Lattice field theory

Problem Sheet 1 - Scalar theory

1. Consider two functions ϕ, χ defined on a lattice of size $L = Na$ with periodic boundary conditions.
$$\phi(x + L\hat{\mu}) = \phi(x), \qquad \chi(x + L\hat{\mu}) = \chi(x)\,, \tag{1}$$
The points on the lattice are labeled as:
$$x_\mu = am_\mu, \quad m = 0, \dots, N-1, \,. \tag{2}$$
Show that:
$$\sum_x \phi(x)\nabla_\mu\chi(x) = -\sum_x \nabla_\mu^*\phi(x)\chi(x)\,. \tag{3}$$

2. Let us consider a D-dimensional lattice of size $L = Na$, the points on the lattice are labeled as in Q1. For N even, we choose to label the lattice momenta are:
$$p_\mu = a^{-1}\frac{2\pi}{N}n_\mu, \quad n = -N/2+1, \dots, N/2\,. \tag{4}$$
For $D = 1$, let us define the matrix:
$$U_{mn} = \exp\left[ixp\right]_{mn} = \exp\left[im\frac{2\pi}{N}n\right]\,.$$
Check that:
$$\frac{1}{N}\sum_m U_{mn}U^*_{mn'} = \bar{\delta}_{nn'}\,, \tag{5}$$
where
$$\begin{aligned}\bar{\delta}_{nn'} &= 0, \quad n \neq n' \mod N\,,\\ &= 1, \quad n = n' \mod N\,.\end{aligned}$$
This result can be generalized to D dimensions, using the notation above:
$$\begin{aligned}\sum_x e^{-i(p-p')\cdot x} &= a^D\sum_m e^{-i(p-p')\cdot x} = \bar{\delta}_{p,p'} \equiv \prod_\mu \left(Na\bar{\delta}_{n_\mu,n'_\mu}\right)\,,\\ \sum_p e^{ip\cdot(x-x')} &= \frac{1}{(Na)^D}\sum_n e^{ip\cdot(x-x')} = \bar{\delta}_{x,x'} \equiv \prod_\mu \left(a^{-1}\bar{\delta}_{m_\mu,m'_\mu}\right)\,.\end{aligned}$$
Notice how the dimensions match in these equations.
Check that for an integrable function f:
$$\begin{aligned}\frac{1}{(Na)^D}\sum_n f(p) &= \left(\frac{\Delta p}{2\pi}\right)^D\sum_n f\left(\Delta p\, n\right)\\ &\xrightarrow{N\to\infty} \int_{-\pi/a}^{\pi/a}\frac{d^Dp}{(2\pi)^D}f(p)\end{aligned}$$

3. Show that

$$\langle\phi(m_4+1)|\exp\left[-\frac{a}{2}\,a^{D-1}\sum_{\vec{m}}\hat{\Pi}(\vec{m})^2\right]|\phi(m_4)\rangle =$$
$$= \kappa\exp\left[-\frac{a}{2}\,a^{D-1}\sum_{\vec{m}}\frac{\big(\phi(\vec{m},m_4+1)-\phi(\vec{m},m_4)\big)^2}{a^2}\right], \qquad (6)$$

where κ is some numerical factor to be determined.

4. *Restoration of rotation invariance.*
The propagator for a free field in two dimensions is given by:

$$G(x-y) = \int \frac{d^2p}{(2\pi)^2}\,\frac{e^{ip\cdot(x-y)}}{m_0^2+\frac{4}{a^2}-\frac{2}{a^2}\cos ap_1-\frac{2}{a^2}\cos ap_2}\,. \qquad (7)$$

Consider the limit where $|x-y|\to\infty$ along some direction specified by a unit vector n. The correlation length $\xi(n)$ is defined by the asymptotic behaviour of G in the direction n:

$$\lim_{t\to\infty} G(tn) \propto e^{-t/\xi(n)}\,. \qquad (8)$$

Show that for $n=(1,0)$:

$$\xi^{-1}=\omega, \quad \cosh\omega = 1+a^2m_0^2/2\,; \qquad (9)$$

and that for $n=(1,1)$:

$$\xi'^{-1}=\sqrt{2}\omega', \quad \cosh\omega' = 1+a^2m_0^2/4\,. \qquad (10)$$

Discuss the limits $am \ll 1$, and $am \gg 1$, and explain their physical significance. Compute the ratio ξ'/ξ for $am\ll 1$, and for $a\to 0$.

5. Compute the free scalar field propagator in a four-dimensional periodic box of size T and L:

$$\langle\phi(x)\phi(0)\rangle = L^{-3}\sum_{\vec{p}}\frac{\cosh[E(\vec{p})(\frac{T}{2}-x_4)]}{2E(\vec{p})\sinh(\frac{T}{2}E(\vec{p}))}\,, \quad E(\vec{p})=\sqrt{\vec{p}^{\,2}+m_0^2}\,. \qquad (11)$$

You can use:

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+a} = -\frac{1}{2a^2}+\frac{\pi}{2a}\frac{\cosh(a(\pi-x))}{\sinh(a\pi)}\,.$$

6. *Reflection positivity.*
Derive reflection positivity for the scalar field theory.

7. Compute the two-point function in the hopping expansion for small κ. Deduce the value of the renormalized mass in the same limit.

8. *Examples of divergent expansion*
Check that the expansions in κ and λ are convergent and divergent respectively:

$$
\begin{aligned}
z(\kappa, \lambda) = & \int_{-\infty}^{+\infty} d\phi \, \exp\left(-\kappa\phi^2 - \lambda\phi^4\right) \\
= & \frac{\lambda^{-1/4}}{2} \sum_{k=0}^{\infty} \frac{\Gamma(k/2 + 1/4)}{k!} (-\kappa\lambda^{-1/2})^k \\
= & \kappa^{-1/2} \sum_{k=0}^{\infty} \frac{\Gamma(2k + 1/2)}{k!} (-\lambda\kappa^{-2})^k \, .
\end{aligned}
$$

9. Consider the interaction term:

$$
\sum_x a^2 \eta_0 \partial_\mu \phi^2(x) \partial_\mu \phi^2(x) \, .
$$

Compute the Feynman rule for the vertex generated by such interaction. Show that it gives a non-zero contribution to the two-point function.