Lecture 3

Gauge fields

3.1 Introduction

The geometric interpretation of gauge theories in the continuum limit suggests a natural way to implement them on a discretized space-time lattice. Starting with a free fermion in the continuum, we assume that it transforms according to a given representation R of a group G, which we call the gauge group:

$$\psi_c(x) \mapsto (\Lambda_R)_{cd}\psi_d(x)\,, \tag{3.1}$$

where Λ_R is the matrix representing the group element in the representation R. We shal denote by d_R the dimension of the representation, so that the color indices $c, d = 1, \ldots, d_R$. The matrix Λ_R can be parametrized as:

$$\Lambda_R = \exp\left[i\omega^a T_R^a\right]\,, \tag{3.2}$$

where T_R^a are the generators of the gauge group. We assume that the generators are Hermitian, and the representation unitary.

The free action:

$$S[\psi,\bar{\psi}] = \int d^D x\, \bar{\psi}(x)\left(\gamma_\mu\partial_\mu + m_0\right)\psi(x) \tag{3.3}$$

is clearly invariant under global transformations Eq. (3.1). However the derivative term is no longer invariant if we make the gauge transformation dependent on the space-time position - these are called *local* gauge transformations.

Imposing invariance under local transformations means that the coordinates in color space are position dependent, and therefore the comparison of two vectors at different points is meaningful only after having performed a *parallel transport* of the two vectors to the same point. The infinitesimal variation of a vector due to parallel transport is described in a geometrical context by the affine connection. The geometrical interpretation of gauge theories identifies the gauge potential $igA_\mu^a(x)T_R^a$ with the affine connection in color space. The covariant derivative describes the infinitesimal change in the vector $\psi_a(x)$ as we move to a neighbouring site $x + dx$; there are two terms, one being the change due to moving the vectors to the same reference frame, and the other due the difference between the two vectors measured in the same reference frame:

$$(D_\mu\psi(x))_c = \left(\partial_\mu\delta_{cd} + igA_\mu^a(x)T_{cd}^a\right)\psi_d(x)\,. \tag{3.4}$$

If A^μ transforms as:

$$A_\mu^a T^a \mapsto \Lambda_R A_\mu^a T^a \Lambda_R^\dagger - \frac{i}{g}[\partial_\mu\Lambda_R]\Lambda_R^\dagger\,, \tag{3.5}$$

then the covariant derivative transforms like the fermion field ψ:

$$D_\mu\psi(x) \mapsto \Lambda_R(x) D_\mu\psi(x)\,, \tag{3.6}$$

and the action:

$$S = \int d^D x \, \bar{\psi}(x) \left[\gamma_\mu D_\mu + m_0 \right] \psi(x) \tag{3.7}$$

is invariant under gauge transformations.

In trying to build a gauge-invariant theory on a discrete space-time, the continuum derivatives are replaced by finite differences. These involve the difference of matter fields at different space-time points, separated by a finite amount given by the lattice spacing. Hence the action contains products of fields at different sites:

$$\bar{\psi}(x)\gamma_\mu \psi(x + a\mu) \, . \tag{3.8}$$

Under a local gauge transformation:

$$\bar{\psi}(x)\gamma_\mu \psi(x + a\mu) \mapsto \bar{\psi}(x)\Lambda_R(x)^\dagger \gamma_\mu \Lambda_R(x + a\mu)\psi(x + a\mu) \, , \tag{3.9}$$

which is clearly not invariant.

According to the geometric picture above, a gauge invariant action can be obtained by parallel transporting the matter field along the lattice link connecting x and $x + a\hat{\mu}$. The parallel transporter along a lattice link is obtained by integrating the affine connection:

$$U_R(x, \mu) = \mathrm{P} \exp \left\{ ig \int_x^{x+a\hat{\mu}} dx_\mu A_\mu^a(x) T_R^a \right\} \, , \tag{3.10}$$

where P indicates a path-ordered integral. $U_R(x, \mu)$ are called link variables, they are the basic degrees of freedom in the lattice formulation of gauge theories. Note that they are elements of the group; this is different from the continuum formulation, where the basic ingredient is the gauge potential, which is an element of the algebra. Under a gauge transformation:

$$U_R(x, \mu) \mapsto \Lambda_R(x) U_R(x, \mu) \Lambda_R(x + a\hat{\mu})^\dagger \, . \tag{3.11}$$

We will sometime denote the gauge-transformed variable by:

$$ccccU_R(x, \mu)^\Lambda \equiv \Lambda_R(x) U_R(x, mu) \Lambda_R(x + a\hat{\mu})^\dagger \, . \tag{3.12}$$

It is easy to check that the introduction of lattice covariant derivatives:

$$\nabla_\mu \psi(x) = \frac{1}{a} \left[U(x, \mu)\psi(x + a\hat{\mu}) - \psi(x) \right] \, , \tag{3.13}$$

$$\nabla_\mu^* \psi(x) = \frac{1}{a} \left[\psi(x) - U(x - a\hat{\mu}, \mu)^\dagger \psi(x - a\hat{\mu}) \right] \, , \tag{3.14}$$

leads to a gauge-invariant action.

In the naive continuum limit $a \to 0$, the lattice covariant derivative reduces to the usual covariant derivative introduced in the continuum, so that the lattice theory is expected to reproduce the usual minimal coupling betweent the gauge potential and the matter fields. Note however that the lattice covariant derivative introduces an infinite number of vertices in the Feynman rules, corresponding to higher-dimensional operators with several gluons interacting with a fermion-antifermion pair. We shall come back to the Feynman rules, and see a few examples of perturbative computations on the lattice later.

3.2 Lattice action

Gauge-invariant operators made of link variables can be used in order to build a kinetic term for the gauge field on the lattice. In the continuum formulation the operator:

$$\mathrm{Tr}\left(G_{\mu\nu}G_{\mu\nu}\right) , \tag{3.15}$$

where

$$\begin{aligned} G_{\mu\nu} &= g G^a_{\mu\nu} T^a & (3.16) \\ &= [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] \ . & (3.17) \end{aligned}$$

is the only gauge-invariant operator of dimension four that can be built. Higher-dimensional operators are irrelevant at the gaussian fixed point, where the continuum limit is defined.

On the lattice, the trace of a product of link variables along a closed path is gauge invariant. The simplest choice is the 1×1 plaquette:

$$P^R_{\mu\nu}(x) = U_R(x,\mu)U_R(x+a\hat{\mu},\nu)U_R(x+a\hat{\nu},\mu)^\dagger U_R(x,\nu)^\dagger \ . \tag{3.18}$$

Because the link variables are group elements, the plaquette can be defined in any representation of the gauge group. In the continuum limit different choices of R yield the same physics, but the phase structure of the lattice theory can be different. Because the theory is usually described in terms of the gauge potential in the continuum, the distinction between representations is not important for the pure gauge theory.

The pure gauge action can be written as:

$$S[U] = \beta \sum_{x,\mu<\nu} \mathrm{ReTr}\left[1 - P_{\mu\nu}(x)\right] \ . \tag{3.19}$$

The coupling β is related to the usual coupling constant g by taking the naive continuum limit:

$$\frac{1}{g^2} = \frac{\beta}{2N} , \tag{3.20}$$

for the link variables in the fundamental representation. As discussed above, the link variables can be defined in an arbitrary representation. Mixed representations can be used for a generic lattice action:

$$S = \sum_R \beta_R \sum_{x,\mu<\nu} \frac{\mathrm{ReTr}_R\left[1 - P^R_{\mu\nu}(x)\right]}{d_R}\,, \tag{3.21}$$

where we have introduced a coupling β_R for each representation, and normalized each term to the dimension d_R of the representation. Tr_R indicates the trace in the R representation. In the naive continuum limit this action reduces to the usual form with:

$$\frac{1}{g^2} = \sum_R \frac{\beta_R t_R}{d_R}, \quad \mathrm{Tr}\left(T^a_R T^b_R\right) = t_R \delta_{ab}\,. \tag{3.22}$$

More complicated closed paths can be used to define the lattice action. They all yield the same classical continuum limit, but the actions differs at the level of higher-dimensional operators. Hence different choices are used to minimize the size of the leading lattice artefacts. Improved actions for gauge theories have been introduced *e.g.* by Luscher-Weisz, and Iwasaki.

3.3 Path integral

As discussed above, the geometric interpretation of gauge theories suggests that the link variables should be used as the fundamental degrees of freedom for the lattice gauge fields. Let us assume here that the link variables are chosen in the fundamental representation of SU(N). The dynamics is specified by the gauge-invariant action, and therefore it is natural to define the path integral as:

$$Z = \int \prod_{x,\mu} dU(x,\mu)\, e^{-S[U]}\,. \tag{3.23}$$

For each link, the integration over $dU(x,\mu)$ extends over the whole group. Since the group is compact, the total volume is finite and the path integral is well defined for a finite lattice. Note that some of the complications that are usually encountered in the continuum do not arise on the lattice precisely because the link variables are group elements rather than elements of the algebra. For the path integral to be gauge-invariant, we need the integration measure dU to be invariant under left- and right-multiplication by an element of the group:

$$dU = dV, \quad V = \Lambda U \Lambda', \quad \forall \Lambda, \Lambda' \in \mathrm{SU(N)}\,. \tag{3.24}$$

The Haar measure satisfies this condition. Let us parametrize the group elements by a set of coordinates in group space: $U = U(\alpha)$, with $\alpha_k \in \mathbb{R}$. A metric g can be defined:

$$g_{kl} = 2\mathrm{Tr}\left(\frac{\partial U}{\partial \alpha_k}\frac{\partial U^\dagger}{\partial \alpha_l}\right)\,. \tag{3.25}$$

The Haar measure is defined as:

$$dU = \kappa \sqrt{\det g} \prod_k d\alpha_k \,, \tag{3.26}$$

where the constant κ is defined by the normalization condition:

$$\int dU = 1 \,. \tag{3.27}$$

Some useful integrals for SU(N):

$$\int dU U_{ab} = 0 \,, \tag{3.28}$$

$$\int dU U_{a_1 b_1} U_{a_2 b_2} \dots U_{a_n b_n} = 0, \quad \text{for } n < N \,, \tag{3.29}$$

$$\int dU U_{ab} U^\dagger_{cd} = \frac{1}{N} \delta_{ad} \delta_{bc} \,, \tag{3.30}$$

$$\int dU U_{a_1 b_1} U_{a_2 b_2} \dots U_{a_N b_N} = \frac{1}{N!} \epsilon_{a_1 a_2 \dots a_N} \epsilon_{b_1 b_2 \dots b_N} \,. \tag{3.31}$$

Having defined the integration measure, we have a well-defined gauge invariant path integral for gauge theories. The expectation value of any operator is defined as usual:

$$\langle O[U] \rangle = Z^{-1} \int dU O[U] e^{S[U]} \,, \tag{3.32}$$

and can be computed by numerical methods. It is clear from the definition above that the expectation value of the operators that are not gauge invariant has to vanish.

3.4 Hilbert space and transfer operator

Following the scheme introduced in previous chapters, let us discuss the construction of the transfer matrix for the lattice gauge. The field on the spatial links $U(\mathbf{x}, m)$, for $m = 1, 2, 3$, are promoted to operators $\hat{U}(\mathbf{x}, m)$ acting on the quantum states $|\psi\rangle$ in the Hilbert space of physical states $\mathcal{H}$. The basis of eigenstates of $\hat{U}(\mathbf{x}, m)$ is denoted $|U\rangle$:

$$\hat{U}(\mathbf{x}, m)_{ab} |U\rangle = U(\mathbf{x}, m)_{ab} |U\rangle \,. \tag{3.33}$$

In order to define the transfer matrix, we need to rewrite the path integral as:

$$\begin{aligned} Z &= \mathrm{Tr}\left(\hat{T}^N\right) \\ &= \prod_{n_4=0}^{N-1} \left[\int \prod_{\mathbf{x},m} dU(x,m) \, \langle U(n_4+1) | \hat{T} | U(n_4) \rangle \right] \,, \end{aligned} \tag{3.34}$$

where $x_\mu = (\mathbf{x}, n_4 a)$, and N is the number of time-slices in the lattice.

The operator $\hat{T}$ is readily identify by separating the contribution to the action coming from the time-like and space-like plaquettes:

$$S[U] = \text{const } - \beta \sum_x \left[\sum_j 2\mathrm{ReTr} P_{j4}(x) + \sum_{i<j} 2\mathrm{ReTr} P_{ij}(x) \right] . \tag{3.35}$$

This expression separates clearly the contributions to the action that come from the field configuration at a single time-slice, from the ones that involve the field configuration at two consecutive times. Using Eq. (3.35), and comparing the expression for the path integral with Eq. (3.34), we see that the transfer matrix can be written as:

$$\begin{aligned} \hat{T} &= e^{\frac{1}{2}a\hat{W}} \hat{T}'_K e^{\frac{1}{2}a\hat{W}} , \\ \hat{W} &= 2\beta \sum_{\mathbf{x}, i<j} \mathrm{ReTr} \hat{P}_{ij}(\mathbf{x}) . \end{aligned} \tag{3.36}$$

Note that in Eq. (3.34) there no integration over the temporal links. The latter integration is hidden in the kinetic operator $\hat{T}'_K$:

$$\langle U' | \hat{T}'_K | U \rangle = \prod_{\mathbf{x},m} \int dU(\mathbf{x},4) \, \exp\left[2\beta \mathrm{ReTr} \left(U(\mathbf{x},m) U(\mathbf{x} + a\hat{m}, 4) U'(\mathbf{x},m)^\dagger U(\mathbf{x},4)^\dagger \right) \right] . \tag{3.37}$$

We can rewrite $\hat{T}'_K$ in a different form, which highlights its physical content. Note that the temporal links $U(\mathbf{x},4)$ in the Eq. (3.37) play the role of a gauge transformation. If we relabel:

$$\Lambda(\mathbf{x}) = U(\mathbf{x},4)^\dagger , \tag{3.38}$$

then the product in the exponent looks very familiar:

$$U(\mathbf{x},4)^\dagger U(\mathbf{x},m) U(\mathbf{x} + a\hat{m}, 4) = \Lambda(\mathbf{x}) U(\mathbf{x},m) \Lambda(\mathbf{x} + a\hat{m})^\dagger . \tag{3.39}$$

Let us define a gauge transformation operator:

$$\hat{D}(\Lambda) | U \rangle = | U^{\Lambda^\dagger} \rangle , \tag{3.40}$$

and the projector onto the gauge-invariant states:

$$\hat{P}_0 = \int \prod_{\mathbf{x}} d\Lambda(\mathbf{x}) \, \hat{D}(\Lambda) . \tag{3.41}$$

We can rewrite the kinetic term in a very simple fashion:

$$\hat{T}'_K = \hat{T}_K \hat{P}_0 = \hat{P}_0 \hat{T}_K \,,$$
$$\langle U'|\hat{T}_K|U\rangle = \prod_{\mathbf{x},m} \exp\left[\beta \mathrm{ReTr}\left(U(\mathbf{x},m)U'(\mathbf{x},m)^\dagger\right)\right] \,. \tag{3.42}$$

The operator $\hat{T}_K$ is positive, and can be written as:

$$\hat{T}_K = e^{a\hat{K}} \,, \tag{3.43}$$

where $\hat{K}$ is a Hermitian operator.

Summarizing, the transfer matrix can be written as the product of a potential part and a kinetic part. Both operators are positive, and hence $\hat{T}$ is also positive, and leads to the definition of a Hamiltonian acting in the Hilbert space of states. In constructing the transfer matrix, the path integral has automatically generated a projector operator $\hat{P}_0$. The physical states must be gauge-invariant. This is the analogue of Gauss' law when quantizing gauge theories in the temporal gauge in the continuum.

3.5 Strong coupling

The strong coupling expansion is an expansion of the path integral in powers of β. It is the analogue of the high-temperature expansion in statistical mechanics, and has typically a non-zero radius of convergence. Unfortunately the strong coupling phase is often separated from continuum physics by some phase transition; in these cases the strong coupling expansion does not provide useful information on the continuum physics. It is nonetheless a useful tool to get quantitative results in a given region of the phase diagram of the lattice theory.

The strong coupling expansion is obtained by expanding the action:

$$\begin{aligned} Z &= \int \prod_{x,\mu} dU(x,\mu)\, \exp\left[\beta \sum_{x,\mu<\nu} \mathrm{Tr}\left(P_{\mu\nu}(x) + P_{\mu\nu}(x)^*\right)\right] \\ &= \int \prod_{x,\mu} dU(x,\mu) \prod_{x,\mu<\nu} \left[\sum_n \frac{\beta^n}{n!} \left(\mathrm{Tr}\left(P_{\mu\nu}(x) + P_{\mu\nu}(x)^*\right)\right)^n\right] \,. \end{aligned} \tag{3.44}$$

The expansion consists of monomials of the trace of the plaquette. Each plaquette appears with one factor of β, so that the leading contribution to a given observable at strong coupling is obtained by considering the term in the expansion with the smallest power of β that yields a non-vanishing contribution according to the integration rules given in Eqs. (3.28)-(3.31).

The static potential The potential between static quarks can be extracted from the propagation in time of a quark-antiquark state. Such a state can be created by acting on the vacuum with a suitable gauge invariant operator:

$$|\phi_{\alpha\beta}(\mathbf{x},\mathbf{y})\rangle = \bar{\psi}_\alpha(\mathbf{x},0)U(\mathbf{x},0;\mathbf{y},0)\psi_\beta(\mathbf{y},0)|0\rangle\,, \tag{3.45}$$

where α and β are spinor indices, $\mathbf{x}$ and $\mathbf{y}$ indicate the spatial positions of the particles, and $U(\mathbf{x},0;\mathbf{y},0)$ is a parallel transporter, which is required to ensure that the operator is gauge-invariant. The asymptotic behaviour of the quark-antiquark correlator in Euclidean time yields the energy of the ground state:

$$\begin{aligned}&G_{\alpha'\beta'\alpha\beta}(\mathbf{x}',\mathbf{y}',T;\mathbf{x},\mathbf{y},0) = \\ &\langle\bar{\psi}_{\alpha'}(\mathbf{x}',T)U(\mathbf{x}',T;\mathbf{y}',T)\psi_{\beta'}(\mathbf{y}',T)\bar{\psi}_\alpha(\mathbf{x},0)U(\mathbf{x},0;\mathbf{y},0)\psi_\beta(\mathbf{y},0)\rangle \propto \\ &\qquad\qquad \propto \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{y}-\mathbf{y}')C_{\alpha'\beta'\alpha\beta}e^{-E(R)T}\,,\end{aligned} \tag{3.46}$$

where $R = |\mathbf{x}-\mathbf{y}|$. In the static limit, *i.e.* in the limit where the mass of the fermions goes to infinity, the fermion propagators become parallel transporter in the temporal direction, and hence:

$$\begin{aligned}&G_{\alpha'\beta'\alpha\beta}(\mathbf{x}',\mathbf{y}',T;\mathbf{x},\mathbf{y},0) \\ &\quad \stackrel{m\to\infty}{\longrightarrow} \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{y}-\mathbf{y}')(P_+)_{\alpha\alpha'}(P_-)_{\beta\beta'}\,e^{-2mT}\langle\mathrm{Tr}\left(\mathrm{P}e^{ig\oint dz_\mu A_\mu(z)}\right)\rangle\,,\end{aligned} \tag{3.47}$$

where the path-ordered line integral extends over a rectangular $R\times T$ path, and the projectors are defined as:

$$P_\pm = \frac{1\pm\gamma_4}{2}\,. \tag{3.48}$$

On the lattice the path-ordered line integral is simply given by the product of the link variables along the rectangular contour. Such a product is called a Wilson loop:

$$W_C[U] = \prod_{(x,\mu)\in C} U(x,\mu)\,. \tag{3.49}$$

The Wilson loop is the parallel transporter around the rectangular loop. Its trace is gauge invariant and is the observable used to compute the static potential:

$$W(R,T) = \langle\mathrm{Tr}\ W_C[U]\rangle \stackrel{t\to\infty}{\longrightarrow} F(R)e^{-V(R)T}\,. \tag{3.50}$$

Note that the contribution of the static fermion mass to the energy drops out when comparing Eq. (3.46) with Eq. (3.47), and the asymptotic behaviour of the Wilson loop is dictated by the interaction energy only.

As an illustration of strong coupling techniques we are going to compute the static quark potential at leading order in β. The expectation is Wilson loop is:

$$W(R,T) = \frac{\int dU \mathrm{Tr} W_C[U] e^{-S[U]}}{\int dU e^{-S[U]}}\,, \tag{3.51}$$

and we need to expand both the numerator and the denominator in powers of β. It is clear from the integration rules in Eqs. (3.28)-(3.31) that the leading contribution to the denominator is obtained by tiling the interior of the Wilson loop with plaquettes. In this way all the link variables along the contour of the Wilson loop are paired with link variables in the opposite direction coming from the plaquettes in the expansion. The links inside the loop are paired because of adjacent plaquettes appearing in the Taylor series. The leading contribution is shown in Fig. (**??**). Hence the leading contribution to the numerator is proportional to β^A, where $A = R \times T$ is the area of the Wilson loop. The leading contribution to the denominator is simply obtained at $O(\beta^0)$, where the path integral is equal to one, leading to:

$$W(R,T) \simeq \beta^{RT}\,, \tag{3.52}$$

and therefore for the potential:

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \log W_C(R,T) = \sigma(\beta) R\,. \tag{3.53}$$

We have obtained a linear potential, characterized by a string tension σ. Lattice gauge theories at strong coupling are confining; and the string tension scales as:

$$a^2\sigma \simeq \log(\beta)\,. \tag{3.54}$$

Unfortunately the strong coupling expansion does not describe the continuum limit of gauge theories, which is reached for $\beta \to \infty$.

3.6 Weak coupling

Lattice gauge theories provide a nonperturbative definition of QFT, which does not rely on expanding the path integral in powers of the coupling constant. However perturbative methods can be applied to evaluate the path integral defined on a lattice. These perturbative calculations are necessary to connect lattice gauge theories to other regularization schemes. Due to asymptotic freedom, different schemes can be related reliably at high energy scales by perturbative computations. Perturbation theory is also useful to study the scaling violations, *i.e.* the behaviour of the theory near the continuum limit, where the bare coupling is small.

The techniques to setup the perturbative expansion are the same as in the continuum. Rather than providing a detailed derivation, we will focus here on the main differences between the continuum and the lattice theory.

Link integration measure Perturbation theory is derived using the gauge potential as the integration variable in the path integral. Therefore we need to express the Haar measure as a function of the elements of the algebra:

$$U(x,\mu) = \exp\left[iaA_\mu(x)\right], \quad A_\mu(x) = gA_\mu^a(x)T^a\,. \tag{3.55}$$

As discussed earlier

$$dU = \prod_{x,\mu} \sqrt{\det\left[\frac{1}{2}M(A_\mu(x))^\dagger M(A_\mu(x))\right]} \times \prod_{x,\mu,a} dA_\mu^a(x)\,, \tag{3.56}$$

where

$$M(A_\mu) = \frac{1-e^{-iA_\mu}}{iA_\mu}\,. \tag{3.57}$$

Hence

$$\begin{aligned} \mathcal{D}U &= \mathcal{D}A\, e^{-S_{\text{meas}}}\,, \\ S_{\text{meas}} &= -\frac{1}{2}\sum_{x,\mu} \operatorname{Tr}\log\left[\frac{2(1-\cos A_\mu(x))}{A_\mu(x)^2}\right]\,. \end{aligned} \tag{3.58}$$

Faddeev-Popov determinant Let us choose a local gauge condition, which is linear in the gauge potential:

$$\mathcal{F}_{x,a}[A,\chi] = \partial_\mu^* A_\mu^a(x) - \chi^a = 0\,. \tag{3.59}$$

We shall follow the standard procedure by Faddeev & Popov to include this condition into the functional integral. The main focus in this section is on highlighting the peculiarities of the lattice formulation. We define $\Delta_{\text{FP}}[a,\chi]$ by:

$$1 = \Delta_{\text{FP}}[A,\chi] \int \mathcal{D}g \prod_{x,a} \delta\left(\mathcal{F}_{x,a}[{}^gA,\chi]\right)\,, \tag{3.60}$$

where $({}^gA)$ is a short-hand notation for the gauge transforms of the group parameters $A_\mu^a(x)$. Inserting this identity in the path integral for the expectation value of an observable $\mathcal{O}$ yields:

$$\begin{aligned} \langle\mathcal{O}\rangle = Z^{-1} &\int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\, \Delta_{\text{FP}}[A,\chi] \\ &\times \left[\prod_{x,a}\delta\left(\mathcal{F}_{x,a}[A,\chi]\right)\right] \mathcal{O}(A,\psi,\bar{\psi}) e^{-S[A,\psi,\bar{\psi}]-S_{\text{meas}}}\,. \end{aligned} \tag{3.61}$$

$\Delta_{\rm FP}$ is defined implicitly by Eq. (3.60). Note that, because of the δ function appearing in Eq. (3.61), $\Delta_{\rm FP}$ only needs to be computed for fields that satisfy the gauge fixing condition Eq. (3.59). The integral in Eq. (3.60) can therefore be restricted to an infinitesimal neighbourhood of the identity $g = e^{i\epsilon^a T^a}$. In this case the gauge transformed A_μ field reads:

$$\delta_\epsilon A_\mu = -iga \sum_b \left(\hat{D}_\mu[A]\right)_{ab} \epsilon^b(x)\,, \tag{3.62}$$

where

$$\hat{D}_\mu[A] = M(A_\mu)^{-1}\partial_\mu + \frac{1}{g}A_\mu(x)\,. \tag{3.63}$$

Note that this is different form the continuum result, where the variation of the gauge field under an infinitesimal gauge transformation is linear in the field itself. The nonlinearity is a lattice artefacts. In the classical continuum limit, the usual result is recovered:

$$g\delta_\epsilon A_\mu^b \stackrel{a\to 0}{\longrightarrow} -\sum_c (D_\mu)_{bc}\epsilon^c(x)\,, \tag{3.64}$$

where D_μ is the usual covariant derivative. You can find the detailed computation in the problem section. Using the expression for $\delta_\epsilon A_\mu$ we obtain:

$$\mathcal{F}_{x,a}[{}^gA,\chi] = -\sum_{y,b} L_{xa,yb}[A]\epsilon^b(y), \quad \text{for } g \approx 1\,, \tag{3.65}$$

$$L_{xa,yb}[A] = \partial^*_\mu \hat{D}_\mu[A]_{ab}\,\delta_{xy}\,. \tag{3.66}$$

The latter expression is the analogue of the more familiar $\partial_\mu D_\mu \delta(x-y)$ expression obtained in the continuum theory. Plugging Eq. (3.65) in Eq. (3.60), and substituing:

$$\mathcal{D}g = \prod_{x,a} d\epsilon(x,a)\,, \tag{3.67}$$

for the integration measure around the identity, we obtain:

$$\begin{aligned}\Delta_{\rm FP}[A,\chi] &= \det(-L)\\ &= \int \mathcal{D}c\mathcal{D}\bar{c}\,\exp\left(-S_{\rm FP}[A,c,\bar{c}]\right)\,.\end{aligned} \tag{3.68}$$

The integration variables $c^a(x), \bar{c}^a(x)$ are Grassman variables which have a color and a spacetime index, but no Dirac index. They are the lattice analogues of the Faddeev-Popov ghosts. The FP action is:

$$S_{\rm FP}[A,c,\bar{c}] = -\sum_{x,a,b} \bar{c}^a(x)\partial^*_\mu \hat{D}_\mu[U]_{ab} c^b(x)\,. \tag{3.69}$$

In order to get rid of the δ functions in the path integral, we can average the expression over the fields χ, with a Gaussian weight:

$$S_{\mathrm{GF}}[U] = \frac{1}{2\alpha} \sum_{x,a} \left(\partial^*_\mu A_\mu\right)^2 . \tag{3.70}$$

The complete expression for the action of the gauge theory coupled to fermions reads:

$$S_{\mathrm{tot}} = S_{\mathrm{G}}[U] + S_{\mathrm{F}}[U,\psi,\bar{\psi}] + S_{\mathrm{GF}}[U] + S_{\mathrm{meas}}[U] + S_{\mathrm{FP}}[U,c,\bar{c}] . \tag{3.71}$$

This action with three light fermions in the fundamental representation of SU(3) defines lattice QCD. Different numbers of flavors, gauge groups, and representations have been used to propose models of strongly-interacting theories beyond the Standard Model.

From the action above, the Feynman rules for the perturbative computation of the path integral can be derived by standard tools. We summarize here the result for Wilson fermions and the Wilson plaquette action.

- The gluon propagator is given by:

$$\frac{1}{\tilde{k}^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right) . \tag{3.72}$$

- The fermion propagator has already been discussed in the previous chapter:

$$\left(\frac{1}{i\sum_\mu \gamma_\mu \hat{p}_\mu} + \tilde{M}(p)\right)_{\alpha\beta} \delta_{ab} . \tag{3.73}$$

- The ghost propagator:

$$\frac{1}{\tilde{k}^2}\delta_{ab} . \tag{3.74}$$

- The gluon-fermion vertex:

$$-ig(2\pi)^4 \delta_P(k+p-p') \left[\gamma_\mu \cos\left(\frac{(p+p')_\mu a}{2}\right) - ir \sin\left(\frac{(p+p')_\mu a}{2}\right)\right]_{\alpha\beta} T^A_{ab} . \tag{3.75}$$

- The 3-gluon vertex:

$$ig(2\pi)^4 \delta_P(k+k'+k'') f_{ABC} \left[\delta_{\nu\lambda}(\widetilde{k''-k'})_\mu \cos\frac{k_\nu a}{2} + \delta_{\mu\lambda}(\widetilde{k-k''})_\nu \cos\frac{k'_\lambda a}{2} + \delta_{\mu\nu}(\widetilde{k'-k})_\lambda \cos\frac{k''_\mu a}{2}\right] . \tag{3.76}$$

- The gluon-ghost vertex:

$$ig(2\pi)^4\delta_P(k+p-p')f_{ABC}\tilde{p}'_\mu \cos(\frac{p_\mu a}{2})\,. \tag{3.77}$$

- The four-gluon vertex has a complicated expression, and we refer the reader to *e.g.* the book by Rothe for a complete expression.

- The two gluon measure insertion:

$$-(2\pi)^4\delta_P(k+k')\,\frac{g^2}{4a^2}\delta_{\mu\nu}\delta_{BC}\,. \tag{3.78}$$

- The 2-gluon-fermion vertex:

$$-\frac{a}{2}g^2(2\pi)^4\delta_P(k+k'+p-p')\,\delta_{\mu\nu}\left\{T^A,T^B\right\}_{ab} \left[r\cos\left(\frac{(p+p')_\mu a}{2}\right)-i\gamma_\mu \sin\left(\frac{(p+p')_\mu a}{2}\right)\right]_{\alpha\beta}\,. \tag{3.79}$$

- The 2-gluon-ghost vertex:

$$-\frac{1}{12}g^2a^2(2\pi)^4\delta_P(k+k'+p-p')\,\left\{T^C,T^D\right\}_{AB}\delta_{\mu\nu}\tilde{p}'_\mu\tilde{p}_\mu\,. \tag{3.80}$$

It is important to remark that the last three diagrams are distinctive of the lattice formulation, and have no continuum analogue. The number of diagrams in lattice perturbation theory is usually larger than in the continuum formulation, and the vertices are more complicated because of the trigonometric functions, so that higher orders in perturbation theory become rapidly very difficult.