

Comparing Resummations

Workshop on Resummation and Parton Showers
IPPP Durham, July 16, 2013

Based on ongoing work with Mao Zeng (to appear) with a nod to
E. Laenen, GS, W. Vogelsang, Phys. Rev. D63 114018 (2001)
and even earlier work.

- A. Threshold resummation
- B. Unity and difference: “D(irect)QCD” & “S(C)ET” evolution-based threshold resummations
- C. Diagrammatic resummation for electroweak annihilation and power corrections
- D. DQCD and SCET for hadrons

Some relatively simple observations that hopefully won't get lost in the formulas.

A. Threshold Resummation

- Electroweak boson production ($E = Z, W, H$) mass M ;

$$A + B \rightarrow E(M) + X$$

$$\tau \equiv \frac{M^2}{S} \quad \text{true threshold : } S \rightarrow M^2 \leftrightarrow \tau \rightarrow 1$$

- Collinear factorization:

$$\begin{aligned} \frac{d\sigma_{AB \rightarrow M}(S, M^2)}{dM^2} &= \sum_{\text{partons } a,b} \frac{\Sigma_{ab}}{S Q^2} \int_0^1 \frac{dx_a}{x_a} \frac{dx_b}{x_b} f_{a/A}(x_a, \mu_f) f_{b/B}(x_b, \mu_f) \\ &\quad \times C_{ab \rightarrow E}(M^2/\hat{s}, M^2/\mu_f^2, \alpha_s(\mu_f)) \\ &= \sum_{\text{partons } a,b} \frac{\Sigma_{ab}}{S Q^2} \int_\tau^1 \frac{dz}{z} \mathcal{L}_{ab}\left(\frac{\tau}{z}, \mu_f\right) C_{ab \rightarrow E}(z, M^2/\mu_f^2, \alpha_s(\mu_f)) \end{aligned}$$

- Partonic luminosity:

$$\mathcal{L}_{ab}\left(\frac{\tau}{z}, \mu_f\right) = \int_0^1 \frac{dx_a}{x_a} \frac{dx_b}{x_b} f_{a/A}(x_a, \mu_f) f_{b/B}(x_b, \mu_f) \delta\left(1 - \frac{\tau/z}{x_a x_b}\right).$$

in terms of

$$z \equiv \frac{\tau}{x_a x_b} \quad \text{partonic threshold : } x_a x_b S \rightarrow M^2 \leftrightarrow z \rightarrow 1$$

- Partonic hard-scattering functions are generally singular at partonic threshold for $b = \bar{a}$: (purely gluonic radiation)

$$C_{a\bar{a}}^{(n)} \sim \alpha_s^n(\mu_f) \left[\frac{\ln^{2n-1}(1-z)}{1-z} \right]_+, \quad \alpha_s^n(\mu_f) \left[\frac{\ln^{2n-2}(1-z)}{1-z} \right]_+, \\ \dots \quad \alpha_s^n(\mu_f) \delta(1-z)$$

- In hadronic scattering these are enhancements, due to incomplete cancellation between virtual and radiated gluons in the cross section *and* in subtraction of soft/collinear gluons.
- (Nearly) two logs per loop: collinear and soft divergences
- Threshold resummation organizes singular distributions for $z \rightarrow 1$. Clearly a good idea.
- The solution turns out to be simple, and based on factorization:

$$C_{ab \rightarrow M}(z, M/\mu_f, \alpha_s(\mu_f)) = \delta_{b\bar{a}} H_{a\bar{a}}(M, \mu_f, \alpha_s(\mu_f)) \times S_{a\bar{a}}(M(1-z)/\mu_f, \alpha_s(\mu_f)) \\ + \mathcal{O}(1-z)$$

- $S(M(1-z))$ is the “soft function”: common to DQCD and SCET analyses

- **Kinematics shows why they're the so similar. For $z \rightarrow 1$, let k_0 be total c.m. energy of radiation, then**

$$1 - z = \frac{2k_0}{M}$$

- **For $z \rightarrow 1$ all radiation is in the soft sector, the “S” of SCET.**
The collinear Lagrangian doesn't play as important role in the z -dependence.
- **The Soft function is a DQCD collinear-subtracted \leftrightarrow SCET renormalized matrix element.**
(GS Nucl.Phys. B281 (1987) G. Marchesini, G. Korchemsky Nucl.Phys. B406 (1993) 225, A. Belitsky Phys.Lett. B442 (1998) 307, T. Becher, M. Neubert, G. Xu JHEP 0807 (2008) 030)

$$S_{a\bar{a}}(2k_0/\mu_f) = M \int \frac{dx^0}{4\pi} e^{ik_0x^0} W_{\text{DY}}(x^0\mu_f, \vec{x} = 0)$$

$$W_{a\bar{a}}(x\mu_f) = \frac{1}{N_c} \langle 0 | \text{Tr} \bar{\mathbf{T}}[\Phi_n^\dagger(x)\Phi_{\bar{n}}(x)] \mathbf{T}[\Phi_{\bar{n}}^\dagger(0)\Phi_n(0)] | 0 \rangle$$

in terms of annihilating, lightlike Wilson lines (in a, \bar{a} color reps):

$$\Phi_n(x) = \text{P exp} \left(ig \int_{-\infty}^0 ds n \cdot A(x + sn) \right)$$

- **For direct photons, similar considerations apply**
(T. Becher & M. Schwartz JHEP 1002 (2010) 040)

- At $z = 1$ infrared divergences are not cancelled.

- Singular $z \rightarrow 1$ behavior is organized by transforms

- Mellin, or Laplace

$$\begin{aligned}\tilde{S}_{a\bar{a}}\left(\ln\frac{M^2}{\tilde{N}^2\mu^2},\alpha_s(\mu)\right) &= \int_0^1 dz z^{N-1} S_{a\bar{a}}(M(1-z),\mu) \\ &= \int_0^\infty d\omega \exp\left(-\frac{N\omega}{M}\right) S_{a\bar{a}}(\omega,\mu) + \mathcal{O}(1/N)\end{aligned}$$

- Correspondence:

$$\left[\frac{\ln^k(1-z)}{1-z}\right]_+ \leftrightarrow \ln^{k+1} N$$

- The problem is posed in the same way in DQCD, SCET.

- They both start with a common, evolution-based resummation, whose basis is in the factorization of soft gluon radiation.

B. Unity and Difference: DQCD & SCET Evolutions

- The N dependence of $\tilde{S}_{a\bar{a}} \left(\ln \frac{M^2}{\bar{N}^2 \mu^2}, \alpha_s(\mu_f) \right)$ is determined by its μ_f dependence:

$$\frac{d}{d \ln \mu} \ln \tilde{S}_{a\bar{a}} \left(\ln \frac{M^2}{\bar{N}^2 \mu^2}, \mu \right) = -2\Gamma_{\text{cusp}}(\alpha) \ln \frac{M^2}{\mu^2 \bar{N}^2} - \Gamma_{\text{DY}}(\alpha)$$

- In terms of the “cusp anomalous dimension”, Γ_{cusp} , and $\Gamma_{\text{DY}} \leftrightarrow \gamma_W$

(A common notation $\bar{N} = N e^{\gamma_E}$ simplifies notation at NLL)

- An equivalent momentum space evolution equation for the log is
(GS, 1987, Belitsky 2000, Becher, Neubert 2006)

$$\begin{aligned} \frac{d}{d \ln \mu} \ln \tilde{S}_{a\bar{a}}(M(1-z)/\mu, \alpha(\mu)) &= \left[-4\Gamma_{\text{cusp}}(\alpha(\mu)) \ln \frac{M}{\mu} - 2\Gamma_{\text{DY}}(\alpha(\mu)) \right] \delta(1-z) \\ &\quad - 4\Gamma_{\text{cusp}}(\alpha(\mu)) \left(\frac{1}{1-z} \right)_+ \end{aligned}$$

The (simpler) moment-space solution:

$$\tilde{S} \left(\frac{M}{\bar{N}\mu_1}, \alpha_s(\mu_1) \right) = \tilde{S} \left(\frac{M}{\bar{N}\mu_2}, \alpha_s(\mu_2) \right) \exp \left\{ \int_{\mu_2}^{\mu_1} \frac{d\rho}{\rho} \left(2\Gamma_{\text{cusp}}(\alpha_s(\rho)) \ln \left(\frac{\rho N}{M} \right) + \Gamma_{\text{DY}}(\alpha_s(\rho)) \right) \right\}$$

- It is at this stage that SCET and QCD methods part ways, in their choices of the lower scale, μ_1 , and then in their use of the inverse Laplace or Mellin transform to derive a physical cross section.

- **DQCD: run up from $\mu_2 = M/\bar{N}$ to $\mu_2 = \mu_f \sim M$, then absorb remaining N -dependence from the lower scale into the exponent**

$$\begin{aligned} \tilde{S} \left(\frac{M}{N\mu}, \alpha_s(\mu) \right) &= S \left(1, \alpha_s(M/\tilde{N}) \right) \exp \left\{ \int_{M/\tilde{N}}^{\mu} \frac{d\rho}{\rho} \left(2\Gamma_{\text{cusp}} \ln \left(\frac{\rho\tilde{N}}{M} \right) + \Gamma_{\text{DY}} (\alpha_s(\rho)) \right) \right\} \\ &= \tilde{S} \left(1, \alpha_s(\mu) \right) \exp \left\{ \int_{M/\tilde{N}}^{\mu} \frac{d\rho}{\rho} \left(2\Gamma_{\text{cusp}}(\alpha_s(\rho)) \ln \left(\frac{\rho\tilde{N}}{M} \right) + \hat{D} (\alpha_s(\rho)) \right) \right\} \end{aligned}$$

- Here, define \hat{D} by

$$\hat{D} (\alpha_s(\rho)) = \Gamma_{\text{DY}} (\alpha_s(\rho)) - \rho \frac{\partial}{\partial \rho} \ln S(1, \alpha_s(\rho))$$

- The function \hat{D} is thus a hybrid function, the sum of an anomalous dimension and a perturbative coefficient.

- A conventional expression is

$$\tilde{S}\left(\frac{M}{N\mu}, \alpha_s(\mu)\right) = \tilde{S}(1, \alpha_s(M)) \exp\left\{\int_{1/\bar{N}}^1 \frac{dy}{y} \left[2 \int_{yM}^M \frac{dq}{q} \Gamma_{\text{cusp}}(\alpha_s(q)) + \hat{D}(\alpha_s(yM)) \right]\right\}$$

- There is a (Landau) singularity for large $N \sim M/\Lambda_{\text{QCD}}$. This can be avoided at the cost of making a choice at the level of power corrections in Λ_{QCD}/M .
- The “minimal” (CMNT) prescription (Catani, Mangano, Nason, Tretadue Nucl.Phys. B478 (1996) 273-310) can be used to avoid the singularity in computing hadronic cross sections and designed specifically to avoid introducing nonperturbative parameters or explicit power corrections. Other prescriptions are available: ‘Borel’ (Forte and Ridolfi) and ‘principle value’ (Contopanagos, GS, Contopanagos, Berger)

- For the SCET, evolve from μ_f down to a fixed “soft” scale, μ_s

$$\tilde{S}\left(\frac{M}{\tilde{N}\mu_f}, \alpha_s(\mu_f)\right) = \tilde{S}\left(\frac{M}{\tilde{N}\mu_s}, \alpha_s(\mu_s)\right) \exp\left\{\int_{\mu_f}^{\mu_s} \frac{d\rho}{\rho} \left(2\Gamma_{\text{cusp}} \ln\left(\frac{\rho\tilde{N}}{M}\right) + \Gamma_{\text{DY}}(\alpha_s(\rho))\right)\right\}.$$

- N -dependence is in the prefactor, the soft function evaluated at scale μ_s and the explicit factor of $\ln N$ that multiplies the cusp anomalous dimension in the exponent.
- Reorganize by separating all the $\ln N$ dependence in the exponent as a multiplicative factor, using it as a generating functional for N -dependence in the soft function at μ_s .
- Gives:

$$\begin{aligned} \tilde{S}_{a\bar{a}}\left(\ln\frac{M^2}{\tilde{N}^2\mu^2}, \mu\right) &= \exp\left[-4\mathcal{S}(\mu_s, \mu) + 2\alpha_{\gamma_{\text{DY}}}(\mu_s, \mu) + \eta \ln\frac{M^2}{\mu_s^2}\right] \\ &\times \tilde{S}_{a\bar{a}}\left(\ln\frac{M^2}{\mu_s^2} + \frac{\partial}{\partial\eta}, \mu_s\right) \exp\left[-\eta \ln\tilde{N}^2\right] \end{aligned}$$

with

$$\begin{aligned} \mathcal{S}(\mu_s, \mu) &= -\int_{\mu_s}^{\mu} d\ln\mu' \Gamma_{\text{cusp}}(\alpha_s(\mu')) \ln\frac{\mu'}{\mu_s} \\ \alpha_{\gamma_{\text{DY}}}(\mu_s, \mu) &= -\int_{\mu_s}^{\mu} d\ln\mu' \Gamma_{\text{DY}}(\alpha_s(\mu')) \end{aligned}$$

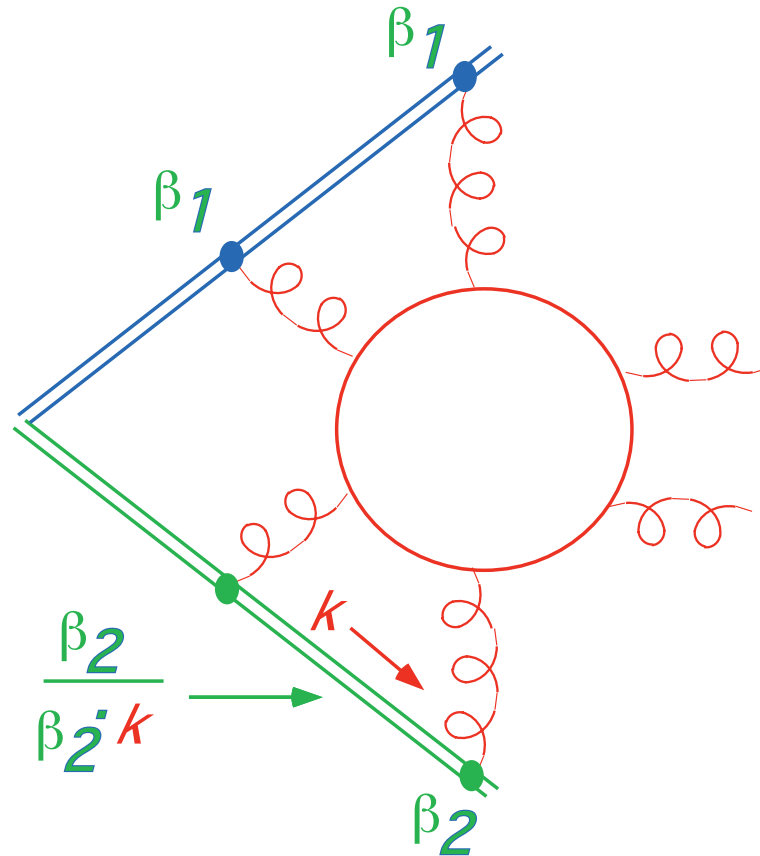
and

$$\eta = 2 \int_{\mu_s}^{\mu} d\ln\mu' \Gamma_{\text{cusp}}(\alpha)$$

- In EFT, the soft scale μ_s is to be chosen to stabilize the cross section when $S_{a\bar{a}}$ is known to low order. The assumption is that the range of z for which $(1 - z)M < \mu_s$ is not phenomenologically important, even though it is full of logs of $1 - z$. **This is termed “dynamical scale generation”**
- Not too surprisingly (given their origin) the two formalisms agree to non-leading logarithms in N at fixed N (Bonvini, Forte, Marzani, Ridolfi 1303.3590 & Nucl.Phys. B861 (2012) 337-360).
- The EFT resummation (Becher, Neubert, Schwartz, Stewart ... 2006 - ...) is even more insensitive to the Landau pole than the minimal prescription in its application to hadronic cross sections, but at the price of abandoning non leading logarithmic dependence
- In a sense, the DQCD minimal prescription tries to “have it both ways”, all orders in N in moment space, and no new nonperturbative parameters. But this is a particular choice of nonperturbative parameters.
- We’ll return to their comparison below, but first have a look at the nonperturbative limit both formalisms are designed to avoid.

C. Diagrammatic resummation and power corrections

- For products of Wilson lines, exponentiation is exact in terms of “webs” (Gardi, Smille, White, Laenen 2009 ... 2013)
- Look for further insight into the “independent emission” – exponentiation – by analogy to soft photons.
- A typical diagram (for final-state sources)



- **Webs and exponentiation for soft contributions to weighted ($e = Q_T$, thrust, and k_0) cross sections** (GS (1981); Gatheral & Frenkel and Taylor, (1981), Mitov, GS, Sung (2009)). **A convolution,**

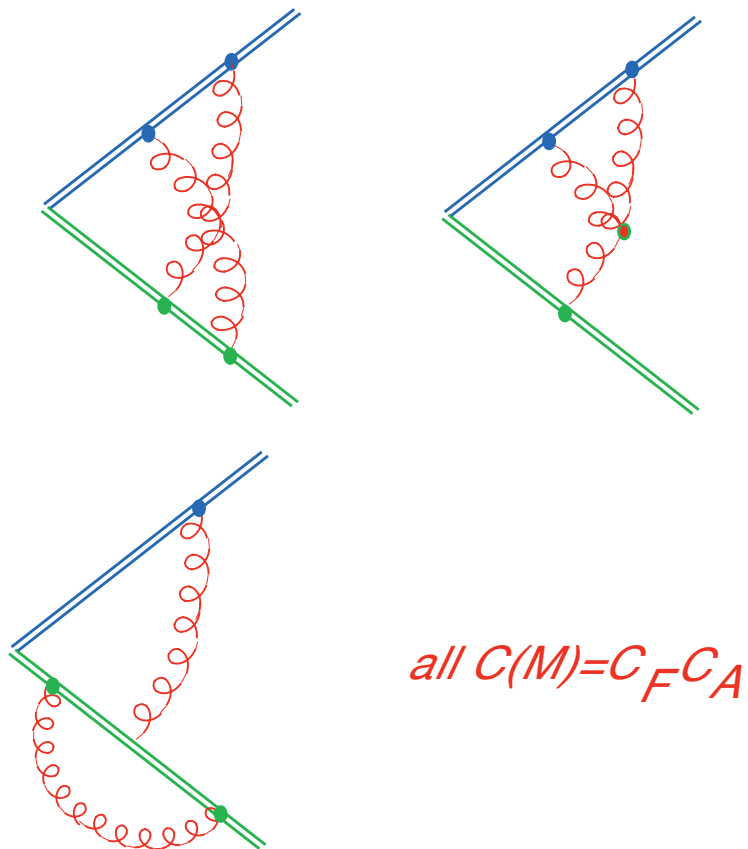
$$\frac{d\sigma}{de_a} = \sum_{n=0}^{\infty} \frac{1}{n!} \int de \delta(e - e_s) \otimes_{\sum e_i = e_a} \prod_{i=1}^n E(e_i)$$

- **With a kernel that becomes the exponent in transform space**

$$E(e) = \sum_{\text{states } n} \mathcal{W}_n(e) = \sum_{\mathcal{M}} C(\mathcal{M}_n) \mathcal{M}_n^2(e)$$

- **The \mathcal{M}_n^2 are the corresponding momentum integrals for the web diagrams.**

- The $C(\mathcal{M}_n)$ are modified color factors for \mathcal{M}_n s.
Examples at α_s^2 :



- Notice that non-planar diagrams contribute in $N_c \rightarrow \infty$ limit

- The webs determine directly determine the exponent under transforms:

$$\tilde{S}_e(N) \equiv \int de e^{-Ne} \frac{dS}{de} = \exp \left[\int de' e^{-Ne'} E(e') \right]$$

- Double logarithmic behavior is encoded in the construction of the webs \mathcal{W} . Subdivergences cancel.
- Each web gives a **single collinear and infrared logarithm up to running coupling effects, just like a single gluon.**
- In a theory with a fixed coupling (SYM ...) a web acts exactly like a single gluon.
- The 2-loop structure of Γ_S is an intriguing suggestion that “web=NP gluon” could generalize to arbitrary hard processes.

- For electroweak annihilation as above, this gives a very specific template for the all-orders form, up to exponentially-suppressed corrections in N :

- Boost invariance along dipole axis \Rightarrow

(Laenen, GS, Vogelsang, 2001)

$$\begin{aligned} \ln C(N, M) &= \sum_N \int d\text{PS}_N \theta(M^2 - k^2) |M^{(\text{eik})}|^2 e^{-N(2k_0/M)} \\ &= \int_0^{M^2} \frac{\rho(\alpha_s(u, \varepsilon))}{u^2} \left[K_0\left(\frac{2Nu}{M}\right) + \ln \frac{u}{M} \right] + \ln \bar{N} \int_0^{M^2} \frac{du^2}{u^2} A(\alpha_s(u, \varepsilon)) \end{aligned}$$

- The “ α_s ” for a single gluon is replaced by:

$$\rho(\alpha_s(u, \varepsilon)) = \Gamma_{\text{cusp}}(\alpha_s(u, \varepsilon)) + \frac{\partial D'}{\partial \ln \mu^2}$$

- $\Gamma_{\text{cusp}} = “A”$ is the same “cusp” anomalous dimension, and $D'(\alpha_s)$ is analogous to the D-term as above.
- The structure of “renormalon” power corrections is given by the low- x expansion of $K_0(x) - \ln(x)$
- Only even powers of $\frac{N}{M}$ occur (as noted for “large- N_f ” by Beneke and Braun (1995))
- This phenomenology has yet to be developed for hadron-hadron scattering, is (mercifully) irrelevant to inclusive Higgs etc. at the LHC, but for large- N_c , all radiation is from dipoles.

D. Cross sections for hadrons

- Does a quantifiable correspondence in moment space breaks down for hadronic cross sections.? (Bonvini, Forte, Ghezzi, Ridolfi, 2012,13) I'll argue here it's "natural" for the two evolution-based resummations to return very similar phenomenological results, at least when the corrections are significant.
- The differing expressions for the cross section
- **DQCD as an inverse transform, to be performed numerically in general**

$$\begin{aligned}
 \frac{d\sigma_{AB \rightarrow M}(S, M^2)}{dM^2} &= \sum_{\text{partons } a,b} \frac{\Sigma_{ab}}{S Q^2} \int_0^1 \frac{dx_a}{x_a} \frac{dx_b}{x_b} f_{a/A}(x_1, \mu_f) f_{b/B}(x_2, \mu_f) \\
 &\quad \times C_{ab \rightarrow E}\left(M^2/\hat{s}, M^2/\mu_f^2, \alpha_s(\mu_f)\right) \\
 &= \sum_{\text{partons } a,b} \frac{\Sigma_{ab}}{S Q^2} \int_{\tau}^1 \frac{dz}{z} \mathcal{L}_{ab}\left(\frac{\tau}{z}, \mu_f\right) C_{ab \rightarrow E}\left(z, M^2/\mu_f^2, \alpha_s(\mu_f)\right)
 \end{aligned}$$

- which can be written as

$$\begin{aligned}
 \frac{d\sigma_{AB \rightarrow E}(\tau, M^2)}{dM^2} &= \sum_a H_{a\bar{a}}(M, \mu_f) \int_{n_0-i\infty}^{n_0+i\infty} \frac{dN}{2\pi i} \tau^{-N} \mathcal{L}_{a\bar{a}}(N, \mu_f) \\
 &\quad \times \exp \left\{ \int_{1/\bar{N}}^1 \frac{dy}{y} \left[2 \int_{yQ}^Q \frac{dq}{q} \Gamma_{\text{cusp}}(\alpha_s(q)) + \hat{D}(\alpha_s(yQ)) \right] \right\}
 \end{aligned}$$

- **SCET: The inverse transform of the explicit N -dependence can be done exactly,**

$$S_{a\bar{a}} \left(\ln \frac{M^2(1-z)^2}{\mu^2}, \alpha_s(\mu) \right) = \exp \left[-4S(\mu_s, \mu) + 2\alpha_{\gamma_{\text{DY}}}(\mu_s, \mu) + \eta \ln \frac{M^2}{\mu_s^2} \right] \\ \times \tilde{S}_{\text{DY}} \left(\ln \frac{M^2}{\mu_s^2} + \frac{\partial}{\partial \eta}, \mu_s \right) (1-z)^{2\eta-1} \frac{1}{\Gamma(2\eta)}$$

- **So that the cross section is given directly as**

$$\frac{d\sigma_{AB \rightarrow M}(S, M^2)}{dM^2} = \int_{\tau}^1 \frac{dz}{z} \mathcal{L}_{ab} \left(\frac{\tau}{z}, \mu_f \right) \\ \times H_{a\bar{a}}(M/\mu, \alpha_s(\mu_f)) S_{a\bar{a}} \left(\ln \frac{M^2(1-z)^2}{\mu^2}, \alpha_s(\mu) \right)$$

- **Which is also adaptable to differential cross sections.**

- **The ‘problem’ in short:**
- In DQCD all N -dependence is kept, even though the inverse transform is then influenced somewhat by a branch cut at large, real N .
- In SCET, higher-order N dependence is sacrificed to get a well-defined transform.
- **Why/when should they return the comparable answers?**

- If the parton luminosity were exactly power-behaved (“single-power approximation”)

$$\begin{aligned}\mathcal{L}_{a\bar{a}}\left(\frac{\tau}{z}\right) &= \text{const} \left[\frac{\tau}{z}\right]^{-s(\tau)} \\ &= \mathcal{L}(\tau) z^{s(\tau)}\end{aligned}$$

the convolution would itself be a moment of the partonic cross section:

$$\begin{aligned}\frac{d\sigma(\tau)}{dM^2} &= \sigma_0 \int_{\tau}^1 \frac{dz}{z} \mathcal{L}\left(\frac{\tau}{z}\right) \omega(z) \\ &= \sigma_0 \mathcal{L}(\tau) \int_{\tau}^1 \frac{dz}{z} z^{s(\tau)} \omega(z) \\ &= \sigma_0 \mathcal{L}(\tau) \tilde{\omega}(s(\tau))\end{aligned}$$

- If this were the case, then when variant resummations for ω agree at the level of moments, they should agree phenomenologically.
- And, perhaps surprisingly, this happens in a wide range of realistic cases.

- The observation: if $\mathcal{L}(\tau/z)$ falls rapidly as a function of z , the single-power approximation will hold for the cross section if $\mathcal{L}(\tau/z)$ becomes negligible before $n \geq 2$ terms in the expansion

$$\ln \mathcal{L} \left(\frac{\tau}{z} \right) = \sum_{n=0} \frac{1}{n!} s_n(\tau) \left(\ln \frac{1}{z} \right)^n$$

become important, with $s_n = d^n \ln \mathcal{L}(\tau') / d \ln^n \tau' |_{\tau'=\tau}$.

- Since the decay distance in $\ln \frac{1}{z}$ is $s_1(\tau)$, we need

$$\frac{1}{n!} \frac{s_n(\tau)}{[s_1(\tau)]^n} \ll 1$$

- For example, model parton distributions like

$$f(x) = C x^{-\gamma} (1-x)^\beta$$

give luminosity functions (Appel, Mackenzie, GS Nucl.Phys. B309 (1988) 259, Becher, Neubert JHEP 0807 (2008) 030)

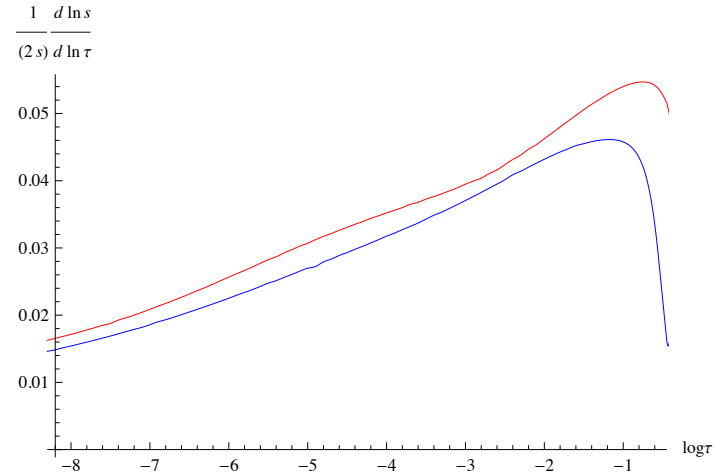
$$\mathcal{L}(\tau') = C^2 B(\beta+1, \beta+1) \tau'^{-\gamma} (1-\tau')^{2\beta+1} F_{2,1}(\beta+1, \beta+1; 2\beta+2, 1-\tau')$$

• For this explicit form one can show: \mathcal{L} is single-power dominated:

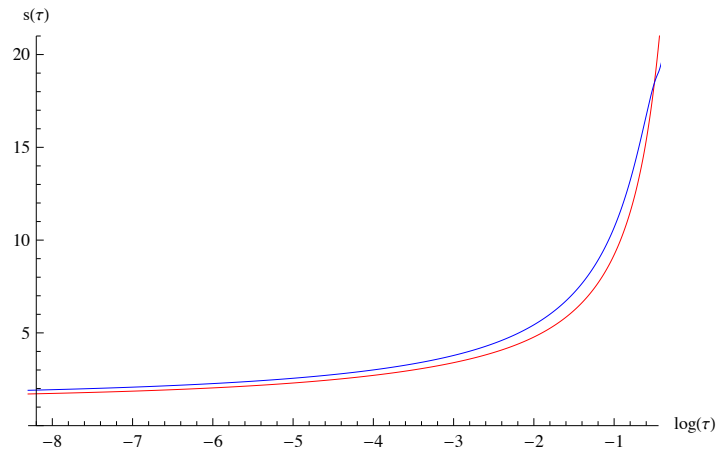
1. for small τ even for moderate $\gamma = \mathcal{O}(1)$

2. for all τ for $\beta \rightarrow \infty$

- And for “real” PDFs (MRST2008 NNLO gluon) at $M=10$ GeV (red) and $M=126$ GeV (blue)



- To be compared to



- the first derivative dominates the second over an adequate region, even when the first derivative is not very large.

- So, for many, probably most cases, the effective power $s_1(\tau)$ determines the dominant moment, and consequently an effective “soft scale”, $\mu_s = M/s_1(\tau)$.
- And we should expect both SCET and DQCD results to be similar, with differences due primarily to “incidental” choices of scales.

Summary

- Commonly-used effective theory and full QCD threshold resummations are based on the same underlying factorization, and differ mainly in how the resulting evolution equations are solved.
- Although both formalisms have been designed to avoid power corrections, such corrections to hard-scattering functions are theoretically accessible in a manner similar to power corrections in event shapes.
- Despite their differences, DQCD and SCET resummations can be expected to give similar phenomenological predictions for a wide range of kinematics and parton distributions.