Problems on Quantum Field Theory

September 7, 2012

Abstract

Problem sets for BUSSTEPP 2012.

1 Grand unification

Consider the running of the SM gauge couplings (in GUT normalisation, i.e. $g_1 = \sqrt{\frac{5}{3}} g_Y$) in an extension of the Standard Model in which these couplings unify (at one loop). Show that the couplings in such a model still unify (at one loop) if an entire scalar or fermionic $SU(5)$ multiplet is considered.

To do so, note that the extra contributions to the beta function coefficients arise from the one-loop correction to the gauge boson two-point functions, which has the form

$$(p_\mu p_\nu - \eta_{\mu\nu} p^2) I_{AB}(p^2),$$

where $I_{AB}(p^2)$ is logarithmically dependent on the cutoff just as in the lecture example, and A, B are gauge indices which can denote an $SU(3)$, an $SU(2)$, or an $U(1)$ generator. Use what you know about the normalisation of the gauge generators from the SMB lectures.

2 Wilson-Fisher fixed point

Consider a real scalar field in D dimensions with the Wilsonian effective Lagrangian

$$\mathcal{L}_{\text{int}}(\Lambda) = \int d^D x (\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\bar{M}^2(\Lambda)\phi^2 + \frac{1}{4!} g(\Lambda)\phi^4)$$

in D Euclidean dimensions.

- Recall the canonical dimensions of the field ϕ, the (mass)^2 M^2, and the quartic coupling g.

- Now consider the case $D = 3$. Classically, M and g do not run under the renormalisation group. Introduce dimensionless parameters m^2 and λ using appropriate powers of the renormalisation group scale Λ. Show that their β-functions are

$$\beta_m^2 = -2m^2, \qquad \beta_\lambda = -\lambda$$

  They display a Gaussian (free) fixed point $(m^2, \lambda)_* = (0, 0)$. Compute the stability matrix and their eigenvalues at the fixed point, and plot typical trajectories in the (m^2, λ) plane in the vicinity of the Gaussian fixed point. What is the physics behind it?

- Quantum corrections modify the β-functions. To 1-loop order and in the local potential approximation (i.e. assuming physics is correctly taken care of by expanding the Wilsonian action in local operators without derivatives, as discussed in the lectures), the β-functions become

$$\beta_{m^2} = -2m^2 - a\lambda, \qquad \beta_\lambda = -\lambda + 6a\lambda^2$$

  and a a numberical factor Explicit computations find that $a > 0$. Show that the β-functions now display two fixed points, the trivial (Gaussian) one and the interacting (Wilson-Fisher) fixed point.

- At the Gaussian fixed point, compute the stability matrix and its eigenvalues. Make a plot of typical trajectories in the (m^2, λ) plane around the Gaussian fixed point and compare with the classical regime. Where does the "tilt" come from?

- At the Wilson-Fisher fixed point, compute the stability matrix and its eigenvalues. Why are the eigenvalues independent of a? Make a plot of typical trajectories in the (m^2, λ) plane including the Gaussian and the Wilson-Fisher fixed point. What is the physics behind it?

- What is the dimensionality of the "infrared critical surface" of the Wilson-Fisher fixed point?

3 Functional methods

Derive the Wilsonian (exact) RG equation in the form given in the lectures

$$\Lambda \frac{dS_{\text{int}}}{d\Lambda} = \frac{1}{2}(2\pi)^4 \int d^4p\, \Lambda \frac{dK}{d\Lambda} \left[\frac{\partial S_{\text{int}}}{\partial \phi(p)} \frac{\partial S_{\text{int}}}{\partial \phi(-p)} - \frac{\partial^2 S_{\text{int}}}{\partial \phi(p) \partial \phi(-p)} \right]$$

Hints:

- The integral

$$\int d^4p\, \Lambda \frac{\partial K}{d\Lambda} \int [d\phi] \frac{\partial}{\partial \phi(p)} \left\{ \left(\phi(p) K^{-1} - \frac{1}{2} (2\pi)^4 \frac{1}{p^2 + m^2} \frac{\partial}{\partial \phi(-p)} \right) \exp^{S_{\rm int}(\Lambda)} \right\}$$

  vanishes, because the functional integral is over a total derivative.

- Evaluate this expression in terms of derivatives of $S_{\rm int}$ with respect to the field.

- Assume that the source $J(\phi)$ has support only at small $p^2 \ll \Lambda$, such that it can be dropped whenever multiplied by the derivative of the cutoff function.

- Finally, compare the expression you get to the one derived in the lectures.