# Exercises to Dark Matter Phenomenology

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## 1 Thermal freeze-out

Depart from the rate equation for the number density $n$ of a particle in the thermal bath of the expanding Universe:

$$\dot{n} + 3Hn = \langle \sigma_{\rm ann} v \rangle \left( n_{\rm eq}^2 - n^2 \right) \tag{1}$$

Here $H$ is the Hubbel parameters $H \equiv \dot{a}/a$, where $a$ is the cosmic scale factor, the dot indicates derivative with respect to time $t$, the superscript "eq" indicates the corresponding quantity in equilibirum, $\langle \sigma_{\rm ann} v \rangle$ is the thermally averaged annihilation cross section times relative velocity.

1. Re-write the differential equation in terms of $Y \equiv n/s$, where $s$ is the entropy density. Use conservation of entropy: $a^3 s = const$.

   Then change from time $t$ to the independent variable $x \equiv m/T$, where $m$ is the mass of the particle and $T$ is the temperature, and show that the differential equation becomes

   $$\frac{dY}{dx} = \frac{1}{3H} \frac{ds}{dx} \langle \sigma_{\rm ann} v \rangle \left( Y^2 - Y_{\rm eq}^2 \right) . \tag{2}$$

   Hint: write

   $$\frac{d}{dt} = \frac{da}{dt} \frac{dx}{da} \frac{d}{dx} \tag{3}$$

   and use the definition of $H$ as well as conservation of entropy.

2. Use the following definitions

   $$H = \sqrt{\frac{8}{3} \pi G_N \rho} \tag{4}$$

   $$\rho = g_{\rm eff} \frac{\pi^2}{30} T^4 \tag{5}$$

   $$s = h_{\rm eff} \frac{2\pi^2}{45} T^3 \tag{6}$$

   $$g_* = \frac{h_{\rm eff}^2}{g_{\rm eff}} \left( 1 + \frac{T}{3h_{\rm eff}} \frac{dh_{\rm eff}}{dT} \right)^2 \tag{7}$$

   to obtain

   $$\frac{dY}{dx} = -\sqrt{\frac{\pi g_*}{45 G_N}} \frac{m}{x^2} \langle \sigma_{\rm ann} v \rangle \left( Y^2 - Y_{\rm eq}^2 \right) . \tag{8}$$

3. Consider now freeze-out of a (moderately) non-relativistic particle at a temperature $T_F = m/x_F$: for $x > x_F$ we have $Y_{\rm eq} \ll Y$. Calculate $Y_\infty$ by integrating eq. 8 from $x_F$ to $\infty$. Hints: assume $Y(x_F) \gg Y_\infty$, and $g_*(x) \approx g_*(x_F) = const.$

   In the non-relativistic limit we can expand $\sigma_{\rm ann} v$ in $v$ as $\sigma_{\rm ann} v \approx a + bv^2$ and $\langle \sigma_{\rm ann} v \rangle \approx a + 6b/x$. Show that

$$Y_\infty \propto \frac{1}{a + 3b/x_F} \tag{9}$$

## 2 DM indirect detection: a $\gamma$-ray line

In many DM models one expects that a $\gamma$-line signal can be induced by the processes $\chi\chi \to \gamma\gamma$, $\chi\chi \to \gamma Z$, and $\chi\chi \to \gamma H$, where $\chi$ denotes the DM particle, $Z$ is the SM $Z$ boson, and $H$ is the Higgs boson with $m_H = 125$ GeV. The relative strength of the three lines will be model dependent.

Assume that experiments searching for $\gamma$ rays from DM annihilations establish the existence of a $\gamma$ ray line with, for example, $E_\gamma = 130$ GeV. Consider the three cases, that the line at 130 GeV comes from the $\gamma\gamma, \gamma Z, \gamma H$ channel. For each case calculate the DM mass and the photon energies where the accompanying lines from the other two channels will occur. Assume that the DM particles annihilate at rest.

## 3 DM direct detection cross section

Consider a fermionic DM particle $\chi$ with an effective DM-quark interaction of the form

$$\mathcal{L} = \sum_q G_q (\bar\chi\chi)(\bar q q) \tag{10}$$

where the sum is over all quark flavours. Assume that $\chi$ is a Dirac particle.

1. Show that the differentail and total cross sections for $\chi$–nucleon scattering, in the non-relativistic limit (i.e., at leading order in the DM velocity $v$) are given by

$$\frac{d\sigma_N}{dE_k} = \frac{G_N^2 m_N}{2\pi v^2}\,, \qquad \sigma_N = \frac{G_N^2 \mu^2}{\pi} \tag{11}$$

   where $E_k$ is the (non-relativistic) kinetic energy of the nucleon, $\mu = m_\chi m_N/(m_\chi + m_N)$ is the reduced mass of the DM–nucleon system, and the effective coupling to the nucleon is defined as

$$2m_N G_N \equiv \sum_q G_q \langle N|\bar q q|N\rangle\,, \tag{12}$$

   where $|N\rangle$, $N = p, n$ is the nucleon state (in standard relativistic normalization, such that $G_N$ has the same dimension as $G_q$).

Hint: The matrix element of the process is given by

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{s_1,s_3}\left|\bar{u}(k_3)u(k_1)\sum_q G_q\langle N|\bar{q}q|N\rangle\right|^2 \tag{13}$$

where $k_1$ and $k_3$ denote the momenta of the incoming and outgoing DM particles.

2. The effective coupling to the nucleon $G_N$ is given in terms of the couplings to the quarks $G_q$ as

$$G_N = \sum_{q=u,s,d} G_q \frac{m_N}{m_q}\xi_q^N + \frac{2}{27}\left(1 - \sum_{q=u,s,d}\xi_q^N\right)\sum_{q=c,b,t} G_q \frac{m_N}{m_q} \tag{14}$$

where $\xi_q^N \equiv m_q\langle N|\bar{q}q|N\rangle/m_N$ and (numbers from Cheng, Chiang, 1202.1292)

$$\begin{aligned} \xi_d^p \approx 0.023\,, \quad \xi_u^p \approx 0.017\,, \quad \xi_s^p \approx 0.054 \\ \xi_d^n \approx 0.034\,, \quad \xi_u^n \approx 0.012\,, \quad \xi_s^n \approx 0.054 \end{aligned} \tag{15}$$

Assume first that the interaction is mediated by the Higgs. In this case the coupings will be proportional to the quark mass:

$$G_q = \frac{\lambda}{m_H^2}\sqrt{2}\frac{m_q}{v} \tag{16}$$

where $v = 250$ GeV is the Higgs VEV and $\lambda$ is the effective coupling constant of the Higgs to the DM particle. Show that $G_p \approx G_n$ and give a physical interpretation of this result. Then assume $m_H = 125$ GeV, $m_\chi = 100$ GeV and calculate the DM–proton scattering cross section as a function of $\lambda$ and compare it to the present limits from experiments.

3. Calculate now DM–nucleon scattering cross section for the case of flavour independet couplings $G_q = G$. Show that also in this case the effective couplings to neutron and proton are approximately equal, i.e., $G_p \approx G_n$. (The light quark masses are $m_u \approx 2.5$ MeV, $m_d \approx 5.0$ MeV, $m_s \approx 100$ MeV.)

# 4 Toy model for fermionic DM matter

Consider a simple toy model for WIMP DM, with a Dirac fermion $\chi$ being the DM particle (mass $m_\chi$) and a mediator (pseudo) scalar particle $\phi$ (mass $m_\phi$) with the following interaction terms:

$$\mathcal{L}_{\text{int}} = \phi\,\bar{\chi}(g_S + ig_P\gamma_5)\chi + \phi\,\bar{f}(c_S + ic_P\gamma_5)f \tag{17}$$

where $f$ is a SM fermion (with $m_f \ll m_\chi$) and $g_S, g_P, c_S, c_P$ are real coupling constants.

1. Calculate the annihilation cross section times relative velocity $\sigma_{\rm ann} v_{\rm rel}$ for $\bar{\chi}\chi \to \bar{f}f$ in the non-relativistic limit to leading order in the DM velocity $v$ for both cases, pure scalar ($g_S$) and pure pseudo-scalar ($g_P$) couplings.

2. Consider the case $c_P = 0$ and calculate the DM–proton scattering cross section by taking $f = q$ and $c_S$ to be independent of $q$. Assuming that $m_\phi$ is large compared to the typical momentum transfer in DM scattering (i.e., $m_\phi \gg$ MeV) you can use the results of exercise 3. Generalize also to the case $g_P \neq 0$.

3. Assume that the relic density of DM is set by thermal freeze out, which fixes the parameters of the model such that

$$\Omega_{\rm DM} h^2 \approx \frac{0.1\,\text{pb}}{\langle \sigma_{\rm ann} v_{\rm rel}^2 \rangle_{x_F}} \approx 0.11 \tag{18}$$

($1\,\text{pb} = 10^{-36}\,\text{cm}^2$), where 0.11 is the number determined by WMAP within about 3% accuracy. For the thermal average use $\langle v^2 \rangle \approx 6/x$ ($x = m/T$), which should be replaced by $3/x_F$ in the expression for the relic abundance (see exercise 1), and at freeze-out $x_F \approx 20$. Take into account that $\chi$ will annihilate into all SM fermions (apart from neutrinos) which are lighter than $\chi$ (assume that $m_b < m_\chi < m_t$).

   (i) Consider first the limit $m_\phi \gg m_\chi$. Is the model viable for $g_P = c_P = 0$? What happens for $g_S = 0$ and $g_P \neq 0$?

   (ii) Discuss qualitatively what happens for $m_\phi \approx 2m_\chi$ and $m_\phi < m_\chi$.

   (iii) Discuss qualitatively the expected strengths of the signals for indirect (from $\bar{\chi}\chi \to \bar{f}f$) for the cases above. Take into account that today typical DM velocities are $v \sim 10^{-3}c$.

   (iv) What would change in the above discussion if the mediator particle $\phi$ was the Higgs? [1]

[1]Remark: This is the so-called Higgs portal scenario. The interaction term $gH\bar{\chi}\chi$ could arise from a non-renormalizable interaction $H^\dagger H\bar{\chi}\chi/\Lambda$ with $g = v/\Lambda$. See e.g., L. Lopez-Honorez, T. Schwetz, J. Zupan, arXiv:1203.2064 for a discussion and references.