Busstepp 2012 – Cosmology
Problems 2

1. Derive the fluid equation

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{1}$$

from the acceleration and Friedmann equations

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a \tag{2}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{3}$$

2. In this question we will introduce the notion of conformal co-ordinates which can often make a problem easier to solve. In equations (2) and (3), change the variable from proper time t to conformal time η using $d\eta = \frac{dt}{a(t)}$ hence

$$\eta = \int^t \frac{dt}{a(t)}.$$

Show that the two equations become

$$a'' = \frac{4\pi G}{3}(\rho - 3p)a^3 - ka \tag{4}$$

$$a'^2 = \frac{8\pi G}{3}\rho a^4 - ka^2 \tag{5}$$

3. We are going to solve for the presence of both matter and radiation in the case of $k = 0$ as well as $k = \pm 1$.

a. Recalling that the dust (matter) component has energy density ρ_m and zero pressure $p_m = 0$, whereas the radiation component has energy density ρ_r and pressure $p_r = \frac{\rho_r}{3}$, use the fluid equation (1) to show that they satisfy

$$\rho_m a^3(t) = \text{const}, \qquad \rho_r a^4(t) = \text{const}.$$

b. Introduce the constants L and M by

$$L \equiv \frac{4\pi G}{3}\rho_r a^4, \qquad M \equiv \frac{4\pi G}{3}\rho_m a^3.$$

Now recalling that the total energy density and pressure is given by $\rho = \rho_m + \rho_r$ and $p = p_m + p_r$ respectively, show that equations (4) and (5) become

$$a'' = M - ka \tag{6}$$

$$a'^2 = 2Ma + 2L - ka^2 \tag{7}$$

c. Now we will concentrate on individual cases. Lets first of all consider a closed universe where $k = +1$. Show that the solution to equation (6) is

$$a(\eta) = M - (A\sin\eta + B\cos\eta) \tag{8}$$

where A and B are constants.

d. To determine the constants A and B, we use the initial condtion $a(\eta = 0) = 0$ and the Friedmann equation (7). Use these to show that

$$A^2 + B^2 = M^2 + 2L, \qquad B = M,$$

hence

$$a(\eta) = M + \sqrt{2L} \sin \eta - M \cos \eta \tag{9}$$

e. Follow the arguments in parts (c) and (d) above to obtain the results for a flat $k = 0$ and negatively curved ($k = -1$) universe:

$$a(\eta) = M + \sqrt{2L} \sin \eta - M \cos \eta \qquad k = +1 \tag{10}$$

$$a(\eta) = \frac{1}{2} M \eta^2 + \sqrt{2L} \eta \qquad k = 0 \tag{11}$$

$$a(\eta) = \sqrt{2L} \sinh \eta + M \cosh \eta - M \qquad k = -1 \tag{12}$$

Given $a(\eta)$, the proper time t can be obtained through $t(\eta) = \int_0^\eta a(\eta) d\eta$. In particular we can use it to obtain the age of the universe for these different cosmologies. For simplicity, let us concentrate on the case of a universe containing only radiation, which means that $M = 0$.

f. Consider the case of $k = 1$, $M = 0$. By using the relationship $t(\eta) = \int_0^\eta a(\eta) d\eta$. with equation (10), show that the scale factor is given by

$$a(t) = \left(2t\sqrt{2L} - t^2\right)^{\frac{1}{2}} \tag{13}$$

Note that the solution goes to zero for two values of t, corresponding to the big bang and big crunch in a universe containing only radiation.

g. We now want to rearrange this to obtain the age of the universe in terms of todays observables H_0 and $\Omega_0 = \frac{\rho_r}{\rho_{\rm cr}}$. Recalling $\rho_{\rm cr} = \frac{3H^2}{8\pi G}$, use the Friedmann equation to show

$$a_0^2 = \frac{1}{(\Omega_0 - 1) H_0^2} \qquad L = \frac{\Omega_0}{2(\Omega_0 - 1)^2 H_0^2},$$

hence show that the current age of a closed universe containing just radiation is

$$t_0 = \frac{1}{(\sqrt{\Omega_0} + 1) H_0} \tag{14}$$

Note that as $\Omega_0 \to 1$ we recover $t_0 H_0 = \frac{1}{2}$ the usual result for a flat radiation dominated universe.