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Tutorials to: Neutrino physics I+II

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Exercise 1: The oscillation phase

Departing from the amplitude for vacuum oscillations

$$\mathcal{A}_{\nu_\alpha \to \nu_\beta} = \sum_i U_{\beta i} U_{\alpha i}^* e^{-i(E_i t - p_i x)} \tag{1}$$

derive the oscillation probability

$$P_{\nu_\alpha \to \nu_\beta} = \left| \mathcal{A}_{\nu_\alpha \to \nu_\beta} \right|^2 = \sum_{jk} U_{\alpha j} U_{\beta j}^* U_{\alpha k}^* U_{\beta k} \exp\left[-i \frac{\Delta m_{kj}^2 x}{2\, E_\nu} \right] . \tag{2}$$

Derive the oscillation phase $\phi_{kj} = \Delta m_{kj}^2 x / (2\, E_\nu)$ in the case of two neutrinos only. Avoid the assumption of equal energy or equal momentum for the neutrino mass states, but use that neutrinos are ultra-relativistic.

Hint: use the definitions

$$\Delta X = X_2 - X_1 \,, \quad \Delta X^2 = X_2^2 - X_1^2 \,, \quad \bar{X} = (X_1 + X_2)/2 \,, \tag{3}$$

which imply $\Delta X^2 = 2\bar{X}\Delta X$, for $X = E, p, m$. Furthermore, use the average velocity $v = \bar{p}/\bar{E}$ and $x \approx vt$.

Think about conceptual problems of this derivation. An overview over a consistent calculation and references can be found in Ref. [1]

Exercise 2: Mass and mixing angle in constant matter

Consider two neutrino flavours and start from the effective Hamiltonian in matter

$$H_{\rm mat} = \frac{1}{2E} U(\theta)\, {\rm diag}(m_1^2, m_2^2)\, U^\dagger(\theta) + {\rm diag}(V, 0) \,, \qquad U(\theta) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \tag{4}$$

with $c = \cos\theta, s = \sin\theta$ and $V = \sqrt{2} G_F N_e$ is the effective matter potential, where N_e is the electron density along the neutrino path, which is assumed to be constant.

Show that

$$H_{\rm mat} = \frac{1}{2E} U(\theta_{\rm mat})\, {\rm diag}(m_{1{\rm mat}}^2, m_{2{\rm mat}}^2)\, U^\dagger(\theta_{\rm mat}) \tag{5}$$

with

$$\sin^2 2\theta_{\rm mat} = \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - A)^2} \tag{6}$$

$$\Delta m_{\rm mat}^2 = \Delta m^2 \sqrt{\sin^2 2\theta + (\cos 2\theta - A)^2} \tag{7}$$

where $A \equiv 2EV/\Delta m^2$, $\Delta m^2 \equiv m_2^2 - m_1^2$ and similar for $\Delta m^2_{\rm mat}$.

Discuss the behaviour of $m^2_{1\rm mat}$, $m^2_{2\rm mat}$, and $\sin^2 2\theta_{\rm mat}$ as a function of A (including its sign).

LBL appearance experiments

The appearance probability in vaccum to second order in θ_{13} and $\Delta m^2_{21}/|\Delta m^2_{31}|$ is given by

$$P_{\mu\to e} \approx s_{23}^2\, S^2\, \sin^2\Delta + \sin 2\theta_{23}\,\tilde\alpha\, S\, \sin\Delta\cos(\Delta \pm \delta_{\rm CP}) + c_{23}^2\,\tilde\alpha^2 \tag{8}$$

with

$$S \equiv \sin 2\theta_{13}\,, \quad \Delta \equiv \frac{\Delta m^2_{31} L}{4E}\,, \quad \tilde\alpha \equiv \sin 2\theta_{12}\,\frac{\Delta m^2_{21} L}{4E}\,, \tag{9}$$

and $s_{23} \equiv \sin\theta_{23}$, $c_{23} \equiv \cos\theta_{23}$. For neutrinos (anti-neutrinos) holds the upper (lower) sign, for $e \to \mu$ transitions exchange $\delta_{\rm CP} \to -\delta_{\rm CP}$.[1] The neutrino mass hierarchy is determined by the sign of Δ. A discussion of LBL oscillation probabilities can be found for example in Ref. [2], see also Ref. [3].

Exercise 3: The $\tilde\alpha^2$-term

Consider an experiment at the first oscillation maximum and estimate the size of the $\tilde\alpha^2$ term in the oscillation probability (third term in Eq. 8). Give the range for $\sin^2 2\theta_{13}$, where this term can be neglected. What does this imply, given the value of $\sin^2 2\theta_{13}$found recently by reactor experiments?

Exercise 4: $\sin^2 2\theta_{13}$-determination in appearance experiments

Consider an experiment at the first oscillation maximum which measures some value for $P_{\mu\to e}$. Suppose this is just a counting experiment and ignore the energy dependence of the signal. To a good approximation this applies to current data from the T2K experiment.

a) Using Eq. 8, estimate the shape of the allowed region for $\sin^2 2\theta_{13}$ as a function of $\delta_{\rm CP}$.

b) How does this shape depend on whether neutrinos or anti-neutrinos are used?

c) Discuss the dependence of the region on θ_{23}.

Hint: use the measured value of $\sin^2 2\theta_{13}$ from reactor experiments and the results of exercise 3 to motivate whether the $\tilde\alpha^2$-term in Eq. 8 has to be considered or not.

[1]Remember: in vacuum the CP-conjugation (exchaning ν with $\overline{\nu}$) is equivalent to the T-conjugation (exchaning initial and final neutrino flavours), as a consequence of CPT invariance.

Exercise 5: The $\text{sign}(\Delta m^2_{31})$-degeneracy

a) Show that in vacuum the relation

$$P_{\mu\to e}(\Delta m^2_{31}, S, \delta_{\rm CP}) = P_{\mu\to e}(-\Delta m^2_{31}, S, \delta'_{\rm CP}) \tag{10}$$

can be fulfilled simultaneously for neutrinos and anti-neutrinos, and independent of the neutrino energy. Determine $\delta'_{\rm CP}$.

b) Consider the case of small matter effect. Without performing any calculations, give an argument why the leading order matter effect correction to Eq. 8 cannot break the $\text{sign}(\Delta m^2_{31})$-degeneracy and similar to Eq. 11, a relation

$$P_{\mu\to e}(\Delta m^2_{31}, S, \delta_{\rm CP}) = P_{\mu\to e}(-\Delta m^2_{31}, S', \delta'_{\rm CP}) \tag{11}$$

still can be satisfied for neutrinos and anti-neutrinos simultaneously.

The hierarchy degeneracy was first noted in Ref. [4], a recent discussion with some analytical considerations can be found in Ref. [5]. The classical paper on the eight-fold degeneracy (including the intrinsic, $\text{sign}(\Delta m^2_{31})$, and octant degeneracies) is Ref. [6].

Exercise 6: Majorana mass term

The charge conjugated field is defined as

$$\psi^c \equiv C\overline{\psi}^T = C\gamma_0\psi^* \tag{12}$$

where the charge conjugation matrix C has the following properties:

$$C^\dagger = C^{-1}\,, \quad C^T = -C\,, \quad C\gamma_\mu^T C^{-1} = -\gamma_\mu\,. \tag{13}$$

a) Show the quivalence of the following notations for the Majorana mass term

$$\frac{m}{2}\psi_L^T C^{-1}\psi_L + \text{h.c.} = -\frac{m}{2}\overline{(\psi_L)^c}\psi_L + \text{h.c.} = -\frac{m}{2}\overline{\psi}\psi \quad \text{with} \quad \psi = \psi_L + (\psi_L)^c\,. \tag{14}$$

b) Show that a Majorana mass matrix has to be symmetric. (Hint: use the anti-commutation rule for fermion fields.)

c) Consider a Lagrangian with one left-handed and one right-handed fermion with mass terms of the following form:

$$\mathcal{L}_M = -m_D\overline{\psi_L}\psi_R + \frac{m_L}{2}\psi_L^T C^{-1}\psi_L + \frac{m_R}{2}\psi_R^T C^{-1}\psi_R\,. \tag{15}$$

Show that this can be cast into the form of a Majorana mass term in the following way:

$$\mathcal{L}_M = \frac{1}{2}\psi^T C^{-1}\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}\psi + \text{h.c.} \quad \text{with} \quad \psi = \begin{pmatrix} \psi_L \\ (\psi_R)^c \end{pmatrix}\,. \tag{16}$$

d) Assume that m_D, m_L, m_R are real (this corresponds to CP conservation). Diagonalize the mass matrix in eq. 16. What are the mass eigenvalues and the mass eigenfields?

e) Consider the two limiting cases (i) $m_L, m_R \ll m_D$ and (ii) $m_L \ll m_D \ll m_R$. In both cases discuss the mass eigenvalues and the mass eigenfields. Give an interpretation of your results.

A discussion along these lines can be found in [7].

References

[1] E. K. .Akhmedov and J. Kopp, "Neutrino oscillations: Quantum mechanics vs. quantum field theory," JHEP **1004** (2010) 008 [arXiv:1001.4815 [hep-ph]].

[2] E. K. Akhmedov, R. Johansson, M. Lindner, T. Ohlsson and T. Schwetz, "Series expansions for three-flavor neutrino oscillation probabilities in matter," JHEP **0404** (2004) 078 [arXiv:hep-ph/0402175].

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[7] S. M. Bilenky, C. Giunti and W. Grimus, "Phenomenology of neutrino oscillations," Prog. Part. Nucl. Phys. **43** (1999) 1 [hep-ph/9812360].