Lecture 4

Topics in Lattice QCD

4.1 Introduction

In this chapter we present some features of lattice gauge theories in more detail. First we discuss the usage of RG equations to understand the behaviour of the static potential. We shall then discuss the scheme dependence of the Λ parameter, and some related issues. Finally we concentrate on lattice QCD with Wilson fermions as an example of a phenomenologically relevant application of lattice gauge theories.

4.2 Static potential

Let us discuss the approach to the continuum limit by analysing the behaviour of the static potential in pure gauge theory. For a given value of the lattice cutoff a and bare coupling g_0, using simple dimensional arguments:

$$V(R; g_0, a) = \frac{1}{a}\hat{V}(\frac{R}{a}; g_0)\,. \tag{4.1}$$

The continuum limit is defined by tuning g_0 as $a \to 0$ so that the potential becomes independent of the cutoff. The independence on the cutoff can be written as:

$$a\frac{a}{da}V(R; g_0, a) = \left[a\frac{\partial}{\partial a} - \beta(g_0)\frac{\partial}{\partial g_0}\right] V(R; g_0, a) = 0\,, \tag{4.2}$$

where we have introduced the beta function:

$$\beta(g_0) = -a\frac{\partial g_0}{\partial a}\,. \tag{4.3}$$

Eq. (4.2) is called a Callan-Symanzik (CS) equation; it encodes the scaling behaviour of physical quantities as the continuum limit is approached.

The static potential:

$$V(R; g_0, a) = -\lim_{T\to 0}\frac{1}{T}\log\langle \mathrm{P}\ \exp\left\{ig_0 \oint dz_\mu A_\mu^A(x) T^A\right\}\rangle \tag{4.4}$$

can be computed in perturbation theory. The CS equation should then hold order by order in perturbation theory. In order to extract some nontrivial information we need to evaluate the potential at two loops. The Feynman rules for the lattice gauge theory were discussed in the previous chapter, the detail of the computation can be found in Ref. [?]. The relevant diagram are shown in Fig. **??**. The result is:

$$V(R; g_0, a) = \frac{C}{4\pi R}\left[g_0^2 + \frac{22}{16\pi^2}g_0^4\log\frac{R}{a} + O(g_0^6)\right]\,. \tag{4.5}$$

C is a numerical constant, which is irrelevant in the following. The partial derivatives that appear in the CS equation can be computed:

$$a\frac{\partial}{\partial a}V(R;g_0,a) = -\frac{C}{4\pi R}\frac{22}{16\pi^2}g_0^4 + \dots\,, \tag{4.6}$$

$$\frac{\partial}{\partial g_0}V(R;g_0,a) = \frac{C}{4\pi R}2g_0 + \dots\,, \tag{4.7}$$

where the dots indicate higher order terms, and hence:

$$\beta(g_0) = -\frac{11}{16\pi^2}g_0^3 + O(g_0^5)\,. \tag{4.8}$$

The behaviour of the lattice spacing as a function of the bare coupling at this order is obtained by integrating Eq. (4.3):

$$a\Lambda_L = \exp\left[-\frac{1}{2\beta_0 g_0^2}\right]\,, \tag{4.9}$$

where $\beta_0 = 11/(16\pi^2)$.

Scaling law Combining the CS equation with dimensional analysis yields an interesting scaling law for the static potential. Let us define a rescaeld, dimensionless potential:

$$\tilde{V}(\frac{R}{a};g_0) = RV(R;g_0,a)\,. \tag{4.10}$$

It can be readily shown that the rescaled potential satisfies the equation:

$$\left[R\frac{\partial}{\partial R} + \beta(g_0)\frac{\partial}{\partial g_0}\right]\tilde{V}(\frac{R}{a};g_0) = 0\,. \tag{4.11}$$

At fixed value of the cutoff a, a change in R can be reabsorbed in a redefinition of the bare coupling g_0. Let us choose some reference scale R_0, the distance R can be measured in units of the reference $R = \lambda R_0$. The scaling equation for the potential becomes:

$$\left[\lambda\frac{\partial}{\partial \lambda} + \beta(g_0)\frac{\partial}{\partial g_0}\right]\tilde{V}(\lambda R_0;g_0,a) = 0\,. \tag{4.12}$$

The equation is solved by the method of characteristics:

$$\tilde{V}(\lambda R_0;g_0,a) = \tilde{V}(R_0;\bar{g}_0(\lambda),a)\,, \tag{4.13}$$

$$\lambda\frac{\partial \bar{g}_0}{\partial \lambda} = \beta\left(\bar{g}_0(\lambda)\right)\,, \quad \bar{g}_0(1) = g_0\,. \tag{4.14}$$

Solving the equation for $\bar{g}_0$ yields:

$$\lambda = \exp\left[-\frac{1}{2\beta_0}\left(\frac{1}{\bar{g}_0^2(\lambda)} - \frac{1}{g_0^2}\right)\right]\,. \tag{4.15}$$

Renormalization-group improved perturbation theory Let us now discuss the improvement of the perturbative computation of the static potential that can be obtained from the renormalization group scaling law. The results above can be rewritten as:

$$V(\lambda R_0; g_0, a) = \frac{1}{\lambda} V(R_0; \bar{g}_0(\lambda), a) \,. \tag{4.16}$$

If we normalize the potential in such a way that:

$$V(a; g_0, a) = C \frac{g_0^2}{4\pi a} \,, \tag{4.17}$$

then

$$V(\lambda a; g_0, a) = C \frac{\bar{g}_0(\lambda)^2}{4\pi \lambda a} \,. \tag{4.18}$$

Choosing $\lambda = R/a$ yields:

$$V(R; g_0, a) = C \frac{\bar{g}_0(R/a)^2}{4\pi R} \tag{4.19}$$

$$= \frac{C}{4\pi R} \frac{g_0^2}{1 - 2\beta_0 g_0^2 \log(R/a)} \,. \tag{4.20}$$

It is clear from this expression that the RG equations have allowed us to resum into the running coupling $\bar{g}_0$ the logarithms that appear in the perturbative expansion. Introducing the scale Λ_L:

$$a\Lambda_L = \exp\left(-\frac{1}{2\beta_0 g_0^2}\right) \,, \tag{4.21}$$

the static potential can be rewritten as:

$$V(R; g_0, a) = -\frac{C}{4\pi R} \frac{1}{2\beta_0 \log(R\Lambda_L)} \,. \tag{4.22}$$

Note that in this expression the lattice cutoff a has disappeared, and we can consider the continuum limit of the above equation. The scale Λ_L is a physical scale since it is related to aphysical quantity like the static potential. It is worthwile to point out that a mass scale has emerged in the renormalization process in a theory that is scale invariant at the classical level. This is a manifestation of the fact that the scale invariance is not preserved by the quantum corrections, as illustrated by the trace anomaly. The reaction of the system to scale variations are described by the beta functions, *i.e.* by the scale dependendence of the couplings. A scale-invariant quantum theory is characterized by a vanishing beta function, *i.e.* by $\beta_0 = 0$ at lowest order in perturbation theory. In this limit we see that:

$$\lim_{\beta_0 \to 0} \Lambda_L = 0 \,, \tag{4.23}$$

the scale generated by the theory vanishes.

4.3 Relation between Λ parameters

The definition of the scale Λ_L discussed above clearly depends on the regularization scheme used to define the theory in the first place. Different schemes lead to different values of the Λ parameter. In physical terms the Λ parameter describes the evolution of the running coupling at high-energies. It is a quantity that characterizes the UV behaviour of the couplings in the theory.

Let us first recall how a mass scale naturally emerges when defining a renormalization scheme. A generic renormalized coupling g is defined at some scale μ as a function of the bare coupling and the cutoff:

$$g = g(g_0, a\mu)\,. \tag{4.24}$$

Inverting this relation yields the bare coupling as a function of the renormalized coupling and the cutoff, $g_0 = g_0(g, a\mu)$. Using this relation, a generic observable $\mathcal{O}$ can be written as:

$$\mathcal{O}(g_0, a) = \mathcal{O}(g_0(g, a\mu), a)\,. \tag{4.25}$$

The renormalization scale μ is introduced in order to define the renormalized coupling, but has no other physical significance. Hence we expect physical observables to be independent of it. In other words, g is tuned to be a function of μ such that $\mathcal{O}$ is independent of the scale. For dimensional reason, this procedure introduces a physical scale Λ such that:

$$g = g(\mu/\Lambda)\,. \tag{4.26}$$

For an asymptotically free theory at high energies,

$$\frac{1}{\mu} = \frac{1}{\Lambda} e^{-1/(2\beta_0 g^2)}\,. \tag{4.27}$$

We are now in a position to discuss the relation of the new scale Λ with the scale discussed in the previous section. Comparing with the expression for Λ_L, we see that:

$$\frac{\Lambda}{\Lambda_L} = a\mu \exp\left[\frac{1}{2\beta_0}\left(\frac{1}{g^2} - \frac{1}{g_0^2}\right)\right]\,. \tag{4.28}$$

The relation between the Λ parameters is obtained by computing g as a function of g_0 in perturbation theory. Any renormalized coupling can be written:

$$g^2 = g_0^2 - 2\beta_0 g_0^4 \log(\mu a/c)\,, \tag{4.29}$$

where the leading term in the expansion *must* be g_0^2, and distinct renormalized couplings will differ by the value of the constant c that appears inside the logarithm. Using Eq. (4.29) we find:

$$\frac{\Lambda}{\Lambda_L} = c\,. \tag{4.30}$$

4.4 More on scheme dependence

It is important to realize that quantities like the beta functions characterize the evolution of the couplings in a given scheme. Changing the definition of the couplings leads to different beta functions. Consider for instance a set of couplings $g_i(\mu)$, and their beta function:

$$\beta_i(g) = \mu \frac{\partial g_i}{\partial \mu} \,. \tag{4.31}$$

The couplings in a different scheme g'_i are related to the original ones by a generic transformation:

$$g_i(\mu) = g_i(g'(\mu'), \mu'/\mu) \,, \tag{4.32}$$

where we require g_i to be an invertible function. Clearly the couplings g_i should not depend on the scale μ':

$$0 = \mu' \frac{d}{d\mu'} g_i(\mu) = \sum_a \frac{\partial g_i}{\partial g'_a} \beta' + \frac{\mu'}{\mu} \frac{\partial g_i}{\partial(\mu'/\mu)} \,. \tag{4.33}$$

Using

$$\beta_i = \mu \frac{d}{d\mu} g_i = -\frac{\mu'}{\mu} \frac{\partial g_i}{\partial(\mu'/\mu)} \,, \tag{4.34}$$

we can rewrite Eq. (4.33):

$$\beta_i = \sum_a \frac{\partial g_i}{\partial g'_a} \beta'_a \,, \tag{4.35}$$

i.e. the beta functions transform covariantly under a change of scheme. Depending on the choice of the mapping between couplings, the beta functions can end up being very different. However the condition that the renormalized couplings have to coincide at lowest order in perturbation theory is sufficient to guarantee the scheme-independence of the first two coefficients of the perturbative expansion of the beta function.

It is also worthwile to point out that the transformation property derived above guarantees that the existence of a fixed point of the theory, *i.e.* a point where the beta function vanishes, is scheme independent. However the value of the coupling at the fixed point depends on the scheme. The existence of IR fixed points has been suggested for a long time, and has been investigated by lattice simulations in recent years. Understanding the nonperturbative properties of IR fixed points is a useful ingredient to build models of strongly-interacting dynamical electroweak symmetry breaking.

Taking one more derivative with respect to g'_b, we obtain:

$$\sum_j \frac{\partial \beta_i}{\partial g_j} \frac{\partial g_j}{\partial g'_b} = \sum_a \left[\frac{\partial^2 g_i}{\partial g'_a \partial g'_b} \beta'_a + \frac{\partial g_i}{\partial g'_a} \frac{\partial \beta'}{\partial g'_b} \right] \,. \tag{4.36}$$

Introducing the matrix S:

$$S_{ia} = \left.\frac{\partial g_i}{\partial g'_a}\right|_* , \tag{4.37}$$

and evaluating everything at the fixed point we obtain:

$$\sum_j L^*_{ij} S_{jb} = \sum_a S_{ia} (L')^*_{ab} \,. \tag{4.38}$$

The latter relation guarantees that the L^* and $(L')^*$ have the same eigenvalues, *i.e.* the critical exponents are scheme-independent.

4.5 Symmetries of QCD

4.5.1 Ward identities

As the dynamics of a field theory is specified by the action, any transformation that leaves the action unchanged is a symmetry of the theory. In a classical field theory, we can associate conserved charges with each continuous symmetry of the theory through Noether's theorem. In a quantum theory the symmetries are encoded in the Ward identities (WI), which are relations between field correlators that are induced by the invariance of the action. Let us consider a symmetry transformation:

$$\phi(x) \mapsto \phi'(x) = \phi(x) + \alpha \delta\phi(x) \,, \tag{4.39}$$

and let $j_\mu(x)$ be the conserved Noether current associated to the transformation at the classical level. Then the Ward identity for the quantized theory reads:

$$\langle \partial_\mu j_\mu(x) \phi(x_1) \ldots \phi(x_n) \rangle = \sum_{k=1}^{n} \delta(x - x_k) \langle \phi(x_1) \ldots \delta\phi(x_k) \ldots \phi(x_n) \rangle \,, \tag{4.40}$$

i.e. the vacuum expectation value of the insertion of the operator ∂j_μ in any correlator of fields vanishes, up to the contact terms on the RHS of Eq. (4.40). Note that this is a property of the field correlators, and is independent of the properties of the vacuum of the theory. In particular the WIs remain valid for a theory that breaks the symmetry spontaneously.

4.5.2 Chiral symmetry

In the massless limit the RH and LH components of Dirac fermions are not coupled. In the presence of n_f fermion species, the theory is invariant under independent transformations of the RH and LH components. The symmetry is therefore $\mathrm{U}(n_f)_L \times \mathrm{U}(n_f)_R$. The $U(1)_V$ yields baryon number conservation, while the $U(1)_A$ is broken by the anomaly, as we discuss below. The remaining $\mathrm{SU}(n_f)_L \times \mathrm{SU}(n_f)_R$ symmetries can be written as vector and axial transformations:

$$\delta\psi = \ \ i\left[\alpha_V^a \frac{\lambda^a}{2} + \alpha_A^a \frac{\lambda^a}{2}\gamma_5\right]\psi\,, \tag{4.41}$$

$$\delta\bar{\psi} = -i\bar{\psi}\left[\alpha_V^a \frac{\lambda^a}{2} - \alpha_A^a \frac{\lambda^a}{2}\gamma_5\right]\,, \tag{4.42}$$

where the $\frac{\lambda^a}{2}$ are the generators of $\mathrm{SU}(n_f)$. The minimal coupling to the gauge field preserves the symmetry, which is broken spontaneously by the dynamics of the strong interactions. Only the vectorial symmetry is preserved. The pattern of symmetry breaking:

$$\mathrm{SU}(n_f)_R \times \mathrm{SU}(n_f)_L \longrightarrow \mathrm{SU}(n_f)_V\,, \tag{4.43}$$

leads to the existence of $n_f^2 - 1$ massless Goldstone bosons, so that the low-energy dynamics of QCD can be successfully described by chiral perturbation theory.

The axial symmetry is explicitly broken by the mass term in the action

$$S_m = \int d^D x\, m_0 \left(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L\,.\right)\,. \tag{4.44}$$

The explicit breaking leads to an extra term in the RHS of the axial WIs, which is due to the variation δS_m of the action under axial transformations:

$$\langle \partial_\mu j_\mu^a(x)\phi(x_1)\ldots\phi(x_n)\rangle = \langle \frac{\delta S_m}{\delta\alpha_A^a(x)}\phi(x_1)\ldots\phi(x_n)\rangle + \sum_{k=1}^{n}\delta(x-x_k)\langle\phi(x_1)\ldots\delta\phi(x_k)\ldots\phi(x_n)\rangle\,. \tag{4.45}$$

4.5.3 Chiral symmetry with Wilson fermions

Let us now consider the patern of symmetry breaking for the lattice theory with Wilson fermions. As already discussed the Wilson term has the same spin structure as a mass term and breaks explicitly the axial $\mathrm{SU}(n_f)$ symmetry:

$$\begin{aligned} S[U,\psi,\bar{\psi}] = a^4 \sum_x \Bigg\{ -\frac{1}{2a}\sum_\mu \big[\bar{\psi}(x)\,(r-\gamma_\mu)\,U(x,\mu)\psi(x+a\hat{\mu}) + \\ + \bar{\psi}(x+a\hat{\mu})U(x,\mu)^\dagger(r+\gamma+\mu)\psi(x)\big] \\ + \bar{\psi}(x)\left(m_0 + \frac{4r}{a}\right)\psi(x)\Bigg\}\,. \end{aligned} \tag{4.46}$$

For $r = 0$ and $m_0 = 0$, S is symmetric under axial transformations, and the associated Noether currents are:

$$A_\mu^a(x) = \frac{1}{2}\left[\bar\psi(x+a\hat\mu)\frac{\lambda^a}{2}\gamma_\mu\gamma_5 U(x,\mu)\psi(x) + \text{h.c.}\right] . \tag{4.47}$$

The variation of the action under an axial transformation is:

$$i\frac{\delta S}{\delta\alpha_A^a(x)} = \nabla_\mu^* A_\mu^a(x) - \bar\psi(x)\left\{\frac{\lambda^a}{2}, m_0\right\}\gamma_5\psi(x) + X^a(x) . \tag{4.48}$$

Let us discuss the origin of the terms on the RHS in turn. The first term is the divergence of the Noether current as usual, the second one is the contribution from the explicit breaking of the axial symmetry due to the bare mass, while the last one is the contribution from the explicit breaking due to the Wilson term. The explicit expression for the latter term is:

$$X^a(x) = -ra\frac{1}{2a^2}\sum_\mu\left[\bar\psi(x)\frac{\lambda^a}{2}\gamma_5 U(x,\mu)\psi(x+a\hat\mu)+ \right.$$
$$\left. + (x\to x-a\hat\mu) - 4\bar\psi(x)\frac{\lambda^a}{2}\gamma_5\psi(x)\right] . \tag{4.49}$$

This is a dimension five operator, and therefore in the classical continuum limit $X^a \to 0$, *e.g.* its matrix element between free quark states vanishes linearly in a.

We will focus here on the large-distance limit of the AWI, where we assume that all the contact terms vanish:

$$\langle\alpha|\nabla_\mu^* A_\mu^a(x)|\beta\rangle = \langle\alpha|\left[\bar\psi(x)\left\{\frac{\lambda^a}{2}, m_0\right\}\gamma_5\psi(x) + X^a(x)\right]|\beta\rangle , \tag{4.50}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are two generic states. When quantum corrections are taken into account the contribution of the operator X^a can no longer be neglected, because the vanishing of the bare vertex as $a \to 0$ can be compensated by power divergent loop integrations. In order to make contact with the known results in the continuum theory we need to renormalize properly the lattice operators. Any divergent contribution from loop integrals will appear in the form of mixing of X^a with lower dimensional operators. Let us define $\bar X^a$ by subtracting such mixing:

$$\bar X^a = X^a + \bar\psi\left\{\frac{\lambda^a}{2}, \bar m\right\}\gamma_5\psi + (Z_A - 1)\nabla_\mu^* A_\mu^a , \tag{4.51}$$

where $\bar m$ and Z_A are coefficients. They are functions of the bare parameters (m_0, r, g_0) that are specified by the renormalization conditions. One of the conditions is the decoupling condition, that guarantees that the matrix element of $\bar X^a$ vanishes in the continuum limit:

$$\langle\alpha|\bar X^a|\beta\rangle \xrightarrow{a\to 0} 0 . \tag{4.52}$$

Using the decoupling condition we see that the lattice AWI yields in the continuum limit:

$$Z_A \langle \alpha | \nabla^*_\mu A^a_\mu | \beta \rangle = \langle \alpha | \bar{\psi} \left\{ \frac{\lambda^a}{2}, m_0 - \bar{m} \right\} \gamma_5 \psi | \beta \rangle \,. \tag{4.53}$$

In order to determine both Z_A and $\bar{m}$ we need to impose a stronger condition, namely that the insertion of $\bar{X}$ vanishes, in the continuum limit, for off-shell correlators, *i.e.*

$$\langle \bar{X}^a \psi(x_1) \bar{\psi}(x_2) \rangle \stackrel{a\to 0}{\longrightarrow} 0 \,. \tag{4.54}$$

The coefficients Z_A and $\bar{m}$ can be computed either in perturbation theory, or nonperturbatively. The one-loop perturbative computation is reported in Ref. [**?**], and can be performed using the technology introduced in the previous chapter. Both coefficients are functions of the bare parameters g_0, m_0 and r. Eq. (4.53) shows that the axial current is conserved in the continuum limit provided that we tune the bare mass to some critical value:

$$m_c - \bar{m}(g_0, m_c, r) = 0 \,. \tag{4.55}$$

Even though Wilson fermions break the axial symmetry explicitly, a massless continuum theory can be defined by fine-tuning the bare fermion mass m_0.

4.6 Index theorem on the lattice

Consider the quantity:

$$\nu = \frac{a}{2} \mathrm{Tr}[\gamma_5 D] \,, \tag{4.56}$$

where D is a GW Dirac operator, and the trace is computed over spin, color, and position. Simple algebraic manipulations yield:

$$\begin{aligned} \nu &= -\frac{1}{2} tr \left[\gamma_5 \left(2 - aD \right) \right] \\ &= -\frac{1}{2} \sum_\lambda v_\lambda^\dagger \gamma_5 (2 - aD) v_\lambda \\ &= -\frac{1}{2} \sum_\lambda (2 - a\lambda) \, v_\lambda^\dagger \gamma_5 v_\lambda \,, \end{aligned} \tag{4.57}$$

where v_λ are the eigenfunctions of the Dirac operator D. As seen in one of the problems that:

$$v_\lambda^\dagger \gamma_5 v_\lambda = 0 \,, \tag{4.58}$$

unless λ is real. Hence only the only values of λ that contribute to the sum in Eq. (4.57) are the real eigenvalues $\lambda = 0, 2/a$. However, for the latter eigenvalue, the factor $(2 - a\lambda)$

vanishes, so that only the zero modes yield a nonvanishing contribution to the sum. The eigenvectors corresponding to $\lambda = 0$ are chiral, and therefore the sum reduces to the difference of the number of zero modes with chirality -1 and the number of zero modes with chirality +1:

$$\nu = n_- - n_+ \,. \tag{4.59}$$

This is a remarkable result: ν is a complicated functional of the gauge fields, however its eigenvalues are integers. This is the lattice version of the index theorem in the continuum. The reason for this is that ν is a topological quantity, called the topological charge It can be defined as the integral of the topological charge density:

$$\begin{aligned} \nu &= \sum_x q(x)\,, \\ q(x) &= \frac{1}{2a^3} \mathrm{Tr}\left[\gamma_5 D(x,x)\right]\,, \end{aligned} \tag{4.60}$$

where the trace is now over spin and color indices.

The topological susceptibility can be defined using ν:

$$\chi = \frac{1}{\Omega}\langle \nu^2 \rangle\,, \tag{4.61}$$

which determines the mass of the η' meson according to the Witten-Veneziano formula.

4.7 Chiral anomaly

Finally it is interesting to discuss the realization of the chiral anomaly in a gauge theory defined on a lattice. As discussed in chapter 2, a naive discretization leads to a doubling of the fermionic spectrum, with an exact matching of RH and LH fermionic states. If it were possible to write a lattice discretization without doublers that does not violate axial symmetry, then we could couple these fermions to the gauge field and obtain a theory without an axial anomaly.

Let us now discuss how GW fermions reproduce correctly the axial anomaly. We start from an axial rotation of the fermion fields:

$$\begin{aligned} \psi' &= \left(1 + i\epsilon M \gamma_5 (1 - \frac{a}{2} D)\right) \psi \\ \bar{\psi}' &= \bar{\psi} \left(1 + i\epsilon M (1 - \frac{a}{2} D)\gamma_5\right)\,, \end{aligned} \tag{4.62}$$

where M is a matrix in flavor space. $M = 1$ corrsponds to a $\mathrm{U}(1)_A$ transformation, while

$M = \lambda^a$ defines a nonsinglet transformation. The change in the integration measure is:

$$\begin{aligned}
\mathcal{D}\psi\mathcal{D}\bar{\psi} &= \mathcal{D}\psi'\mathcal{D}\bar{\psi}' \det\left[1 + i\epsilon M\gamma_5(1 - \frac{a}{2}D)\right] \det\left[1 + i\epsilon M(1 - \frac{a}{2}D)\gamma_5\right] \\
&= \mathcal{D}\psi'\mathcal{D}\bar{\psi}' \det\left[1 + i\epsilon M\gamma_5(1 - \frac{a}{2}D)\right]^2 \\
&= \mathcal{D}\psi'\mathcal{D}\bar{\psi}' \left\{1 + 2i\epsilon \mathrm{Tr}\left[M\gamma_5(1 - \frac{a}{2}D)\right] + O(\epsilon^2)\right\} \\
&= \mathcal{D}\psi'\mathcal{D}\bar{\psi}' \left\{1 + 2i\epsilon \mathrm{Tr}_F M \sum_x \mathrm{Tr}\left[\gamma_5(1 - \frac{a}{2}D(x,x))\right] + O(\epsilon^2)\right\} . \qquad (4.63)
\end{aligned}$$

Choosing $M = \mathbb{1}$ yields:

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \mathcal{D}\psi'\mathcal{D}\bar{\psi}' \left(1 - 2i\epsilon n_f \nu + O(\epsilon^2)\right) . \qquad (4.64)$$

We have recovered the expression of the anomaly in the fermionic measure that reproduces the usual continuum result.

4.8 Final comments

Lattice QCD provides a nonperturbative definition of the path integral, which allows to compute physical predictions in the strongly coupled regime via numerical simulations. This is a powerful tool, which is now providing robust evidence that QCD is indeed the theory of strong interactions at the hadronic scale, *i.e.* for energies of the order of the proton mass.

The theoretical framework is clean, and there has been an impressive amount of work in improving the nonperturbative formulation of QFT. We have seen spectacular progress in handling chiral fermions, nonperturbative renormalization, and algorithms for simulations.

On the phenomenological side, numerical simulations have provided fundamental input in flavor physics both in the pion/kaon sector and in the heavy flavor sector. Lattice results currently play a major role in the determination of the CKM matrix elements.

On the more theoretical side, there has been progress in studying strongly-interacting models of new physics beyond the Standard Model, lattice supersymmetry, and extra-dimensional models. A robust theoretical foundation is mandatory to explore these new territories.

Field theory on the lattice has contributed directly to the development of new, powerful parallel computers, and efficient algorithms on sophisticated hardware architecture. The subject provides an exciting research topic at the crossroads of quantum field theory, statistical mechanics, and computational physics.