Supersymmetric Lattice Gauge Theories

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Lattice SUSY - the traditional problems and how to dodge them

Simple two dimensional model: continuum and lattice

$\mathcal{N} = 4$ Super Yang-Mills

Non-perturbative physics: phase diagram of $\mathcal{N} = 4$ YM

Acknowledgments

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Motivation

Why should one care about SUSY ?

- SUSY theories: toy models for understanding confinement, chiral symmetry breaking, ...
- AdS/CFT – super YM theories may tell us about gravity ..

Why lattice SUSY ?

- Would like a $\boxed{\text{non-perturbative}}$ definition of SUSY theories; like lattice QCD for QCD
- Many phenomena eg susy breaking, corrections to SUGRA approx in AdS/CFT involve strongly coupled, non-perturbative physics – ideal ?? for $\boxed{\text{lattice}}$

LHC/BSM era: might expect lots of lattice groups working on supersymmetry ...

Indeed supersymmetric theories were studied in early days of LGT

But significant theoretical problems in lattice SUSY

Stopped work for many years ....

Barriers to Lattice Supersymmetry

- $\{Q,\overline{Q}\}=\gamma_\mu p_\mu$. No p_μ on lattice. Equivalently: no Leibniz rule for difference ops: $\Delta(fg)\neq(\Delta f)g+f\Delta g$
- Classical SUSY breaking leads to (many) SUSY violating ops in quantum theory. Couplings must be adjusted with cut-off $(1/a)$ to achieve SUSY in continuum limit -fine tuning.
- Discretization of Dirac equation: Lattice theories contain unwanted fermions (doublers) which do not decouple in continuum limit. Consequence: no. fermions $\neq$ no. bosons
- Lattice gauge fields live on lattice links and take values in group. Fermions live on lattice sites and (for adjoint fields) live in algebra ....

Putting SUSY on a lattice

Goals of any successful SUSY lattice formulation:

- Reduce/eliminate fine tuning. In particular: keep $m_{\text{scalars}} << 1/a$.
- Keep exact gauge invariance. Lesson of lattice QCD (Wilson)
- Avoid fermion doubling...
- More symmetrical treatment of bosons and fermions...?
- Avoid sign problems. After integration over fermions is effective bosonic action real ? Monte Carlo simulation requires this ...

New formulations exist with all these features

Key new idea - twisting

- Rewrite continuum theory in twisted variables.
- Exposes a single scalar supersymmetry $\mathcal{Q}$ whose algebra is simple: $\mathcal{Q}^2 = 0$. Bonus: $S =\sim \mathcal{Q}\Lambda$.
- Key: this SUSY can be retained on discretization: easy to build invariant lattice action.
- Hope (prove ?) that exact lattice symmetry reduces/eliminates fine tuning
- See that $\boxed{\text{all}}$ fields will live on links and take values in algebra.
- Structure of fermionic action dictated by exact SUSY - would doublers all physical

Toy example: $(2,2)$ Super Yang-Mills in two dimensions

$$S=\int F(A)^2+\sum_{i=1}^{2}\psi^i\gamma.D\psi-\sum_{i=1}^{2}B^i\Box B^i+\mathrm{interactions}$$

Global symmetry $SO_{\mathrm{Lorentz}}(2)\times SO_{\mathrm{flavor}}(2)$

$$\psi^i_\alpha\to R_{\alpha\beta}\psi^j_\beta(F^T)^{ji}$$

Under $\boxed{\text{diagonal subgroup } R=F}$ fermions like matrix - Ψ!
Decompose on products of γ matrices:

$$\Psi=\eta I+\psi_i\sigma_i+\chi_{12}\sigma_1\sigma_2\ \mathrm{twisted\ fields}\ \eta,\psi_\mu,\chi_{12}$$

$$\boxed{\text{Scalar fermion } \eta \text{ ... implies scalar SUSY } \mathcal{Q} \text{ with } \mathcal{Q}^2=0}$$

What about the bosons ?

Twisting: decompose all fields on twisted rotation-flavor group

- Gauge field A_μ is singlet: remains vector under twisted group
- Original theory had 2 scalars B_i in vector rep of group (flavor $\rightarrow$ R symmetry) – become vectors under twisted rotations!!
- In fact all bosons in twisted theory packaged as complex gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$

Twisted action

Can write action in twisted form

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right)$$

with

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu,\ \overline{\mathcal{D}}_\mu = \partial_\mu + \overline{\mathcal{A}}_\mu\ \mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$$

All fields in adj rep ($X = \sum_A T^A X^A$ with AH generators)
Important:

- Complex gauge fields but just U(N) gauge symmetry
- In flat space - twisting just a $\boxed{\text{change of variables}}$

Scalar supersymmetry

As promised:

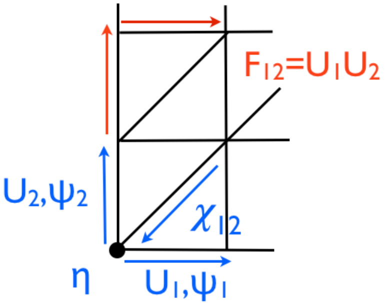
$$
\begin{aligned}
\mathcal{Q}\ \mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\ \psi_\mu &= 0 \\
\mathcal{Q}\ \overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\ \chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\ \eta &= d \\
\mathcal{Q}\ d &= 0
\end{aligned}
$$

- $\mathcal{Q}^2 = 0$
- Looks like BRST ? (side story - connection to topological FT)
- 4 original susys appear as $(\mathcal{Q}, \mathcal{Q}_\mu, \mathcal{Q}_{12})$

Lattice construction

LATTICE ?

Twisted N=(2,2) Lattice



- Lattice fields live on links
- Gauge transform like endpoints: $\psi_1(x) \rightarrow G(x)\psi_1(x)G^\dagger(x+1)$
- $\mathcal{Q}$ SUSY same as continuum

$$\mathcal{F}_{12} = U_1(x)U_2(x+1) - U_2(x)U_1(x+2)$$

Derivatives

- $\boxed{\text{Precise}}$ dictionary exists to translate $\mathcal{D}_\mu$ to gauge covariant difference ops. Eg

$$\mathcal{D}^+_\mu f_p(x) = \mathcal{U}_\mu(x) f_p(x+\mu) - f_p(x)\mathcal{U}_\mu(x+p)$$

  New field lives on link $(x \to x+p+\mu)$
- Naive continuum limit $\mathcal{U}_\mu(x) = I + a\mathcal{A}_\mu(x) + \ldots$:

$$\mathcal{D}^+_\mu f_p(x) \to \frac{1}{a}\left(f_p(x+\mu) - f_p(x)\right) + [\mathcal{A}_\mu, f_p] + \mathcal{O}(a)$$

Lattice action

Do $\mathcal{Q}$ variation and integrate out auxiliary field d:

$$\begin{aligned} S_{\rm exact} &= \sum_{\mathbf{x}} {\rm Tr} \left(\mathcal{F}^{\dagger}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_{\mu}\mathcal{U}_{\mu}\right)^2 \right. \\ &- \left. \chi_{\mu\nu}\mathcal{D}^{(+)}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}^{(-)}_{\mu}\psi_{\mu} \right) \end{aligned}$$

Comments:

- Boson action real, pos semidef. Collapses to Wilson plaquette for unitary $\mathcal{U}$
- Fermion action: Kähler-Dirac form: map to (reduced) staggered fermions for $g=0$. Hence: $\boxed{\text{no doubling!}}$

Gauge invariance

- All terms local, correspond to closed loops and hence are lattice gauge invariant
- $\mathcal{U}_\mu$'s non compact! $\mathcal{U}_\mu=\sum_B T^B\mathcal{U}_\mu^B$ – flat measure $\int\prod D\mathcal{U}_\mu D\overline{\mathcal{U}}_\mu$. Nevertheless, still gauge invariant - Jacobians resulting from gauge transformation of $\mathcal{U}$ and $\overline{\mathcal{U}}$ cancel.
- Bigger question: how to generate correct naive continuum limit
  Requires that can expand (suitable gauge)
  $\mathcal{U}_\mu = I + \mathcal{A}_\mu(x) + \ldots....?$

Naive continuum limit

- Need $\mathcal{U}_a = I + \mathcal{A}_\mu(x) + \ldots$. Here, unlike lattice QCD, unit matrix does not come from expanding group element $e^{\mathcal{A}}$ but from the vev of a dynamical field! - $\mathrm{Tr}\,\mathrm{Im}\,\mathcal{U}_\mu(x)$ - a scalar field in untwisted theory
- Ensure by adding gauge invariant potential

$$\delta S = \mu^2 \sum_{x,a} \left(\frac{1}{N} \mathrm{Tr}\left(\mathcal{U}_\mu(x)^\dagger \mathcal{U}_\mu(x)\right) - 1 \right)^2$$

  Writing $\mathcal{U}_\mu = (I + B_\mu)e^{A_\mu}$ yields $< Tr(B_\mu) >= 0$ and mass μ for $Tr(B_\mu)$.
- Breaks $\mathcal{Q}$ SUSY softly. All breaking terms must vanish for $\mu \to 0$ (exact $\mathcal{Q}$).

$\mathcal{N} = 4$ SYM

Lattice twisting prescription works if no. susys $= 2^D$. In $D = 4$ unique lattice theory:

$$\boxed{\mathcal{N} = 4 \text{ Yang-Mills}}$$

4 Majorana fermions, 6 scalars B_i and 1 gauge field A_μ. $SO(6)$ flavor (R) symmetry. 16 real supersymmetries.

Twist:

- Decompose fields under twisted rotation group $SO(4)' = \mathrm{diag}(SO_R(4) \times SO_{lorentz}(4))$
- Fermions $\to$ matrix $\Psi = (\eta, \psi_\mu, \chi_{\mu\nu}, \overline{\psi}_\mu, \overline{\eta})$
- Bosons $\to$ $\mathcal{A}_\mu = A_\mu + iB_\mu,\ B_5, B_6$
- Scalar nilpotent supercharge $\mathcal{Q}$

More elegantly ...

Compactly expressed as dimensional reduction of 5D theory:

- 16 fermions: $\Psi = (\eta, \psi_m, \chi_{mn}), m, n = 1 \ldots 5$
- 10 bosons as 5 complex gauge fields $\mathcal{A}_m, m = 1 \ldots 5$

Remarkably: $\mathcal{N} = 4$ action (almost) of same form as 2D case!

$$S = \mathcal{Q} \int \left(\chi_{ab} F_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - 1/2\eta d\right) + S_{\rm closed}$$

with

$$S_{\rm closed} = -\frac{1}{4} \int \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$

Transition to lattice

- Basic idea, prescription for derivatives, $\mathcal{Q}$-supersymmetry all same as 2D
- Only new aspect: want lattice with 5 equivalent basis vectors in 4D

$$\boxed{A_4^* \text{ lattice}}$$

Analog of A_2^* lattice in 2D (triangular lattice)

- Clearly basis vectors not independent:

$$\sum_{a=1}^{5} \mathbf{e}^a = 0$$

Requirement of exact lattice SUSY determines the lattice!

Quantum corrections ...

Can show:

- Lattice theory renormalizable: only counterterms allowed by exact symmetries correspond to terms in original action
- Effective potential at $\mu = 0$ vanishes to all orders in p. theory.
  Flat directions in lattice theory survive quantum correction
- At one loop: divergence structure matches continuum
  - No fine tuning needed to restore full SUSY in continuum limit.
  - Vanishing beta function!

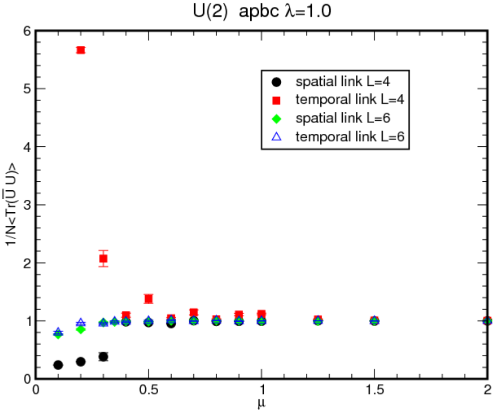
New Physics lies beyond p. theory

Need to explore theory at strong coupling
First step: Phase diagram of lattice theory ...

Simulations

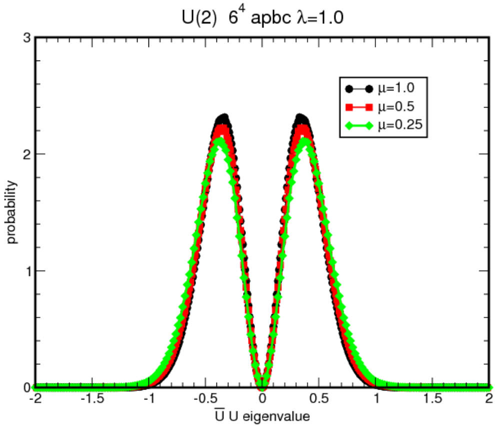
- Integrate fermions $\to \mathrm{Pf}(M)$. Realize as $\det\left(M^\dagger M\right)^{\frac{1}{4}}$
- Standard lattice QCD algs may be used: RHMC with Omelyan, multi time step evolution. GPU acceleration for inverter (speedup: 5-10 over single core code for $L=8^3\times 16$)
- Phase quenched approximation should be ok: analytical argument, numerical evidence ...
- First step: phase structure .... $U(2), L^4, apbc,\ L=4,6,8$
  - Fix the unit matrix vev ? Instabilities from flat directions ?
  - Supersymmetry realized ?
  - String tension, chiral symmetry breaking ?
  - Phase transitions ?

Setting the vev



Classical vev stable $<\mathcal{U}_a \mathcal{U}_a> = I$ for $\mu > \mu_*$ with μ_* decreasing for increasing L

$SU(2)$ Flat directions - I

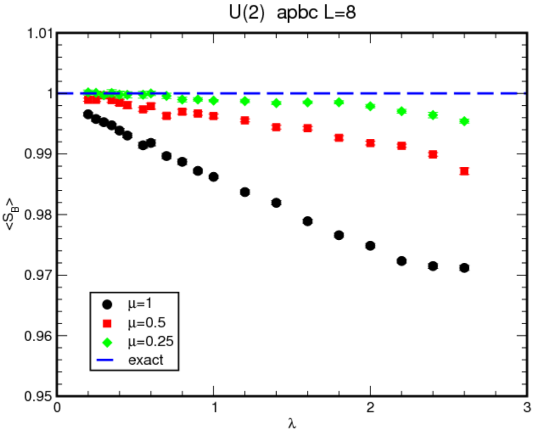


(Localized) distribution of scalars insensitive to μ

Comments

- Common statement: "Moduli space is not lifted in $\mathcal{N}=4$ by quantum corrections ..."
  Why is scalar distribution not flat as $\mu \to 0$?
- Pfaffian vanishes on flat directions since we integrate fermion zero modes. Formally this zero cancels against infinity from boson zero modes but latter are lifted at non-zero μ.
- Thus configurations corresponding to flat directions make no contribution to lattice path integral.
- Small fluctuations around flat directions cost increasing action as move away from origin in field space - large scalar eigenvalues suppressed.

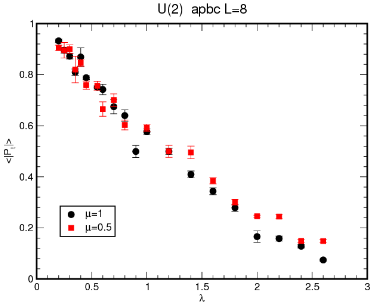
Test of exact supersymmetry

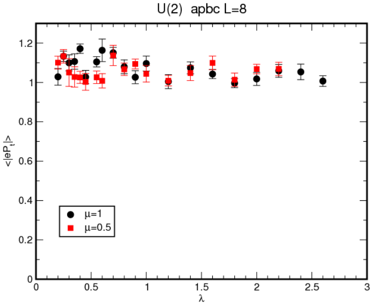


For $\mu \to 0$ S_B given by simple $\mathcal{Q}$ Ward identity.

Phase structure - Polyakov lines

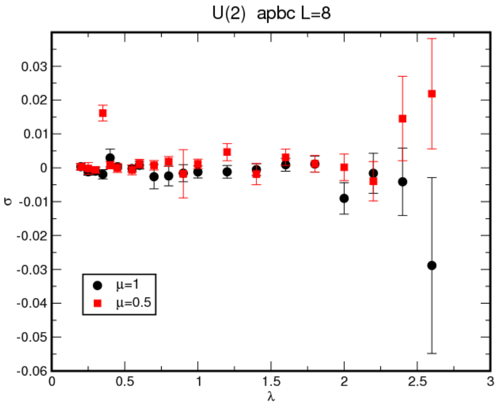
$$P = \prod \mathcal{U}_t \qquad P = \prod \exp\left(\mathcal{U}_t - \mathrm{Tr}\,\mathcal{U}_t\right)$$





Traceless modes yield λ-indep Polyakov line
Consistent with SUSY: $\mathcal{Q}\overline{\mathcal{U}} = 0$

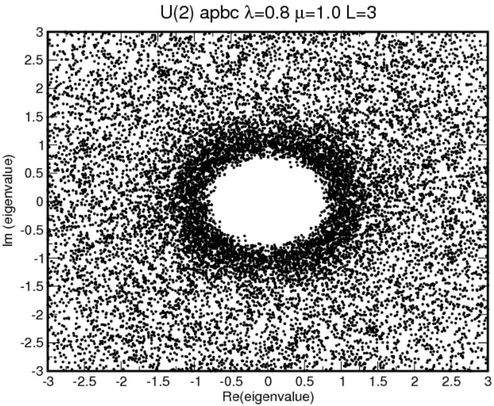
Phase structure - String tension



Extract by fitting $W(R, T)$ to $e^{-V(R)T}$ and extract σ

Vanishing for all λ

Chiral symmetry breaking - or lack of it ..



Eigenvalues excluded from origin: insensitive to μ and λ

Summary

- Simulations of $\mathcal{N}=4$ YM look promising: gauge invariance and (some) SUSY can be preserved. No instabilities from flat directions, no sign problem.
- Prelim investigations show no sign of any phase transitions as vary λ. String tension small and static quark potential best fit with simple Coulomb term. No chiral symmetry breaking. Evidence for single, deconfined phase. VERY DIFFERENT FROM TYPICAL LGT!
- Consistent with pert theory: 1 loop calc shows $\beta_{\rm latt}(\lambda)=0$
- Has already been used to test holographic dualities for dimensionally reduced $\mathcal{N}=4$ YM and black p-branes in string theory (with Toby Wiseman)

Things to do ...

- Establish phase structure definitively ... using large lattices (better quark potentials), push to stronger λ, smaller μ.
- Examine the spectrum: evidence of SUSY, compute anomalous dims. Compare to known results in $\mathcal{N} = 4$ (eg Konishi/supergravity multiplet), Maldecena loop.
- Check restoration of full SUSY: study broken SUSY Ward identities, determine how much tuning needed.
- Need to understand how to take continuum limit; in QCD send $\beta \to \infty$ with ever increasing L. In CFT g is parameter - does *not* determine lattice spacing. Continuum physics by increasing L. But how to tune $\mu(g, L)$?

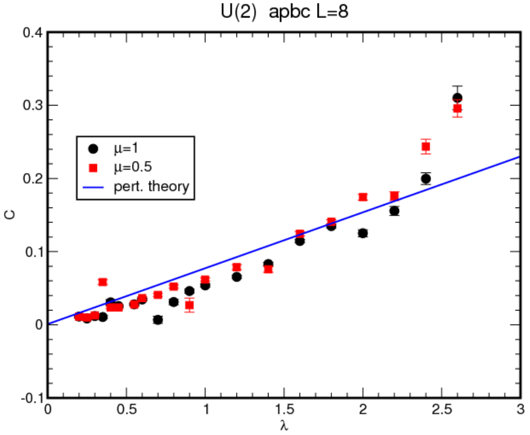
Exciting time - lots to do !!

The end

Sketch of $\Gamma_{\rm eff}=0$

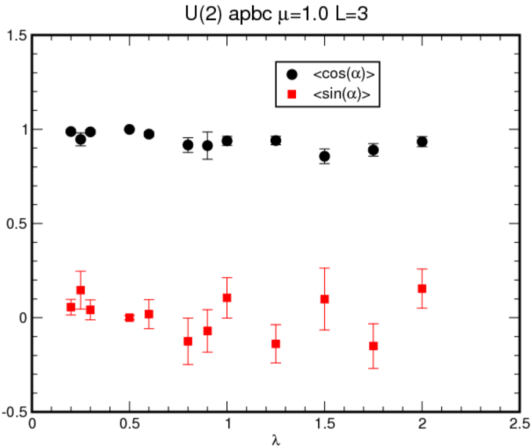
- Classical vacua $\boxed{\text{constant commuting complex matrices } \mathcal{U}_\mu}$
- Expand to quadratic order about generic vacuum $\mathcal{U}_b(x)=I+\mathcal{A}_b^c+a_b(x)$. Integrate
- Bosons $\det^{-5}\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{D}_a^{(+)}\right)$
- Ghosts+Fermions: $\det\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{D}_a^{(+)}\right)+\left(Pf\left(M_F\right)\stackrel{Maple}{=}\det^4\left(\overline{\mathcal{D}}_a^{(-)}\mathcal{D}_a^{(+)}\right)\right)$
- Thus $Z_{\rm pbc}=1$ at 1-loop. $\mathcal{Q}$-exact structure – result good to all orders! $\boxed{\text{Exact quantum moduli space}}$
- Witten index: all states cancel except vacua. Counting indep of g.

Coulomb fits

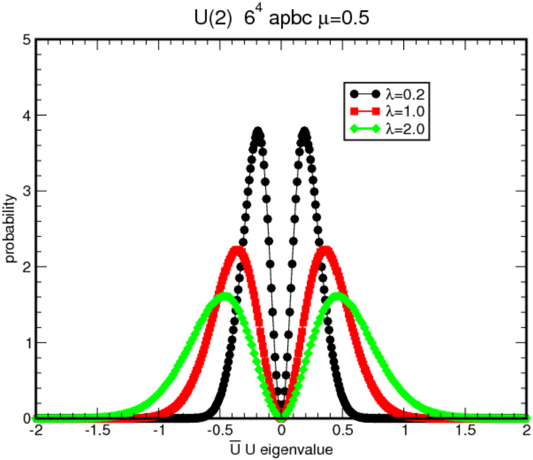


Fits are good and consistent with p theory .. no sign of Maldacena $\sqrt{\lambda}$ behavior

Pfaffian phase



$SU(2)$ Flat directions - II



Localized distribution for all λ