Lecture 1

Scalar fields

1.1 Introduction

The path integral formulation provides an elegant way to quantize a field theory. Introducing a source field J, the generating functional for a theory in D dimensions is defined as:

$$Z[J] = \int \mathcal{D}\phi \exp\left\{ iS[\phi] + i \int d^D x\, J(x)\phi(x) \right\} . \tag{1.1}$$

By analogy with statistical mechanics, we will sometimes refer to Z as the partition function for the system. The analogy with statistical mechanics will become more obvious after rotating to Euclidean space, as discussed below. The field correlators, which are called Wightman functions in Minkowski space, are the functional derivatives of $Z[J]$ with respect to the source:

$$W(x_1, \ldots, x_n) = \langle \phi(x_1) \ldots \phi(x_n) \rangle = (-i)^n \frac{\delta}{\delta J(x_1) \ldots \delta J(x_n)} Z[J] . \tag{1.2}$$

All physical quantities are obtained from the field correlators in Eq. (1.39), which are the fundamental building blocks of a quantum field theory (QFT). Starting from the Wightmann functions, the Hilbert space of physical states, and the S-matrix of the theory can be reconstructed.

Källen-Lehmann representation As an illustration, let us discuss how the particle content of the theory is encoded in the two-point functions. The KL representation makes the link between the spectrum of the theory and the correlators explicit.

The physical states are eigenstates of the Hamiltonian. If the theory is invariant under spatial translations, we have:

$$\left[\hat{H}, \hat{P}_k\right] = 0. \tag{1.3}$$

Hence $\hat{H}$ and $\hat{P}_k$ can be diagonalized simultaneously, *i.e.* the states of the theory are characterized as eigenstates of the Hamiltonian and the momentum operators:

$$\hat{P}_k |\alpha, \mathbf{p}\rangle = p_k |\alpha, \mathbf{p}\rangle , \tag{1.4}$$

$$\hat{H} |\alpha, \mathbf{p}\rangle = E_\alpha(\mathbf{p}) |\alpha, \mathbf{p}\rangle , \tag{1.5}$$

where α is a label for the states in the theory. Note that α ranges over multi-particle states as well. The energy of the state is:

$$E_\alpha(\mathbf{p})^2 = \mathbf{p}^2 + m_\alpha^2 . \tag{1.6}$$

The total momentum is $\mathbf{p}$, while m_α is the energy of the state in the center-of-mass frame. We adopt a relativistic normalization of the states:

$$\langle \alpha, \mathbf{p} | \alpha', \mathbf{p}' \rangle = \delta_{\alpha\alpha'} 2E_\alpha(\mathbf{p}) (2\pi)^{D-1} \delta(\mathbf{p} - \mathbf{p}') , \tag{1.7}$$

which leads to the completeness relation:

$$\mathbb{1} = |0\rangle\langle 0| + \sum_\alpha \int \frac{d^{D-1}p}{(2\pi)^{D-1} 2E_\alpha(\mathbf{p})} \, |\alpha, \mathbf{p}\rangle\langle\alpha, \mathbf{p}| \, . \tag{1.8}$$

Let $x^0 > 0$, the time-ordered two-point correlator can be written as:

$$\begin{aligned} \langle T\phi(x)\phi(0)\rangle &= \langle\phi(x)\phi(0)\rangle & (1.9)\\ &= \langle 0|e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}\cdot x}\phi(0)|0\rangle & (1.10)\\ &= \sum_\alpha \int \frac{d^{D-1}p}{(2\pi)^{D-1} 2E_\alpha(\mathbf{p})} e^{-ip\cdot x} \, |\langle 0|\phi(0)|\alpha, \mathbf{p}\rangle|^2 \Big|_{p^0=E_\alpha(\mathbf{p})} & (1.11)\\ &= i \sum_\alpha \int \frac{d^D p}{(2\pi)^D} e^{-ip\cdot x} \frac{|\langle 0|\phi(0)|\alpha, \mathbf{p}\rangle|^2}{p^2 - m_\alpha^2 + i\epsilon} \, . & (1.12) \end{aligned}$$

The state $|\alpha, \mathbf{p}\rangle$ can be written as:

$$|\alpha, \mathbf{p}\rangle = \Lambda(\mathbf{p})|\alpha, 0\rangle \, , \tag{1.13}$$

where we have denoted by $\Lambda(\mathbf{p})$ the Lorentz transformation that boosts the system from being at rest to having momentum $\mathbf{p}$. Using Eq. (1.13), we find:

$$\begin{aligned} \langle 0|\phi(0)|\alpha, \mathbf{p}\rangle &= \langle 0|\Lambda(\mathbf{p})^{-1}\phi(0)\Lambda(\mathbf{p})|\alpha, 0\rangle & (1.14)\\ &= \langle 0|\phi(0)|\alpha, 0\rangle \, , & (1.15) \end{aligned}$$

where we used the fact that the field $\phi(x)$ is a scalar under Lorentz transformations.

Inserting Eq. (1.15) in Eq. (1.12) yields the Källen-Lehman representation for the two-point function:

$$\langle T\phi(x)\phi(0)\rangle = \int_0^\infty \frac{d\mu^2}{2\pi} \, \rho(\mu^2)\Delta(x;\mu^2) \, . \tag{1.16}$$

$\Delta(x;\mu^2)$ is the free propagator for a particle of mass μ^2:

$$\Delta(x;\mu^2) = i \int \frac{d^D p}{(2\pi)^D} \frac{e^{-ip\cdot x}}{p^2 - \mu^2 + i\epsilon} \, , \tag{1.17}$$

and $\rho(\mu^2)$ is the spectral density:

$$\rho(\mu^2) = \sum_\alpha Z_\alpha (2\pi)\delta(\mu^2 - m_\alpha^2) + \text{cont} \, . \tag{1.18}$$

Note that each single-particle states contributes a δ function to the spectral density. They correspond to single poles in momentum space, and yield an exponential decay of the two-point functions in Euclidean space:

$$\langle\phi(x_4, \vec{x})\phi(0)\rangle = \sum_\alpha \int \frac{d^{D-1}p}{(2\pi)^{D-1} 2E_\alpha(\mathbf{p})} e^{-E_\alpha(\mathbf{p})x_4} \, e^{-i\mathbf{p}\cdot\vec{x}} \, |\langle 0|\phi(0)|\alpha, 0\rangle|^2 \, . \tag{1.19}$$

Lattice regularization In order to be able to compute the field correlators we need to solve two problems that are hidden in Eq. (1.1).

1. $Z[J]$ is usually defined in perturbation theory, by expanding the partition function in powers of the coupling constants. As this expansion is often at best asymptotic, a nonperturbative definition is required for strongly-interacting theories.

2. The path integral is plagued by ultraviolet (UV) divergences, which need to be regulated, and renormalized. This problem was realized immediately in the early days of QFT, and the renormalization procedure was fully developed in the works of Tomonaga, Schwinger, Feynman, and Dyson.

The formulation of field theories on a space-time Euclidean lattice solves both problems. The space-time points are the vertices of a regular lattice, with lattice size a, so that the coordinates of the lattice points are $x_\mu = n_\mu a$, where n_μ is an integer. The path integral in the partition function becomes a multi-dimensional integral. If the system is put in a finite box, we are left with a finite-dimensional integral, which is mathematically well-defined. The inverse of the lattice spacing a^{-1} acts as an UV cutoff. The continuation of the time variable to imaginary values was first introduced by Dyson, Wick, Schwinger, and Symanzik; the integrand in the path integral is real and bounded, thereby simplifying both the theoretical and numerical studies. Moreover the Euclidean formulation makes the equivalence between QFT and statistical mechanics particularly evident, and has allowed notable progress in both fields over the years. Most of the physical information can be extracted directly from the Euclidean theory. When necessary, correlators can be analytically continued to Minkowski space.

1.2 Euclidean lattice theory

Let us start by defining the scalar theory on a Euclidean lattice. In this Section we introduce the notation used in the rest of the lectures. We will follow quite closely the presentation in the book by Jan Smit. [1]

The dynamical variables in a lattice scalar theory are the fields at each spatial point $\phi(\mathbf{x}, t)$. In QFT they are promoted to operators $\hat{\phi}(\mathbf{x})$ acting in the Hilbert space of physical states. We can define the coordinate representation by introducing a basis of eigenstates of

[1] J. Smit, "Introduction to Quantum Fields on a Lattice", Cambridge lecture notes in Physics, 2002

the fields:

$$\hat{\phi}(\mathbf{x})|\phi\rangle = \phi(\mathbf{x})|\phi\rangle \,, \tag{1.20}$$

$$|\phi\rangle = \prod_{\mathbf{x}} |\phi(\mathbf{x})\rangle \,, \tag{1.21}$$

$$\langle\phi'|\phi\rangle = \prod_{\mathbf{x}} \delta\left(\phi'(\mathbf{x}) - \phi(\mathbf{x})\right) \,. \tag{1.22}$$

Note that in the equations above we have suppressed the dependence of the fields on time. The latter can be reinstated when needed with little effort. As time evolves the state vectors are denoted $|\phi(t)\rangle$.

The lattice coordinates are:

$$x_\mu = n_\mu a; \qquad n_\mu = 0, \ldots, N-1 \,, \tag{1.23}$$

where a is the lattice spacing, and the size of the box is $L = Na$. The index μ ranges from 1 to D, and we shall usually identify x_D with the Euclidean time direction. In D dimensions, the total volume is $\Omega = L^D$. We will also refer to the spatial volume $V = L^{D-1}$. When needed we shall identify a "temporal" direction and call T its extension in physical units. [2] A sum over the whole volume is written as:

$$\sum_x = a^D \sum_n \,. \tag{1.24}$$

If we take the limit $a \to 0$, at fixed L, for a smooth function $f(x)$:

$$\sum_x f(x) \to \int_0^L d^D x \, f(x) \,. \tag{1.25}$$

Derivatives can be discretized as:

$$\nabla_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\hat{\mu}) - \phi(x)\right] \,, \tag{1.26}$$

$$\nabla^*_\mu \phi(x) = \frac{1}{a}\left[\phi(x) - \phi(x - a\hat{\mu})\right] \,. \tag{1.27}$$

The lattice formulation yields an unambiguous definition the path integral:

$$Z = \int \mathcal{D}\phi \; e^{-S_E[\phi]} \,, \tag{1.28}$$

[2] Note however that in Euclidean space the identification of a temporal direction is arbitrary.

where:

$$\int \mathcal{D}\phi = \prod_x \int d\phi(x)\,, \tag{1.29}$$

$$S_E[\phi] = \sum_x \left[\frac{1}{2}\nabla_\mu\phi(x)\nabla_\mu\phi(x) + \frac{1}{2}m_0^2\phi(x)^2 + \frac{1}{4!}g_0\phi(x)^4\right]\,. \tag{1.30}$$

The bare couplings m_0^2 and g_0 define the theory at the scale of the UV cutoff. In eq. (1.30) the couplings are dimensioful quantity. When discussing the RG flows, we will use dimensionless couplings, *i.e.* we will rescale the dimensionful ones by the appropriate power of the lattice spacing. Note that the scalar field has mass dimension

$$[\phi] = \frac{D-2}{2}\,, \tag{1.31}$$

and hence the action in Eq. (1.30) is dimensionless. The choice of the discretized action S_E is such that it yields the classical action in continuum space-time as $a \to 0$.

We can readily determine the symmetries of the lattice action S_E. Similarly to the case of the continuum theory, this action is symmetric under the transformation $\phi \to -\phi$. However, the Lorentz invariance of the continuum theory is replaced by the D-dimensional hypercubic group. We shall comment again on the consequences of this property.

1.3 Transfer matrix

Let us consider a symmetric lattice, we identify one direction with the time direction, and interpret the path integral as the quantum amplitude for a field configuration to evolve from time 0 to time $T = Na$. For a lattice with periodic boundary conditions in time, we want to identify this quantum amplitude with the trace of the transfer matrix:

$$\begin{aligned} Z &= \left(\prod_x \int d\phi(x)\right) \langle\phi(N)|\hat{T}|\phi(N-1)\rangle \ldots \langle\phi(1)|\hat{T}|\phi(0)\rangle \\ &= \mathrm{Tr}\,\hat{T}^N\,. \end{aligned}$$

We start by rewriting the Euclidean action in Eq. (1.30) as:

$$S_E[\phi] = a\sum_{n_4} a^{D-1}\sum_{\mathbf{n}} \frac{1}{2a^2}\left[\phi(\mathbf{n},n_4+1) - \phi(\mathbf{n},n_4)\right]^2 + a\sum_{n_4} V[\phi(n_4)]\,, \tag{1.32}$$

where

$$V[\phi(n_4)] = a^{D-1}\sum_{\mathbf{n}}\left[\sum_{k=1}^{3}\frac{1}{2}\left(\nabla_k\phi(\mathbf{n},n_4)\right)^2 + \frac{1}{2}m_0^2\phi(\mathbf{n},n_4)^2 + \frac{1}{4!}g_0\phi(\mathbf{n},n_4)^4\right]\,. \tag{1.33}$$

Note that each term in the sum over n_4 in Eq. (1.32) only depends on the field configuration at time n_4 and n_4+1. Hence the matrix elements of the operator $\hat{T}$ between field eigenstates is:

$$\langle\phi(n_4+1)|\hat{T}|\phi(n_4)\rangle = \exp\left\{-a\,a^{D-1}\sum_{\mathbf{n}}\frac{1}{2a^2}\left[\phi(\vec{n},n_4+1)-\phi(\vec{n},n_4)\right]^2\right\}\times$$
$$\exp\left\{-\frac{a}{2}\left(V[\phi(n_4+1)]-V[\phi(n_4)]\right)\right\}\,,$$

where $|\phi(n_4)\rangle$ is the field eigenstate at time $t=n_4a$. It is straightforward to check that the operator $\hat{T}$ can be written as:

$$\hat{T} = \exp\left[-\frac{a}{2}V(\hat{\phi})\right]\,\exp\left[-\frac{a}{2}\,a^{D-1}\sum_{\mathbf{n}}\hat{\Pi}(\vec{n})\right]\,\exp\left[-\frac{a}{2}V(\hat{\phi})\right]\,, \tag{1.34}$$

where $\hat{\Pi}(\vec{n})$ is the conjugate momentum operator such that

$$\left[\hat{\phi}(\vec{n}),\hat{\Pi}(\vec{n}')\right] = i\frac{1}{a^{D-1}}\delta_{\mathbf{n},\mathbf{n}'} \equiv i\bar{\delta}_{\mathbf{x}\mathbf{x}'}\,. \tag{1.35}$$

The discrete delta function $\bar{\delta}_{\mathbf{x}\mathbf{x}'}$ is normalized in order to have the same dimensions as the Dirac delta in the continuum theory. With the normalizations adopted here:

$$\sum_x \bar{\delta}_{x,0} = 1\,, \tag{1.36}$$

so that the lattice equations should be very similar to the continuum ones. The Hamiltonian is defined from the transfer matrix:

$$\hat{T} = e^{-a\hat{H}} = 1 - a\hat{H} + O(a^2)\,, \tag{1.37}$$

$$\hat{H} = a^{D-1}\sum_{\mathbf{n}}\left[\frac{1}{2}\hat{\Pi}(\mathbf{n})^2 + \frac{1}{2}\sum_j \partial_j\hat{\phi}(\mathbf{n})\partial_j\hat{\phi}(\mathbf{n}) + \frac{1}{2}m_0^2\hat{\phi}(\mathbf{n})^2 + \frac{1}{4!}g_0\hat{\phi}(\mathbf{n})^4\right]\,. \tag{1.38}$$

It is straighforward to show that the operator $\hat{T}$ defined above is indeed positive, and therefore the Hamiltonian is a well-defined hermitean operator.

1.4 Expectation values of operators

The same manipulations can be performed for a generic field correlator; they lead to an expression for the correlator as a trace of field operators and powers of the transfer matrix. For instance for a tow-point function we have:

$$\langle\phi(\mathbf{x},t)\phi(\mathbf{y},0)\rangle = Z^{-1}\int\mathcal{D}\phi\,e^{-S_E[\phi]}\,\phi(\mathbf{x},t)\phi(\mathbf{y},0) = \frac{\mathrm{Tr}\,\left[\hat{T}^{T-t}\,\hat{\phi}(\mathbf{x})\,\hat{T}^t\,\hat{\phi}(\mathbf{y})\right]}{\mathrm{Tr}\,\hat{T}^N}\,. \tag{1.39}$$

1.5 Free scalar field

A number of interesting results can be derived by analysing the free theory ($g_0 = 0$). In particular, we will obtain some simple examples of properties that characterize a lattice field theory. We shall assume in this section that $m_0^2 > 0$. The action in this case is a quadratic form:

$$S_E[\phi] = \sum_x \left[\frac{1}{2}\nabla_\mu\phi(x)\nabla_\mu\phi(x) + \frac{1}{2}m_0^2\phi(x)^2\right] \tag{1.40}$$

$$= \frac{1}{2}\sum_{x,y}\phi(x)K_0(x,y)\phi(y)\,. \tag{1.41}$$

The kernel for the free action is

$$K_0(x,y) = a^{-2}\sum_z\left[\sum_\mu(\bar{\delta}_{z+a\hat{\mu},x} - \bar{\delta}_{z,x})(\bar{\delta}_{z+a\hat{\mu},y} - \bar{\delta}_{z,y}) + (m_0^2a^2)\bar{\delta}_{z,x}\bar{\delta}_{z,y}\right]\,. \tag{1.42}$$

Here and in the following equations, ths suffix "0" indicates that the field correlators are computed in the free field theory. The generating functional

$$Z[J] = \int \mathcal{D}\phi \exp\left(-S_E[\phi] + \sum_x J(x)\phi(x)\right)$$

$$= \int \mathcal{D}\phi \exp\left(-\sum_{xy}\phi(x)K_0(x,y)\phi(y) + \sum_x J(x)\phi(x)\right) \tag{1.43}$$

is computed by the usual shift of variables, thereby reducing the integral to a Gaussian one. The result is

$$Z[J] = \left(\det a^{2+D}\,K_0\right)^{-1/2}\,\exp\left[-\frac{1}{2}\sum_{x,y}J(x)\Delta_0(x,y)J(y)\right]\,, \tag{1.44}$$

where Δ_0 is the free lattice propagator, *i.e.* the inverse of the kernel K_0:

$$\sum_z K_0(x,z)\,\Delta_0(z,y) = \bar{\delta}_{x,y}\,. \tag{1.45}$$

The propagator is easily computed in momentum space, where the kernel for the kinetic term K_0 becomes diagonal:

$$K_0(p,q) = \sum_{x,y} e^{-ip\cdot x - iq\cdot y}K_0(x,y) \tag{1.46}$$

$$= K_0(p)\bar{\delta}_{p,-q}\,. \tag{1.47}$$

A simple computation yields:

$$K_0(p) = \sum_\mu \frac{4}{a^2} \sin^2\left(\frac{p_\mu a}{2}\right) + m_0^2 \equiv \Delta_0(p)^{-1}\,, \tag{1.48}$$

where $\Delta_0(p)$ is the propagator in momentum space. Taking the limit $a \to 0$, we recover the continuum expression:

$$\lim_{a\to 0} \Delta(p) = \frac{1}{p^2 + m_0^2} + \mathcal{O}(a^2)\,. \tag{1.49}$$

Note that by taking the continuum limit we have restored the $O(4)$ symmetry, *i.e.* the Euclidean version of the continuum Lorentz symmetry.

Particle content We can now discuss the physical spectrum of the free lattice theory by computing the two-point function in position space and comparing with the KL representation that we have introduced at the beginning of this chapter.

The two-point correlator in a lattice with infinite temporal extent is given by:

$$\Delta_0(x) = \langle \phi(x)\phi(0) \rangle = \int_{-\pi/a}^{\pi/a} \frac{dp_4}{2\pi} \sum_{\mathbf{p}} e^{ip\cdot x}\, \Delta_0(p)\,. \tag{1.50}$$

Thus

$$\Delta_0(x) = \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\, \mathcal{I}(\mathbf{p}, m_0 a)\,, \tag{1.51}$$

where:

$$\mathcal{I}(\mathbf{p}, m_0 a) = a^2 \int_{-\pi/a}^{\pi/a} \frac{dp_4}{2\pi} \frac{e^{ip_4 t}}{2b - 2\cos p_4 a}\,. \tag{1.52}$$

The dimensionless function b is defined as:

$$b(\mathbf{p}, m_0 a) = 1 + \frac{1}{2}\left(m_0^2 a^2 + \sum_j 4 \sin^2 \frac{p_j a}{2} \right) > 1 \quad \text{for } m_0^2 > 0\,. \tag{1.53}$$

Introducing the dimensionless variables $n_4 = t/a$, and $\hat{p}_4 = p_4 a$, the function $\mathcal{I}$ becomes:

$$\mathcal{I} = a \int_{-\pi}^{\pi} \frac{d\hat{p}_4}{2\pi} \frac{e^{i\hat{p}_4 n_4}}{2b - 2\cos \hat{p}_4}\,. \tag{1.54}$$

The integral is computed by a change of variable, $z = e^{i\hat{p}_4}$:

$$\mathcal{I} = -a \oint \frac{dz}{2\pi i} \frac{z^{n_4}}{z^2 - 2bz + 1}\,, \tag{1.55}$$

where the integral in the complex z-plane extends along the unit circle. The latter integral is easily computed by examining the residue of the integrand at the poles. The integrand has got poles at:

$$z_\pm = b \pm \sqrt{b^2 - 1}\,. \tag{1.56}$$

Only the pole at z_- is inside the unit circle, and therefore contributes to the integral. Let us introduce $\omega > 0$ by writing $z_- = e^{-\omega}$. Then:

$$e^{-\omega} = b - \sqrt{b^2 - 1} \tag{1.57}$$

$$e^{\omega} = b + \sqrt{b^2 - 1}\,, \tag{1.58}$$

and hence:

$$b = \cosh\omega \tag{1.59}$$

$$\omega = \log\left(b + \sqrt{b^2 - 1}\right)\,. \tag{1.60}$$

Computing the residue allows us to rewrite Eq. (1.51) as:

$$\Delta(\mathbf{x}, t) = \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x} - \omega t}\,\frac{1}{2\sinh\omega}\,. \tag{1.61}$$

Note that the pole at $z = z_-$ corresponds to a pole at $p_4 = i\omega$. Comparing with the KL representation discussed in the paragraph above, we see that the spectrum of the free lattice theory contains a single particle state with energy $\omega(\mathbf{p})$. In the continuum limit $am_0 \ll 1, ap_i \ll 1$, we recover Lorentz invariance and the usual dispersion relation:

$$\omega \to (\sqrt{\mathbf{p}^2 + m_0^2})a\,. \tag{1.62}$$

1.6 Wick rotation

The free field theory is a simple laboratory, which is useful to explore properties of field theories. The Wick rotation from Euclidean to Minkowski space-time, and the resulting analytical continuation of the field correlators, is nicely illustrated by the behaviour of a free field in the continuum limit $a \to 0$. In this case, the two-point correlator in Euclidean space can be written:

$$\Delta(x) = \int \frac{d^Dp}{(2\pi)^D}\, e^{ip\cdot x}\,\frac{1}{p^2 + m_0^2}\,. \tag{1.63}$$

Separating the integral in the Euclidean p_D direction from the integral in the other $D-1$ spatial directions yields:

$$\Delta(x) = \int \frac{d^{D-1}p}{(2\pi)^{D-1}}\, e^{i\mathbf{p}\cdot\mathbf{x}} \int_{-\infty}^{+\infty} \frac{dp_D}{(2\pi)}\, \frac{e^{ip_D x_D}}{p^2 + m_0^2}\,, \tag{1.64}$$

where the integral over p_D goes from $-\infty$ to $+\infty$ along the real axis. Let us now focus precisely on the integration over p_D. We can extend the integral to complex p_D, and consider the closed contour in Fig. 1.1. The vertical part of the integration contour is shifted by some angle ϵ in order to avoid the poles of the integrand on the imaginary axis. Since the contour does not contain these poles, the total integral has to vanish.

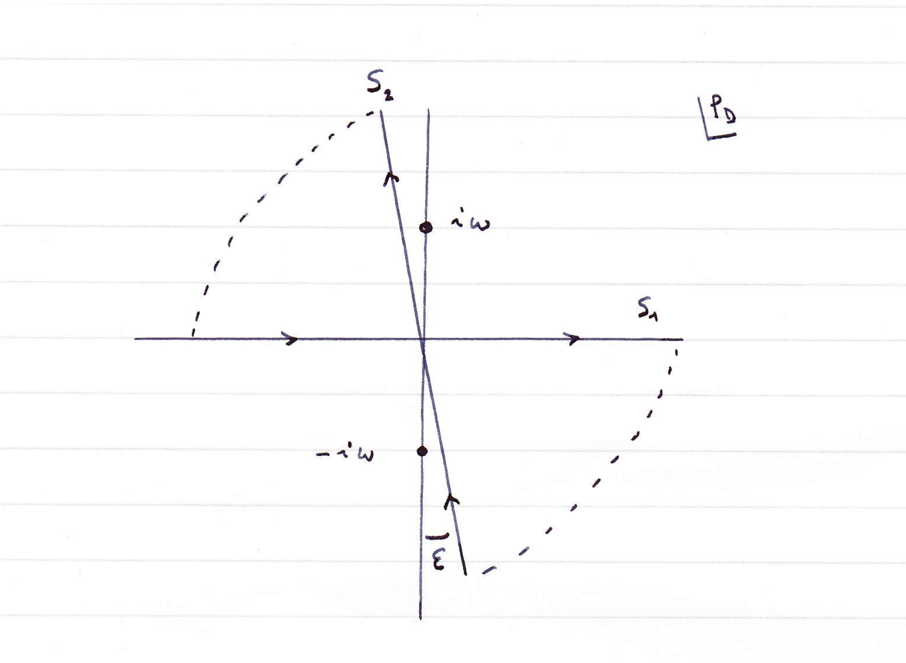

Figure 1.1: Integration contour defining the Wick rotation. Note that the poles of the integrand at $\pm\omega$ are not included inside the contour. The paths S_1 and S_2 are indicated.

The integral along the two arcs at infinity vanishes, and therefore we find:

$$\left(\int_{S_1} + \int_{S_2}\right) \frac{dp_D}{(2\pi)} \frac{e^{ip_D x_D}}{p^2 + m_0^2} = 0\,. \tag{1.65}$$

Note that we have denoted by S_1 the integration path along the horizontal axis, and by S_2 the integration path along the "almost" vertical axis. The complex variable p_D along the S_2 path can be parametrized as:

$$p_D = ie^{i\epsilon} p_0\,, \tag{1.66}$$

where p_0 is a real variable ranging from $-\infty$ to $+\infty$.

A straightforward computation yields:

$$-\int_{S_2} \frac{dp_D}{(2\pi)} \frac{e^{ip_D x_D}}{p^2+m_0^2} = -i \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \frac{e^{-i(ip_0)x_0}}{-p_0^2+\mathbf{p}^2+m_0^2-i\epsilon} \tag{1.67}$$

$$= i \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \frac{e^{-ip_0 x_0}}{p^2-m_0^2+i\epsilon}\,, \tag{1.68}$$

where we have defined $x_0 = ix_D$, and all the scalar products in the last line are to be computed with a Minkowskian metric. The expression in the last line is precisely the time-ordered product

$$\langle T\phi(x)\phi(0)\rangle = \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \frac{e^{-ip_0 x_0}}{p^2-m_0^2+i\epsilon} \tag{1.69}$$

for a free field theory in Minkowski space. Hence, we obtain the final result:

$$\Delta(x) = i\Delta_M(x_M)\,, \tag{1.70}$$

where quantities in Minkowski space are denoted by a subscript M. Eq. (1.70) shows explicitly how the two-point time-ordered correlator in Minkowski space is obtained by analytical continuation of the Euclidean one. This analytical continuation is usually called a Wick rotation:

$$x_M^\mu = (x^0, \mathbf{x}), \quad x_\mu = (\mathbf{x}, x_D), \quad x_D = ix_0\,, \tag{1.71}$$

$$p_M^\mu = (p^0, \mathbf{p}), \quad p_\mu = (\mathbf{p}, p_D), \quad p_D = ip_0\,. \tag{1.72}$$

1.7 Euclidean vs Minkowski space

The result in the previous section is a simple example of the relations between field correlators in Euclidean and Minkowski space. Let us now discuss the general case.

A QFT in Minkowski space is characterized by: [3]

1. a Hilbert space of states, $\mathcal{H}$, which contains the vacuum state $|0\rangle$;
2. a unitary representation of the Poincaré group $U(\Lambda, a)$ acting on $\mathcal{H}$;
3. the spectrum of the momentum four-vector P^μ being contained in the forward light-cone:

$$\overline{V}_+ = \left\{q^\mu : q^0 \geq 0, q^2 \geq 0\right\}\,. \tag{1.73}$$

[3] For more details and references you can consult the book by Montvay and Münster, "Quantum Fields on a Lattice", Cambridge Monographs on Mathematical Physics, 1994. A succint presentation is also available in P. Hernandez, lectures in "Modern Perspective in Lattice QCD", Oxford University Press, 2011.

4. the vacuum state $|0\rangle$ being the only vector invariant under the Poincaré group.
5. the field ϕ being an operator-valued distribution, acting on $\mathcal{H}$.
6. the field transforming under Poincaré as:

$$\phi(\Lambda x + a) = U(\Lambda, a)\phi(x)U(\Lambda, a)^{-1}\,. \tag{1.74}$$

7. locality:

$$[\phi(x), \phi(y)] = 0\,, \quad \text{for } (x-y)^2 \leq 0\,. \tag{1.75}$$

The n-point correlation functions $W(x_1, \ldots, x_n)$ defined in Eq. (1.39) contain all the physical information about the QFT, and allow the Hilbert space to be reconstructed. The correlators in Minkowski space are called Wightman functions.

Under the conditions listed above, the Wightman functions can be analytically continued to complex values of the time coordinates x_k^0. The Euclidean correlators, also called Schwinger functions, are defined as:

$$S(x_1, \ldots, x_n) = W(\ldots; -ix_k^4, \mathbf{x}_k; \ldots)\,. \tag{1.76}$$

They are analytic functions for all points, as long as:

$$x_i \neq x_j\,, \quad \text{for } i \neq j\,. \tag{1.77}$$

The Schwinger functions are computed on the lattice for real values of the Euclidean time x_k^D. The Wightman functions are obtained as boundary values of the Schwinger functions:

$$W(x_1, \ldots, x_n) = \lim_{\substack{\epsilon_k \to 0 \\ \epsilon_k - \epsilon_{k+1} > 0}} S(\ldots; \mathbf{x}_k, ix_k^0 + \epsilon_k; \ldots), \quad x_k \in \mathbb{R}^4\,. \tag{1.78}$$

The time-ordered correlators in Minkowski space are instead obtained from the Wick rotation as discussed in the previous section.

To guarantee that Schwinger functions can be continued to Minkowski space, they have to satisfy the Osterwalder-Schrader positivity (or reflection positivity) condition. Reflection positivity replaces the Hilbert space positivity and the spectral condition introduced above for a theory in Minkowski space. Checking reflection positivity for a theory defined on the lattice is crucial to ensure that a positive-definite Hamiltonian can be constructed.

1.8 Reflection positivity

Consider a product of fields at positive Euclidean times:

$$O(x_1, \ldots, x_n), \quad x_k^D > 0, \quad \text{for all } k\,. \tag{1.79}$$

Let us define the reflection operator

$$\theta : \quad x_\mu = (\mathbf{x}, x_D) \mapsto \tilde{x}_\mu = (\mathbf{x}, -x_D)\,; \tag{1.80}$$

its action on the correlator is given by:

$$\theta\left[O(x_1, \ldots, x_n)\right] = O(\tilde{x}_1, \ldots, \tilde{x}_n)\,. \tag{1.81}$$

The theory obeys reflection positivity if:

$$\langle \theta[O^\dagger]\, O\rangle \leq 0, \quad \text{for all } O. \tag{1.82}$$

1.9 Continuum limit

Let us now discuss the relation between the theory defined on a space-time lattice, and the continuum theory. The lattice formulation of a field theory regulates the UV divergencies by introducing a sharp cut-off in momentum space $\Lambda = a^{-1}$; all momenta are limited within the first Brillouin zone, *i.e.* $|ap_\mu| < \pi$. For given bare parameters, the Schwinger functions are well-defined and can be computed. As the bare parameters are changed, we explore the phase diagram of the lattice theory. The continuum limit of the lattice theory is obtained by tuning the bare parameters in such a way that the lattice theory describes the continuum physics at energy scales μ well below the cutoff scale, $\mu a \ll 1$. The Euclidean formulation clearly emphasizes the analogy between QFT and statistical mechanics; the renormalization group formalism introduced by Wilson illustrates the close relationship between the renormalization of QFT and critical phenomena in statistical mechanics.

If a limit exists where $a \to 0$ and physical scales m_{phys} stay finite, then clearly

$$\lim_{a\to 0} a m_{\text{phys}} \to 0\,. \tag{1.83}$$

In statistical mechanics the mass is related to the inverse correlation length $m_{\text{phys}} = \xi^{-1}$, and therefore in the continuum limit we expect:

$$\lim_{a\to 0} \xi/a \to \infty\,, \tag{1.84}$$

i.e. the correlation length in units of the lattice spacing has to vanish. The points in the phase diagram of a statistical system where the correlation length diverges are called critical points. The continuum limit of a QFT corresponds to the critical points of the lattice model. It is a well-known fact that the long-distance physics of a system near criticality is dictated by the symmetries and dimensionality of the system rather than by the details of the Hamiltonian. This phenomenon is known under the name of universality. In QFT universality ensures that the low-energy predictions become independent of the details of the discretization at the lattice scale.

Universality is best understood in terms of the renormalization group flow introduced by Wilson. Let us start from a generic action, defined at the scale of the lattice cutoff by including all the local couplings that are consistent with the symmetries of the lattice theory:

$$S_E[\phi] = \sum_\alpha g_\alpha(a) O_\alpha[\phi] \,, \tag{1.85}$$

where g_α denote bare dimensionless couplings in the lattice action, and $O_\alpha[\phi]$ are functional polynomials in the field and its derivatives.

When interested in the long-distance dynamics, the UV modes between the cutoff scale a and some larger distance scale $a' = sa$, $(s > 1)$ can be integrated out, while requiring that the low-energy physics remains constant. This results in a redefinition of the couplings and a rescaling of the fields, *i.e.* in a new action:

$$S'_E[\phi'] = \sum_\alpha g_\alpha(a') O_\alpha[\phi'] \,, \tag{1.86}$$

where $\phi' = Z(s)\phi$ is the rescaled field. Note that the action S'_E has the same form as the old one because we started from an action that included all possible interactions terms. Integrating out the degress of freedoms yields a redefinition of the parameters only. Had we started from a subset of interactions, new terms would have been generated while integrating out the UV modes. Correlators of the field ϕ' computed in the theory with a cutoff a' and bare couplings $g'_\alpha = g_\alpha(a')$ yield the same low-energy physics as the correlators of the original theory. The same procedure can be repeated for an arbitrary number of steps, with the cutoff scale being decreased by a factor s in each step. We obtain in this way a series of couplings $g_\alpha^{(n)} \equiv g_\alpha(s^n a)$. The transformation:

$$g_\alpha^{(n)} \mapsto g_\alpha^{(n+1)} = R_\alpha(g^{(n)}) \,, \tag{1.87}$$

is called a renormalization group transformation. The limit where $s = e^t$, and $t \to 0$, defines an infinitesimal transformation, which generates a flow in the space of couplings. The rate of change of each coupling is encoded in the so-called beta functions:

$$-\frac{d}{dt} g_\alpha = \beta_\alpha(g) \,. \tag{1.88}$$

Note that there is one beta function for each coupling, and the beta functions depend in principle on all couplings g_α. This defines a flow in an infinite-dimensional space of couplings, since we started from an action that contained all possible couplings that are compatible with symmetry. Fortunately things become simpler in the vicinity of critical points.

Since the physics is left unchanged, while the lattice spacing is increased by a factor s at each iteration, the correlation length in lattice units decreases along an RG trajectory. Critical points must correspond to fixed points of the trajectories:

$$g_\alpha^* = R_\alpha(g^*) \,. \tag{1.89}$$

Note that *at* a fixed point, the action is invariant as the cutoff is varied, *i.e.* the action describes a scale-invariant theory.

The behaviour of the RG flow in the neighbourhood of a fixed point can be obtained by linearizing the flow equations. Let us define the deviation from the fixed point:

$$\delta g_\alpha = g_\alpha - g^*_\alpha \,, \tag{1.90}$$

then:

$$\delta g'_\alpha = R^*_{\alpha\beta} \delta g_\alpha \,; \tag{1.91}$$

$$R^*_{\alpha\beta} = \left.\frac{\partial R_\alpha}{\partial g_\beta}\right|_{g=g^*} \,. \tag{1.92}$$

Note that the matrix R^* depends on the fixed point. It is clear from the linearized flow equation that the behaviour of the couplings near criticality is dictated by the eigenvalues of R^*. For eigenvalues $\lambda_i < 1$ the flow is attractive, while for eigenvalues $\lambda_i > 1$ the flow will diverge away from the fixed point. If we consider the differential flow, we find:

$$\frac{d}{dt}\left(\delta g_\alpha\right) = L^*_{\alpha\beta}\left(\delta g_\beta\right) \,. \tag{1.93}$$

The eigenvalues of the linearized flow e^i_α satisfy:

$$(L^*)^t_{\alpha\beta} e^i_\beta = \lambda_i e^i_\alpha \,. \tag{1.94}$$

The evolution equations for the couplings

$$u_i = e^i_\alpha \left(\delta g_\alpha\right) \tag{1.95}$$

are decoupled ODEs:

$$\frac{d}{dt} u_i = y_i u_i \,, \tag{1.96}$$

and yield a power-law behaviour for the evolution of the couplings:

$$u_i(s) = u_i(1) s^{y_i} \,. \tag{1.97}$$

Note that this is the same behaviour one would obtain in Eq. (1.91), after the identification:

$$\lambda_i = s^{y_i} \,. \tag{1.98}$$

Couplings that correspond to $y_i > 0$ are therefore relevant, and viceversa. The irrelevant couplings converge to their fixed point value u^*_i as s increases, *i.e.* the low-energy dynamics becomes independent from the value of the irrelevant couplings as we increase the separation $s = 1/(\mu a)$ between the UV cutoff (a^{-1}) and the scale that we want to probe in the theory (μ). The set of points that flow into the fixed point defines a hypersurface in the space of couplings called the critical surface. On the critical surface the correlation length is infinite.

For the lattice theory to describe the desired physics at low-energies, we need to tune the *relevant* operators only at the scale of the cutoff. Hence the reason for universality, and for being able to reproduce the continuum theory without having to do an infinite amount of fine-tuning of the bare couplings.

1.10 Weak coupling

Let us now discuss the continuum limit of the scalar theory in the weak coupling regime, *i.e.* when the partition function can be evaluated reliably in perturbation theory. All computations are performed in the symmetric phase, where the vacuum expectation vacuum of the field vanishes. The Feynman rules can be derived by standard techniques from the Euclidean action:

1. each line is associated to a lattice propagator;
2. each four-point vertex yields a factor of $-g_0$;
3. momentum is conserved modulo 2π at each vertex;
4. loop momenta are integrated in the first Brillouin zone.

It is customary to rewrite the lattice action in terms of two new couplings κ and λ:

$$S_E[\phi] = \sum_x \left\{ -2\kappa \sum_\mu \phi(x)\phi(x+a\hat{\mu}) + \phi(x)^2 + \lambda \left[\phi(x)^2 - 1\right] \right\} . \tag{1.99}$$

Note that this action is obtained from the original one by the following redefinition of the fields and couplings in four dimensions:

$$a\phi(x) \mapsto \sqrt{2\kappa}\phi(x) \, , \tag{1.100}$$

$$a^2 m_0^2 \mapsto \frac{1-2\lambda}{\kappa} - 8 \, , \tag{1.101}$$

$$g_0 \mapsto \frac{6\lambda}{\kappa^2} \, . \tag{1.102}$$

All the new couplings and fields are dimensionless. This form of the action emphasizes the relation with statistical mechanics. The kinetic term gives rise to a next-neighbour coupling, whose strength is given by the hopping parameter κ. The quartic coupling g_0 is proportional to λ; in the $\lambda \to \infty$ limit, we get the constraint $\phi(x)^2 - 1 = 0$, and the system reduces to the four-dimensional Ising model. On the other hand, in the limit $\lambda \to 0$ we recover the free field theory discussed above. Perturbative calculations are performed in a neighbourhood of this limit, by expanding the solution in a power series in λ. There is another interesting limit where the system can be solved analytically; this is the limit where $\kappa \to 0$ and the fields at different sites are decoupled. The partition function is simply the product of single-site partition functions. Expanding the partition function in a power series in κ is called a hopping parameter expansion (or high-temperature expansion by analogy with statistical mechanics).

Propagator As discussed before, the particle content of the theory can be obtained from the poles in two-point function. Let us introduce:

$$C(t, \mathbf{p}) = \sum_{\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} \Delta(t, \mathbf{x}) \tag{1.103}$$

$$= \int \frac{dp_4}{2\pi} e^{ip_4 t} \Delta(\mathbf{p}, p_4)\,. \tag{1.104}$$

The full propagator in momentum space has poles at $p_4 = \pm iE(\mathbf{p})$, and therefore the physical mass can be defined as:

$$m = E(0)\,. \tag{1.105}$$

With the new normalization for the field introduced in Eq. (1.100) we denote the residue at the pole by:

$$\frac{Z_3}{4m\kappa}\,, \tag{1.106}$$

so that the zero-momentum propagator for the original field ϕ is:

$$[2\kappa\Delta(0, p_4)]^{-1} = \frac{1}{Z_3}(p_4^2 + m^2) + O\left((p_4 + m^2)^2\right)\,. \tag{1.107}$$

For small momenta we can expand

$$E(\mathbf{p}) = m + \frac{\mathbf{p}^2}{2m_*} + O(p^4)\,. \tag{1.108}$$

The coefficient of the quadratic term defines the kinetic mass m_*.

Vertex functions The vertex functions (1PI correlators) are the functional derivatives of the quantum action $\Gamma[\varphi]$, the latter being the Legendre transform of the generator of connected diagrams $W[J]$:

$$\Gamma^{(n)} = \left. \frac{\partial \Gamma}{\partial \varphi_1 \ldots \partial \varphi_n} \right|_{\varphi=0}, \tag{1.109}$$

where

$$\Gamma[\varphi] + W[J] = \sum_x \varphi(x) J(x), \quad \varphi(x) = \frac{\partial W}{\partial J(x)}. \tag{1.110}$$

The explicit expressions for the vertex functions for $n = 2, 3, 4$ in momentum space are given respectively by:

$$\Gamma^{(2)}(p) = \Delta(p)^{-1}, \tag{1.111}$$

$$\Gamma^{(3)}(p_1, p_2, p_3) = S(p_1, p_2, p_3)\Delta(p_1)^{-1}\Delta(p_2)^{-1}\Delta(p_3)^{-1}, \tag{1.112}$$

$$\begin{aligned}\Gamma^{(4)}(p_1, p_2, p_3, p_4) = \{&S(p_1, p_2, p_3, p_4) \\ &- S(p_1, p_2, -p_1 - p_2)S(p_3, p_4, -p_3 - p_4)\Delta(p_1 + p_2)^{-1} \\ &- S(p_1, p_3, -p_1 - p_3)S(p_2, p_4, -p_2 - p_4)S(p_1 + p_3)^{-1} \\ &- S(p_1, p_4, -p_1 - p_4)S(p_2, p_3, -p_2 - p_3)\Delta(p_1 + p_4)^{-1}\} \\ &\times \Delta(p_1)^{-1}\Delta(p_2)^{-1}\Delta(p_3)^{-1}\Delta(p_4)^{-1}\end{aligned} \tag{1.113}$$

The renormalized couplings can be defined from the vertex functions:

$$\Gamma_R^{(n)}(p_1, \ldots, p_n) = \left(\frac{Z_R}{2\kappa}\right)^{n/2} \Gamma^{(n)}(p_1, \ldots, p_n), \tag{1.114}$$

$$\Gamma_R^{(2)}(p) = m_R^2 + p^2 + O(p^4) \tag{1.115}$$

$$\Gamma_R^{(4)}(0, 0, 0, 0) = g_R. \tag{1.116}$$

The renormalized couplings are obtained from the so-called susceptibilities and moments:

$$\chi_n = \sum_{x_1, \ldots, x_n} S(x_1, \ldots, x_n), \tag{1.117}$$

$$\mu_2 = \sum_x x^2 \Delta(x), \tag{1.118}$$

by the following relations:

$$m_R^2 = \frac{8\chi_2}{\mu_2}\,, \tag{1.119}$$

$$Z_R = 2\kappa\frac{8\chi_2^2}{\mu_2}\,, \tag{1.120}$$

$$g_R = \frac{64}{\mu_2^2}\left[\chi_4 - 3\frac{\chi_3^2}{\chi_2}\right]. \tag{1.121}$$

Perturbation theory We have now all the tools in place to discuss the continuum limit in perturbation theory. The contributions to $\Gamma^{(2)}$ up to $O(g_0^2)$ are shown in Fig. 1.2. They yield:

$$\frac{1}{2\kappa}\Delta(p)^{-1} = (a^2\hat{p}^2 + a^2m_0^2) + \frac{g_0}{2}J_1(am_0) - \frac{g_0^4}{4}J_1(am_0)J_2(am_0) - \frac{g_0^2}{6}I_3(am_0) + \ldots\,, \tag{1.122}$$

where

$$J_n(am_0) = \int_q \Delta(aq)^n\,, \tag{1.123}$$

$$I_3(am_0, ap) = \int_{q_1,q_2} \Delta(aq_1)\Delta(aq_2)\Delta(ap - aq_1 - aq_2)\,. \tag{1.124}$$

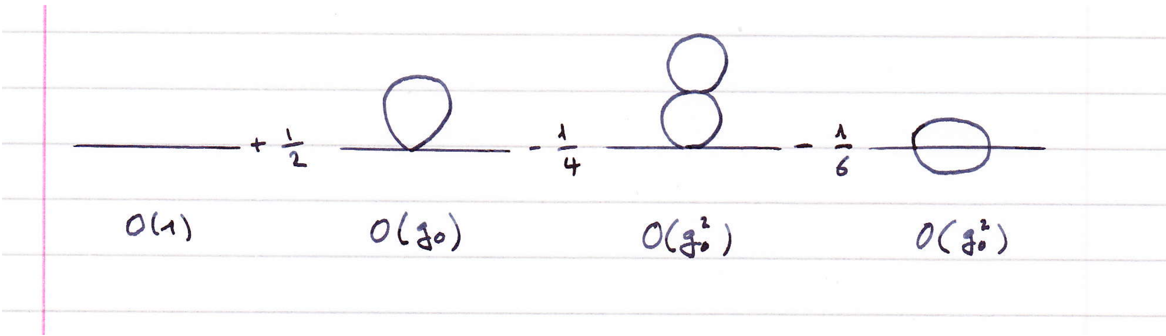

Figure 1.2: Contribution to the two-point vertex function up to two loops.

We have denoted by $\Delta(aq)$ the lattice propagator for the scalar field with bare mass m_0:

$$\Delta(q) = \frac{1}{4\sum_\mu \sin^2(aq_\mu/2) + a^2m_0^2}\,. \tag{1.125}$$

At one-loop there is no momentum-dependent term in the expression for $S(p)$, and hence:

$$Z_R = 1 + O(g_0^2)\,, \tag{1.126}$$
$$a^2 m_R^2 = a^2 m_0^2 + \frac{g_0}{2} J_1(am_0) + O(g_0^2)\,. \tag{1.127}$$

The four-point vertex is computed by evaluating the diagrams in Fig. 1.3. The final result is:

$$\left(\frac{1}{2\kappa}\right)^2 \Gamma^{(4)}(p_1,p_2,p_3,p_4) = g_0 - \\ - \frac{g_0^2}{2}\left[I_2(am_0, ap_1+ap_2) + I_2(am_0, ap_1+ap_3) + I_2(am_0, ap_1+ap_4)\right] + \ldots\,, \tag{1.128}$$

where we introduced

$$I_2(am_0, ap) = \int_q \Delta(aq)\Delta(ap-aq)\,. \tag{1.129}$$

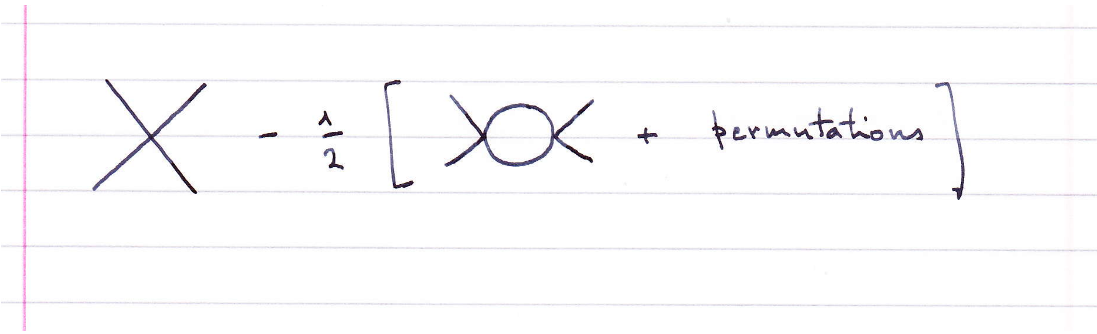

Figure 1.3: Contribution to the four-point vertex function up to one loop.

Eq. (1.128) yields:

$$g_R = g_0 - \frac{3}{2} g_0^2 J_2(am_0) + O(g_0^3)\,. \tag{1.130}$$

Note that these relations involve dimensionless couplings like g_0 and am_0. It is clear from these results that the renormalized couplings are related to the bare parameters via the loop integrals J_n and I_n. As already mentioned, when the theory is defined on the lattice all UV divergencies are regulated, and these integrals are convergent.

In order to expose the divergencies, let us focus on the a dependence of the terms that appear in the above relations. For the renormalized mass we obtain:

$$m_R^2 = m_0^2 + \frac{g_0}{2}\frac{1}{a^2} J_1(am_0) + \dots . \tag{1.131}$$

For small x:

$$J_1(x) = r_0 + x^2 \left[\frac{1}{16\pi^2} \log y^2 + r_1 + O(y^2) \right] , \tag{1.132}$$

with $r_0 = 0.154\dots$ and $r_1 = -0.030\dots$.
Hence we obtain:

$$m_R^2 = m_0^2 + \frac{1}{a^2}\frac{g_0 r_0}{2} + \frac{g_0}{32\pi^2} m_0^2 \log(am_0) + \frac{g_0}{2} r_1 m_0^2 + \dots . \tag{1.133}$$

Eq. (1.133) shows explicitly the quadratic and logarithmic divergencies in the mass, which need to be cancelled by appropriately divergent counterterms in the action. On the other hand Eq. (1.127) tells that the bare mass m_0 has to be fine-tuned in order for the system to be at criticality, *i.e.* in order to have $am_R \to 0$:

$$(am_0)^2 \to \frac{g_0}{2} r_0 + O(g_0^2) . \tag{1.134}$$

we have obtained a relation between the bare parameters appearing in the lattice action, which defines a critical line in the space of bare theories. Expressed in terms of the hopping parameter we get:

$$\kappa \to \kappa_c = \frac{1}{8} + \left(3r_0 - \frac{1}{4} \right) \lambda + O(\lambda^2) . \tag{1.135}$$

To approach a fixed point of the RG, we need to tune the bare mass, which is a relevant operator.

Let us now consider the renormalized charge. From the small-x behaviour of $J_2(x)$, we find:

$$g_R = g_0 + \frac{3}{32\pi^2} g_0^2 \log(a^2 m_0^2) + \frac{3}{2} g_0^2 (\frac{1}{16\pi^2} + r_1) + \dots , \tag{1.136}$$

where the logarithmic divergence is clearly visible.

The relations between g_R, m_R and g_0, m_0 can be inverted, and yields:

$$m_0^2 = m_R^2 - \frac{1}{a^2} J_1(am_R) + O(g_R^2) , \tag{1.137}$$

$$g_0 = g_R + \frac{3}{2} g_R^2 J_2(am_R) + O(g_R^3) . \tag{1.138}$$

The results for the vertex functions as functions of m_R and g_R are all finite in the limit $a \to 0$.

Finally let us discuss in this framework the fate of higher-dimensional interactions. Note that in the bare action at the cutoff scale we have not introduced higher-dimensional operators, *e.g.* there is no ϕ^6 term. However these interactions are generated by loop corrections. The six-point vertex function at one-loop is given by the diagram in Fig. 1.4:

$$\left(\frac{1}{2\kappa}\right)^3 \Gamma^{(6)}(0,\ldots,0) = -15 g_0^3 J_3(am_0) + O(g_0^4)\,. \tag{1.139}$$

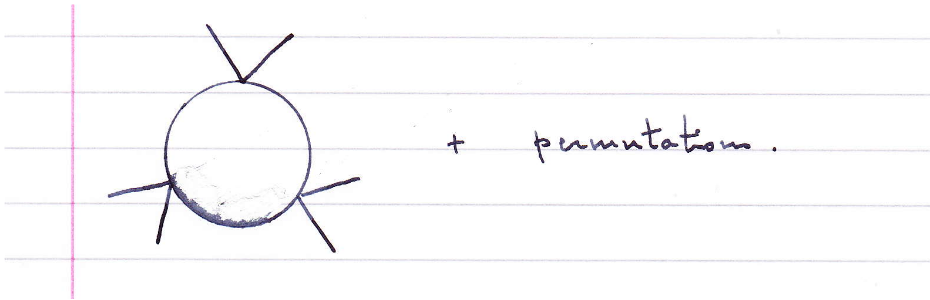

Figure 1.4: Contribution to the six-point vertex function up to one loop.

Had we started with a non-zero value η_0 for this coupling, we would have:

$$\left(\frac{1}{2\kappa}\right)^3 \Gamma^{(6)}(0,\ldots,0) = \eta_0 - 15 g_0^3 J_3(am_0) + O(g_0^4)\,. \tag{1.140}$$

The coupling η_0 in four dimensions has dimension $[\eta_0] = -2$. If we require the low-energy physics to stay invariant as we change the cutoff, we obtain:

$$\eta_0 - a^2 15 g_0^3 J_3(am_0) = \eta_0' - a'^2 15 g_0'^3 J_3(am_0)\,. \tag{1.141}$$

The evolution of the dimensionless couplings can be read from the equation above by considering the appropriate dimensionless combinations:

$$\begin{aligned}\left(a'^{-2}\eta_0'\right) &= s^{-2}\left(a^{-2}\eta_0\right) + 15\left[\frac{g_0'^3}{(a'm_0')^2} - s^{-2}\frac{g_0^3}{(am_0)^2}\right] + O(g_0^4) \\ &= s^{-2}\left(a^{-2}\eta_0\right) + O(g_0^4)\,.\end{aligned} \tag{1.142}$$

We see explicitly in this example that the value of η_0 flows to zero as s^{-2} independently of the initial value at the cutoff scale. The six-point coupling in four dimensions is irrelevant.

1.11 Triviality of the continuum limit

The perturbative computations above yield a relation between the renormalized couplings, the bare ones, and the UV cutoff of the theory. The dependence of the renormalized coupling on the cutoff for fixed bare coupling is encoded in the Callan-Symanzik equation, which in our case can be obtained from Eq. (1.136):

$$\beta(g_R) = \frac{3}{16\pi^2} g_0^2 + O(g_0^3) = \frac{3}{16\pi^2} g_R^2 + O(g_R^3)\,. \tag{1.143}$$

Higher orders in perturbation theory yield:

$$\beta(g_R) = \beta_0 g_R^2 + \beta_1 g_R^3 + \ldots\,. \tag{1.144}$$

The first two coefficients do not depend on the regularization scheme:

$$\beta_0 = \frac{3}{16\pi^2}, \quad \beta_1 = -\frac{17}{3(16\pi^2)^2}\,. \tag{1.145}$$

The behaviour of the renormalized coupling as we approach the continuum limit can be obtained by integrating the Callan-Symanzik equation:

$$a = C \exp\left[-1/(\beta_0 g_R)\right]\, g_R^{-\beta_1/\beta_0^2} \left(1 + O(g_R)\right)\,. \tag{1.146}$$

Hence:

$$\lim_{a\to 0} g_R(a)\Big|_{g_0} \sim \lim_{a\to 0} \frac{1}{\log a} = 0\,. \tag{1.147}$$

Perturbation theory suggests that g_R vanishes when we take the continuum limit. This property of the scalar field theory in four dimensions goes under the name of triviality. Of course it would be desirable to have a nonperturbative proof of triviality, *i.e.* a proof that does not rely on being in a neighbourhood of $g_0 = 0$. Lüscher and Weisz studied this problem in a series of papers, and did not find any evidence of a nontrivial fixed point.

1.12 Symanzik effective theory

As the lattice theory approaches a continuum limit, the separation between the cutoff scale and the physical scales becomes increasing large. In this regime, we expect to be able to describe the lattice theory with a continuum effective theory, *i.e.* a theory defined directly in the continuum, with higher dimensional (irrelevant) operators added to the action in order to mimick the cutoff effects. All operators that are compatible with the symmetries of the lattice action should be added to the low-energy effective theory.

Once again we can discuss this phenomenon in the simple context of the free field theory. For a complete discussion, the reader is referred to the recent lectures by P. Weisz. [4]

The lattice propagator was derived in Eq. (1.48):

$$\Delta(p) = \frac{1}{\sum_\mu \frac{4}{a^2}\sin^2\left(\frac{p_\mu a}{2}\right) + m_0^2}\,. \tag{1.148}$$

For small momenta ($p_\mu a \ll 1$):

$$\sum_\mu \frac{4}{a^2}\sin^2\left(\frac{p_\mu a}{2}\right) = p^2 - \frac{1}{12}a^2\sum_\mu p_\mu^4 + O(a^4)\,. \tag{1.149}$$

Let us consider now the low-energy effective theory. In the continuum limit the only operators of dimension D that are symmetric under the hypercubic symmetry group of the lattice theory are:

$$S[\phi] = \int d^D x \frac{1}{2}\left[\partial_\mu\phi\partial_\mu\phi + m_0^2\phi^2\right]\,. \tag{1.150}$$

If we want to reproduce the $O(a^2)$ effects in Eq. (1.149), we need to add operators of dimension $D+2$ to the action. Again taking into account the symmetry of the lattice action, and using the equations of motions to reduce the admissible operators, we end up with a single term:

$$a^2 S_1 = -ca^2\int d^D x \sum_\mu \partial_\mu^2\phi\partial_\mu^2\phi\,. \tag{1.151}$$

This term is not symmetric under the continuum Euclidean $O(4)$ group, but is symmetric under the hypercubic group. The two-point function computed with the effective action is:

$$\begin{aligned}\Delta(p) &= Z^{-1}\int \mathcal{D}\phi e^{-S-a^2S_1}\phi(p)\phi(0) \\ &= \frac{1}{p^2+m_0^2} - a^2\langle S_1\phi(p)\phi(0)\rangle + O(a^4)\,.\end{aligned} \tag{1.152}$$

You can check that if you choose $c = 1/12$, then the propagator in the continuum effective theory reproduces the lattice one at the chosen order in powers of a.

Symanzik conjectures that there exists a continuum low-energy description for a large class of interacting lattice theories.

The discussion above shows why the continuum O(4) symmetry is automatically recovered in the continuum limit without the need for fine tuning. Indeed the first operators that are invariant under the hypercubic group, but not under O(4) are irrelevant operators of dimension 6, and their contribution is suppressed by powers of the cutoff. A symmetry that arises naturally in the low-energy limit is called an emergent symmetry.

[4] P. Weisz, lectures in "Modern Perspective in Lattice QCD", Oxford University Press, 2011.

1.13 Numerical simulations

First-principle results beyond perturbation theory can be obtained by evaluating the Schwinger functions numerically with numerical simulations. The analogy between Euclidean QFT and statistical mechanics is fully exploited in numerical simulations, where the path integral is computed by importance sampling. The field correlators:

$$\langle \mathcal{O}[\phi] \rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{-S_E[\phi]} \mathcal{O}[\phi] \,, \tag{1.153}$$

are computed by generating an ensemble of field configurations $\{\phi_i\}$ distributed according to the Boltzmann weight:

$$p[\phi_i] \propto e^{-S_E[\phi_i]} \,. \tag{1.154}$$

The expectation value is then computed by taking the average over the ensemble:

$$\langle \mathcal{O} \rangle = \bar{\mathcal{O}} = \frac{1}{N_{\text{cnfg}}} \sum_{i=1}^{N_{\text{cnfg}}} \mathcal{O}[\phi_i] + O(1/\sqrt{N_{\text{cnfg}}}) \,, \tag{1.155}$$

where we have denoted by $\bar{\mathcal{O}}$ the average over configurations.

Markov processes are used in order to generate the ensemble with the correct probability distribution. These are recursive procedures that generate the field configurations according to some specified algorithm, which yields asymptotically the desired probability distribution. It is possible to show under some very general assumptions that the probability distribution of the states along the Markov chain converges exponentially to the equilibrium distribution. The convergence is characterized by a number of step τ that are needed for the system to "thermalize". After thermalization, the configurations can be used to compute expectation values according to the prescription in Eq. (1.155). The number τ is known as the exponential autocorrelation of the Markov chain.

The configurations generated by a Markov chain are correlated by construction. As a result the variance of the average over configuration is larger than it would be for independent configurations:

$$\text{Var}[\bar{\mathcal{O}}] = \text{Var}[\mathcal{O}] \left(\frac{2\tau_{\mathcal{O}}}{N_{\text{cnfg}}} \right) \,. \tag{1.156}$$

$\tau_{\mathcal{O}}$ is called the integrated autocorrelation time, and depends on the observable under consideration. We refer *e.g.* to M.Lüscher's lectures [5] for a quantitative definition of $\tau_{\mathcal{O}}$. The error on the estimator is reduced like $N_{\text{cnfg}}^{-1/2}$. The variance of the observable

$$\text{Var}[\mathcal{O}] = \langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle \,, \tag{1.157}$$

[5] M. Lüscher, lectures in "Modern Perspective in Lattice QCD", Oxford University Press, 2011.

is a property of the QFT and has no relation with the Markov chain used to generate the ensemble of configurations. The correlation between configurations increases the variance of the estimator $\bar{\mathcal{O}}$, and depends strongly on the simulation algorithm.

Numerical simulations are necessarily performed on finite lattices. We will assume in the rest of this section that the lattice has a finite spatial volume $V = L^3$, and infinite extent in the Euclidean time direction. We can get some feeling for the dynamics of finite-volume effects by revisiting the perturbative computations discussed in the section above.

Clearly the fact that the theory is defined in a finite volume changes the integrals over spatial momenta into sums over multiples of $2\pi/L$. Therefore we can write the renormalized mass at one-loop in perturbation theory with little effort:

$$(am_R(L))^2 = (am_0)^2 = \frac{g_0}{2} J_1(am_0, L/a) + O(g_0^2)\,, \tag{1.158}$$

where the loop integral has been replaced by:

$$J_n(am, L/a) = \frac{1}{V} \sum_{\mathbf{p}} \int \frac{dp_4}{2\pi} (\hat{p} + a^2 m_0^2)^{-n}\,. \tag{1.159}$$

Hence the difference in the renormalized mass:

$$(am_R(L))^2 - (am_R)^2 = \frac{g_0}{2} \left[J_1(am_0, L/a) - J_1(am_0, \infty)\right] + O(g_0^2)\,. \tag{1.160}$$

These finite-volume shifts can be re-expressed in terms of the renormalized couplings, using the asymptotic behaviour of J_1 and J_2 for large L/a:

$$\delta(am_R(L))^2 = \frac{g_R}{2} 6(am_*^2)(2\pi m_* L)^{-3/2} e^{-mL} \left[1 + O(L^{-1})\right]\,, \tag{1.161}$$

$$\delta g_R(L) = -\frac{3}{2} g_R^2 \frac{3}{2} (m_* L)(2\pi m_* L)^{-3/2} e^{-mL} \left[1 + O(L^{-1})\right]\,. \tag{1.162}$$

The infinite-volume limit is approached exponentially for both quantities. The mass approaches the thermodynamic limit from above, while the renormalized coupling does it from below.