43}$$

where the lattice action is obtained by block-spin transformation:

$$e^{-S_{\mathrm{lat}}}[\chi, \bar{\chi}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[ -\sum_x \alpha(\bar{\chi}(x) - \bar{\psi}_{\mathrm{lat}}(x))(\chi(x) - \psi_{\mathrm{lat}}(x)) - S[\psi, \bar{\psi}] \right] \,, \tag{2.44}$$

and $S[\psi, \bar{\psi}]$ is the usual, chirally symmetric, continuum action for massless fermions. Note the Gaussian damping that we introduced in order to have a convergent integral in $d\chi$, $d\bar{\chi}$. The coefficient $\alpha$ clearly has dimensions of mass. This term introduces an explicit breaking of chiral symmetry. Performing the integral over $\psi$ and $\bar{\psi}$ defines the lattice action $S_{\mathrm{lat}}$. If $S[\Psi, \bar{\psi}]$ is quadratic, then so is $S_{\mathrm{lat}}$, which we write as:

$$S_{\mathrm{lat}} = \sum_{x,y} \bar{\chi}(x) D(x, y) \chi(y) \,. \tag{2.45}$$

The blocking procedure *defines* the lattice Ginsparg-Wilson Dirac operator $D$. Let us assume that we have $n_f$ degenerate massless flavors in the continuum, we shall now look at the properties of the lattice action under chiral rotations of the $\chi$ fields:

$$\begin{aligned} \chi &\mapsto e^{i\epsilon^a \gamma_5 \lambda^a} \chi \,, \\ \bar{\chi} &\mapsto \bar{\chi} e^{i\epsilon^a \gamma_5 \lambda^a} \,. \end{aligned} \tag{2.46}$$

Rotating the integration variables in Eq. (2.44) yields:

$$\exp\left[ -\bar{\chi} e^{i\epsilon^a \gamma_5 \lambda^a} D \; e^{i\epsilon^a \gamma_5 \lambda^a} \chi \right] = \\ = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \; \exp\left[ -\sum_x \alpha(\bar{\chi}(x) - \bar{\psi}_{\mathrm{lat}}(x)) e^{i2\epsilon^a \gamma_5 \lambda^a} (\chi(x) - \psi_{\mathrm{lat}}(x)) - S[\psi, \bar{\psi}] \right] \,. \tag{2.47}$$

In the equation above we used the fact that the continuum massless action is invariant under chiral rotations, and therefore remains unchanged as we rotate the integration variables. It is well-known that the integration measure is not invariant under chiral U(1) rotations; the non-invariance of the measure leads to the U(1) anomaly in the Fujikawa derivation. However if the fermion field is not coupled to a gauge field, the anomaly vanishes. We shall return to this formula for the case of the $\mathrm{U}(1)_A$ symmetry later on when we discuss chiral symmetries in QCD.

Expanding Eq. (2.47) at first order in $\epsilon$:

$$
\begin{aligned}
&\left(-\sum_x \bar{\chi}(x)\ \{\gamma_5, D\}\,\lambda^a\,\chi(x)\right) e^{-\bar{\chi}D\chi} = \\
&\qquad = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left[-\sum_x \alpha(\bar{\chi}(x)-\bar{\psi}_{\mathrm{lat}}(x))2\gamma_5\lambda^a(\chi(x)-\psi_{\mathrm{lat}}(x))\right] \\
&\qquad\quad \times \exp\left[-\sum_x \alpha(\bar{\chi}(x)-\bar{\psi}_{\mathrm{lat}}(x))(\chi(x)-\psi_{\mathrm{lat}}(x)) - S[\psi,\bar{\psi}]\right] \\
&\qquad = -\frac{2}{\alpha}\left(\sum_x\left[\bar{\chi}(x)D\gamma_5 D\lambda^a\chi(x)\right]\right) e^{-\bar{\chi}D\chi}\,. \qquad (2.48)
\end{aligned}
$$

Let us now set $\alpha = 2/a$, multiply the above equation by $\lambda^a$ and take the trace, we obtain the Ginsparg-Wilson equation:

$$\{\gamma_5, D\} = aD\gamma_5 D\,. \qquad (2.49)$$

This condition replaces condition 4. in the Nielsen-Ninomiya theorem. A lattice Dirac operator that satisfies the Ginsparg-Wilson relation will automatically yield an action that is chirally symmetric in the continuum limit. Note that the explicit breaking of chiral symmetry in Eq. (2.49) is proportional to a higher-dimensional operator, which is irrelevant in the continuum limit near a Gaussian fixed point.

The only known solution to the Ginsparg-Wilson equation is the *overlap* operator:

$$D = 1 + \gamma_5\epsilon\left(H(m)\right)\,, \qquad (2.50)$$

where $\epsilon(x)$ is the sign function, and $H$ is the Hermitian Wilson-Dirac operator $H(m) = \gamma_5\left(D_{\mathrm{w}} + m\right)$.

An alternative expression for the overlap operator is obtained by rewriting the sign function:

$$
\begin{aligned}
D &= 1 + \gamma_5\frac{H(m)}{\sqrt{H(m)^2}} \\
&= 1 + \frac{D_{\mathrm{w}}+m}{\sqrt{(D_{\mathrm{w}}+m)^\dagger(D_{\mathrm{w}}+m)}}\,. \qquad (2.51)
\end{aligned}
$$

Without entering into the details, we should mention here that for the overlap operator to represent a single chiral fermion, we need the bare mass in the Hermitian Dirac operator to be in the range:

$$-2 < am < 0\,. \tag{2.52}$$

The overlap operator can be derived from domain-wall fermions, which we do not have time to cover in this notes. Many physical features of overlap fermions become much more transparent in that formulation.

**Locality** Eq. (2.51) suggests that the overlap operator connects a given lattice site with all other sites, the fermion lagrangian is no longer involving only finite-range interactions. It is therefore legitimate to wonder whether the operator satisfies the locality requirement, or whether we may expect singularities in momentum space. The locality of the overlap was proved analytically for sufficiently smooth gauge configurations. Its space-dependence was also studied numerically, showing that the strength of the coupling decreases exponentially with the distance between points. This is sufficient to guarantee locality of the continuum theory.

## 2.8 Lattice chiral symmetry

Overlap fermions share many appealing features of continuum fermions. In particular the fermion mass undergoes a multiplicative renormalization only. The chiral limit of overlap fermions is obtained by tuning the *bare* mass to zero. Likewise the renormalization of composite operators is greatly simplified compared to the case of Wilson fermions. In the continuum these nice properties are direct consequences of the fact that the theory is chirally symetric in the massless limit.

Lüscher realized that a Dirac operator satisfying the GW relation leads to a fermionic action that has an exact symmetry at finite lattice spacing, *i.e.* before the continuum limit is taken. The exact lattice symmetry is given by:

$$\psi \mapsto \left[1 + \epsilon\gamma_5 \left(1 + \frac{a}{2} D\right)\right] \psi\,, \tag{2.53}$$

$$\bar\psi \mapsto \bar\psi \left[1 + \epsilon\gamma_5 \left(1 + \frac{a}{2} D\right)\right]\,. \tag{2.54}$$