PHYSICS AT THE TERA SCALE

Helmholtz Alliance

Parton Showers + NLO

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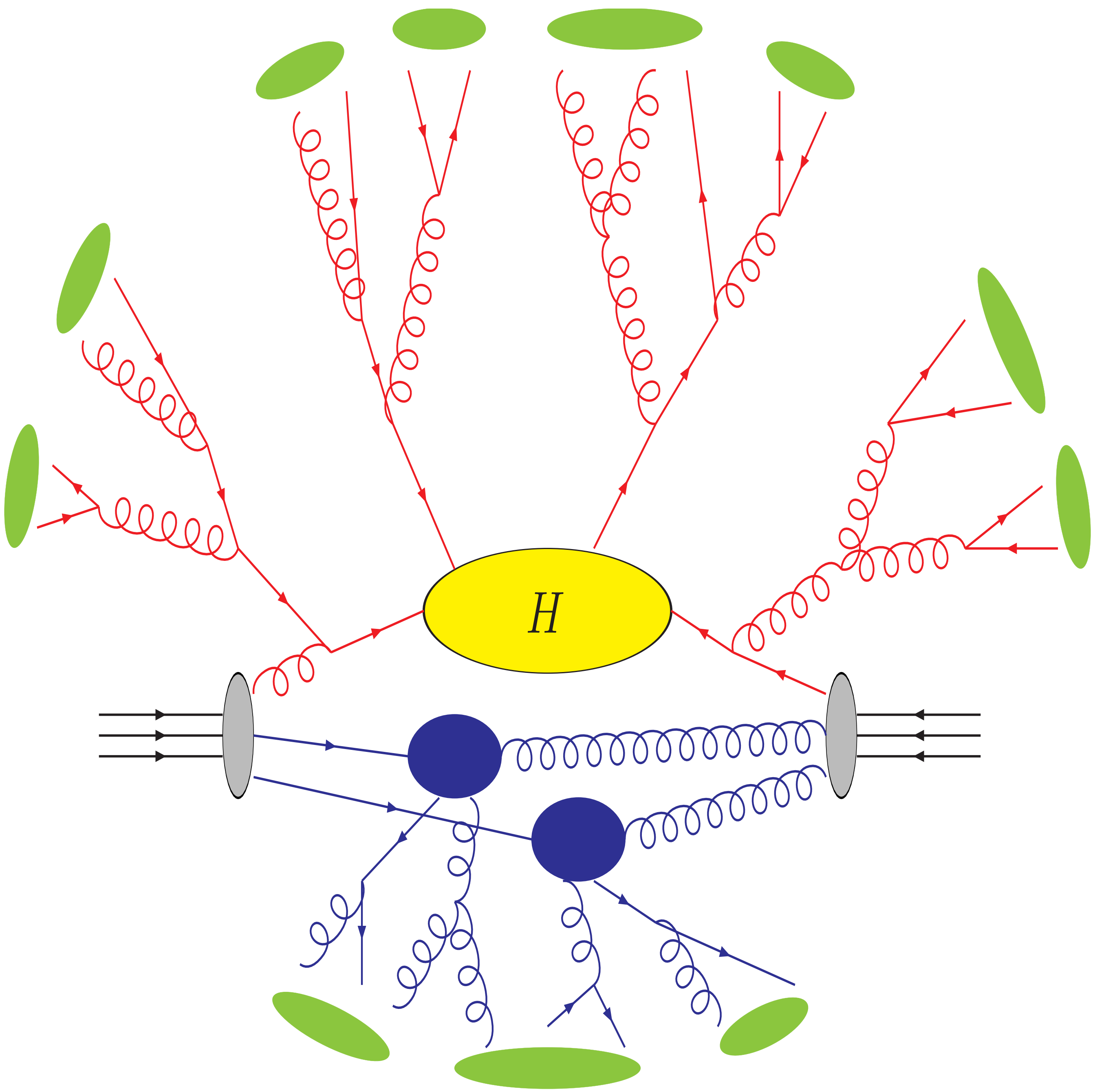
DESY

Many thanks to Dave Soper

Introduction

From theory point of view an event at the LHC looks very complicated



1. Incoming hadron (gray bubbles)
   - Parton distribution function
   - Multi parton distribution functions
2. Hard part of the process (yellow bubble)
   - Matrix element calculation, cross sections at LO, NLO, NNLO level
3. Radiation (red graphs)
   - Parton shower calculation
   - Partonic decay
   - Matching to NLO, NNLO
4. Underlying event (blue graphs)
   - Models based on multiple interaction
   - Diffraction
5. Hardonization (green bubbles)
   - Universal models
   - Hadronic decay
   - ....

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$$\sigma[F] = \sum_m \int [d\{p,f\}_m] \overbrace{f_{a/A}(\eta_\mathrm{a},\mu_F^2)\, f_{b/B}(\eta_\mathrm{b},\mu_F^2)}^{\textit{parton distributions}} \frac{1}{2\eta_\mathrm{a}\eta_\mathrm{b} p_A\cdot p_B}$$

$$\times \big\langle \mathcal{M}(\{p,f\}_m) \big| \underbrace{F(\{p,f\}_m)}_{\textit{observable}} \underbrace{\big| \mathcal{M}(\{p,f\}_m) \big\rangle}_{\textit{matrix element}}$$

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$$\sigma[F] = \sum_m \int \left[d\{p,f\}_m\right] \mathrm{Tr}\{\underbrace{\rho(\{p,f\}_m)}_{\text{density operator in color} \otimes \text{spin space}} F(\{p,f\}_m)\}$$

The fully exclusive final state is described by the QCD density operator, that is the basic object in the Monte Carlos

$$\rho = \sum \rho(\{p,f\}_m) \Leftrightarrow |\rho) = \sum |\rho(\{p,f\}_m))$$

Statistical Space

The density operator is

$$\rho(\{p,f\}_m) = \big|\mathcal{M}(\{p,f\}_m)\big\rangle \frac{f_{a/A}(\eta_\mathrm{a},\mu_F^2)\, f_{b/B}(\eta_\mathrm{b},\mu_F^2)}{2\eta_\mathrm{a}\eta_\mathrm{b}\, p_A\!\cdot\! p_B} \big\langle\mathcal{M}(\{p,f\}_m)\big|$$

$$= \sum_{s,c,s',c'} \big|\{s',c'\}_m\big\rangle \big(\{p,f,s',c',s,c\}_m\big|\rho\big) \big\langle\{s,c\}_m\big|$$

In the statistical space it is represented by a vector

$$\big|\rho\big) = \sum_m \frac{1}{m!} \int \big[d\{p,f,s',c',s,c\}_m\big] \underbrace{\big|\{p,f,s',c',s,c\}_m\big)} \big(\{p,f,s',c',s,c\}_m\big|\rho\big)$$

Basis vector in the statistical space

The probability to have momenta and flavor $\{p,f\}_m$ and be in this color and spin state.

How to Design Parton Showers?

Mandatory design principles

1. Shower generates events and calculates cross sections approximately using the soft and collinear factorization of the QCD amplitudes (tree and 1-loop level).
2. The emissions are strongly ordered.
3. The ordering must control the goodness of the soft and collinear approximations.
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Normalization

5. Shower doesn't change the normalization. This is the unitarity condition.

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Some technical choices

6. Everything that makes the implementation simpler
    - leading color approximation
    - spin averaging
    - angular ordering (loosing full exclusiveness of the event)
    - Catani-Seymour momentum mapping
    - ....

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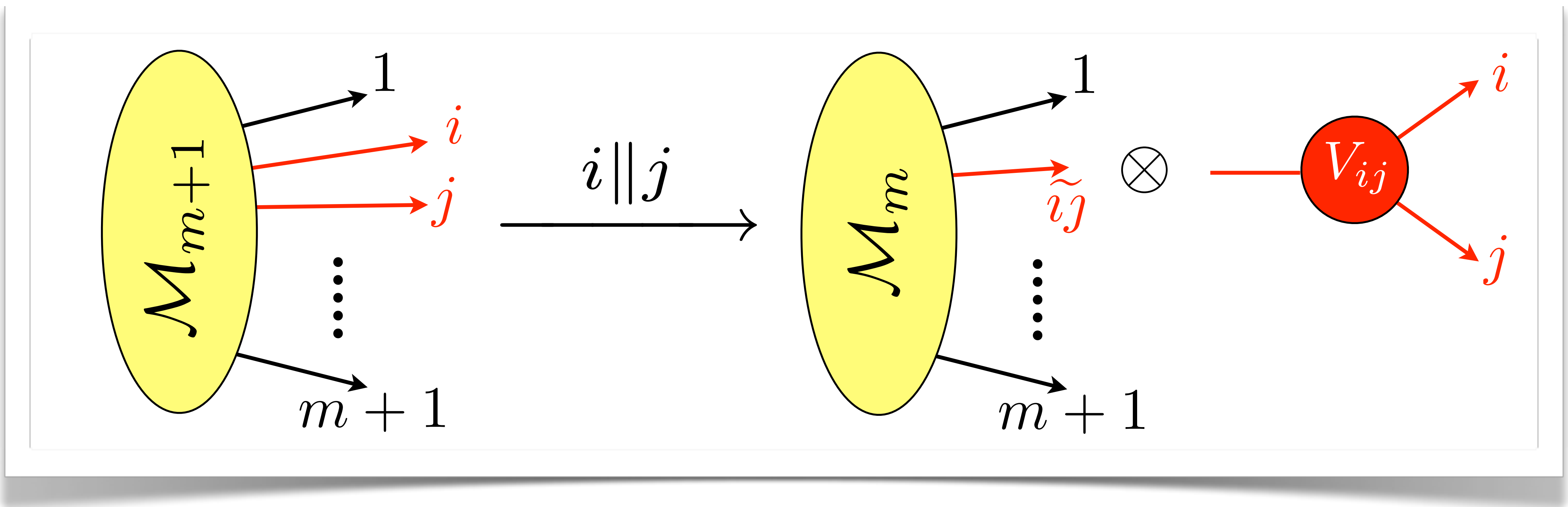
Factorization: Collinear limit

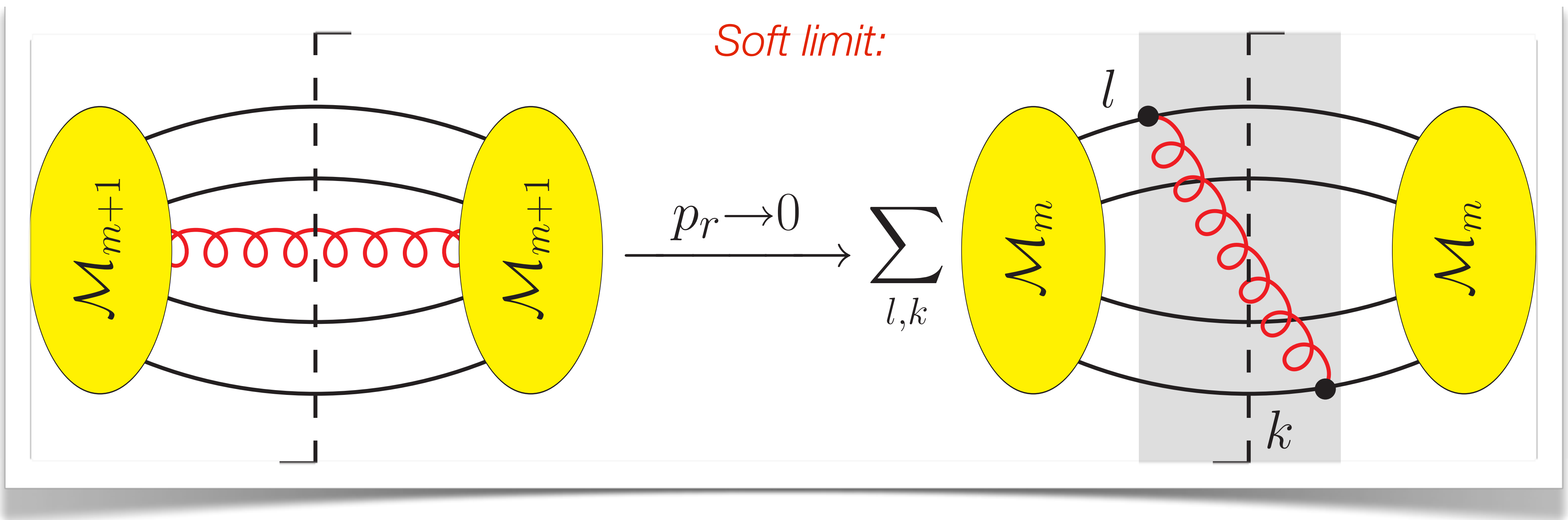
The QCD matrix elements have universal factorization property when two external partons become collinear

Collinear limit:



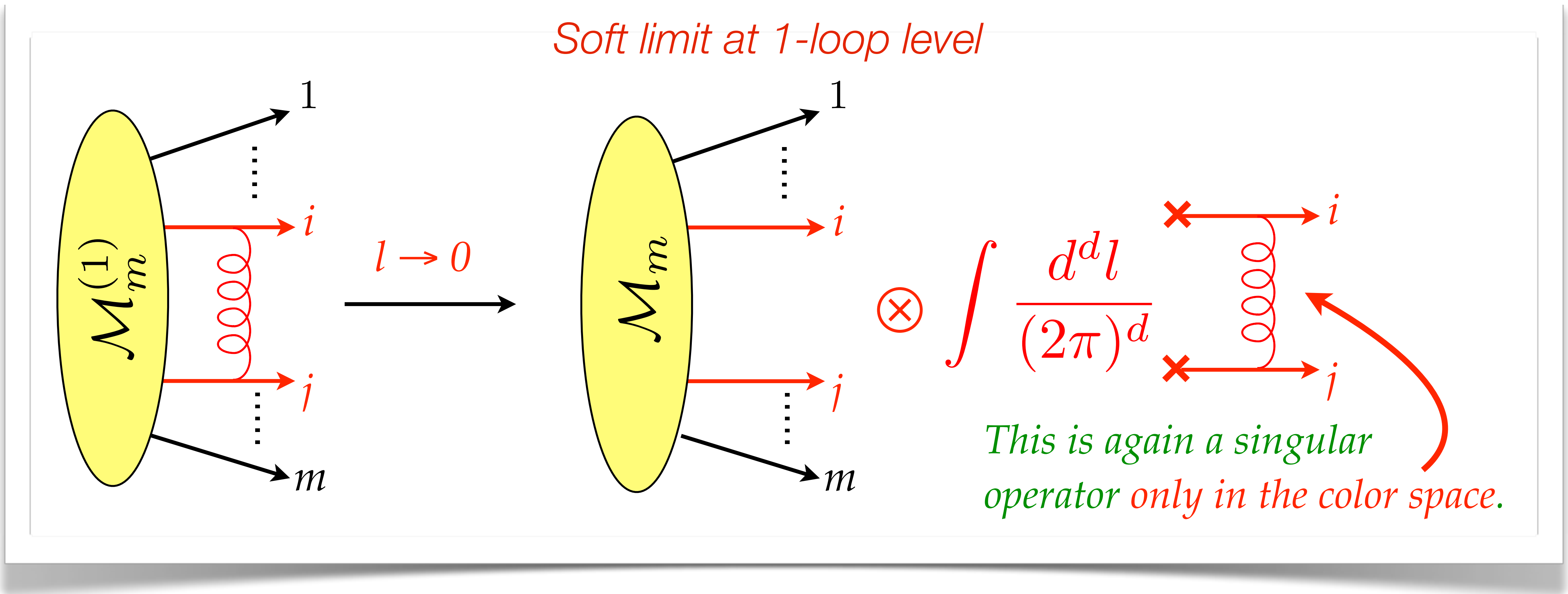
Factorization: Soft limit

The QCD matrix elements have universal factorization property when an external gluon becomes soft

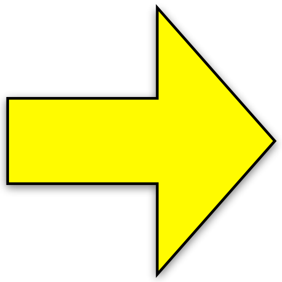


Factorization: Soft limit (1-loop)

There is another type of the unresolvable radiation, *the virtual (loop graph) contributions.* We have *universal factorization properties* for the loop graphs. E.g.: in the soft limit, when the loop momenta become soft we have



$$\mathcal{M}_m^{(1)} \xrightarrow{l \to 0} \mathcal{M}_m \otimes \int \frac{d^d l}{(2\pi)^d}$$



The splitting operators can be obtained from these factorization rules.

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1. *Fixes the general structure of the splitting kernels.*

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Approx. of the Density Operator

Real radiation

$$|\rho^{\mathrm{R}}_{\infty}) \approx \int_t^{\infty} d\tau\, \mathcal{H}_I(\tau) |\rho(t))$$

Virtual radiation

$$|\rho^{\mathrm{V}}_{\infty}) \approx -\int_t^{\infty} d\tau\, \mathcal{V}_I^{(\epsilon)}(\tau)\, |\rho(t))$$

Here we impose strong ordering. Only the softer or more collinear radiation are allowed.

Some of the real emissions are not resolvable. Having a snapshot of the system at shower time t'

$$|\rho^{\mathrm{R}}_{\infty}) \approx \underbrace{\int_t^{t'} d\tau\, \mathcal{H}_I(\tau)\, |\rho(t))}_{\text{Resolved emissions}} + \underbrace{\int_{t'}^{\infty} d\tau\, \mathcal{V}_I^{(\epsilon)}(\tau)\, |\rho(t))}_{\text{Unresolved emissions}}$$

Resolved emissions

Unresolved emissions
This is a singular contribution

Combining the real and virtual contribution we have got

$$|\rho^{\mathrm{R}}_{\infty}) + |\rho^{\mathrm{V}}_{\infty}) = \int_t^{t'} d\tau\, [\mathcal{H}_I(\tau) - \mathcal{V}_I(\tau)] |\rho(t))$$

This operator dresses up the physical state with one real and virtual emissions those *are softer or more collinear than the hard state*. Thus the emissions are ordered.

Shower Operator

Now we can use this to build up physical states by considering all the possible way to go from t to t'.

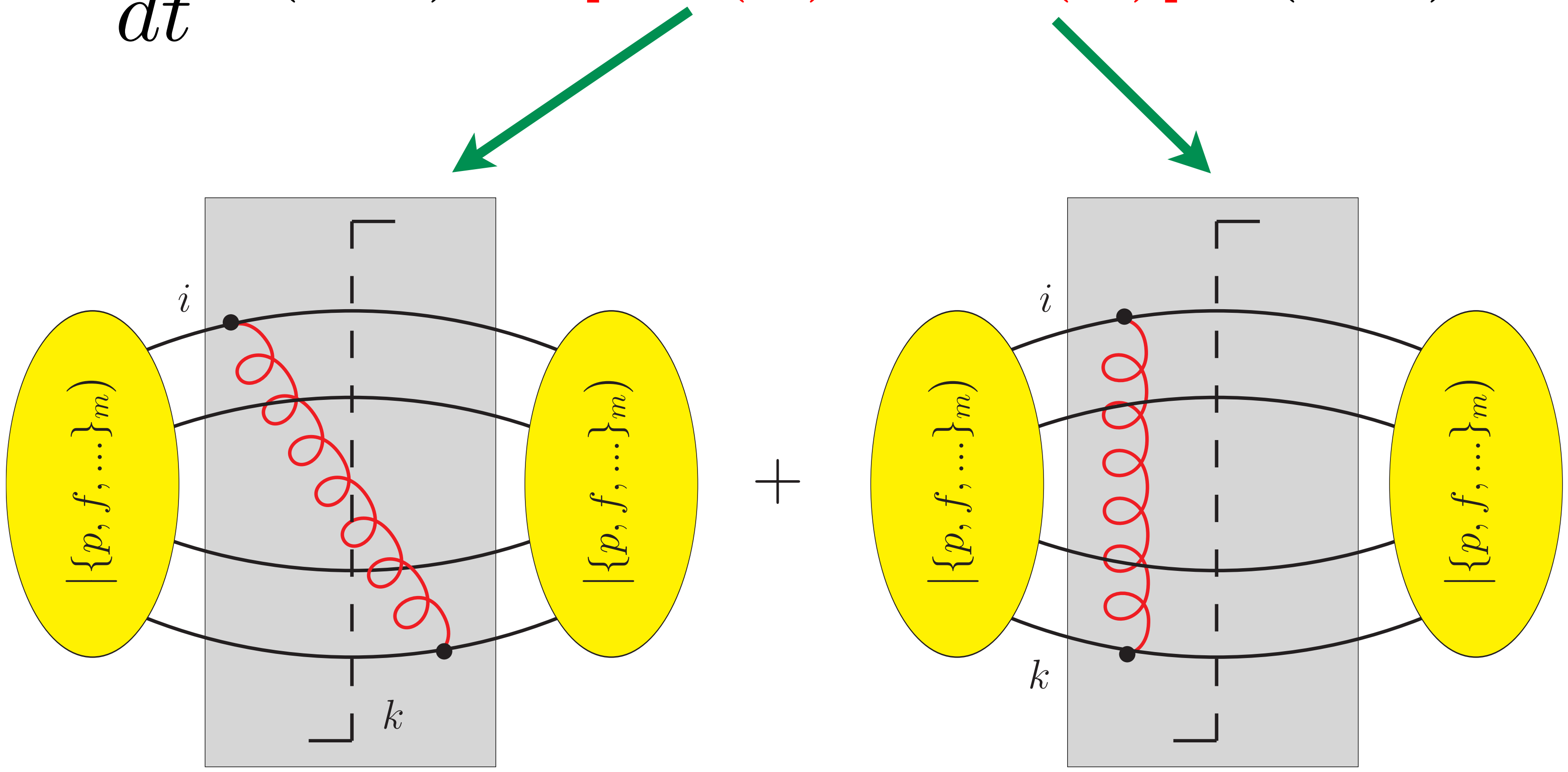
$$\begin{aligned}\big|\rho(t')\big) = \big|\rho(t)\big) &+ \int_t^{t'} d\tau\, [\mathcal{H}_I(\tau) - \mathcal{V}_I(\tau)]\big|\rho(t)\big) \\ &+ \int_t^{t'} d\tau_2\, [\mathcal{H}_I(\tau_2) - \mathcal{V}_I(\tau_2)] \int_t^{\tau_2} d\tau_1\, [\mathcal{H}_I(\tau_1) - \mathcal{V}_I(\tau_1)]\big|\rho(t)\big) \\ &+ \cdots \\ = \underbrace{\mathbb{T}\exp\left\{\int_t^{t'} d\tau\, [\mathcal{H}_I(\tau) - \mathcal{V}_I(\tau)]\right\}}_{\mathcal{U}(t',t)\ \textit{shower evolution operator}} &\big|\rho(t)\big)\end{aligned}$$

$$\big|\rho(t')\big) = \mathcal{U}(t',t)\big|\rho(t)\big)$$

Evolution Equation

The evolution operator obeys the following equation

$$\frac{d}{dt}\mathcal{U}(t',t) = [\mathcal{H}_I(t') - \mathcal{V}_I(t')]\,\mathcal{U}(t',t)$$



resolved radiations *unresolved radiation*

Evolution Equation

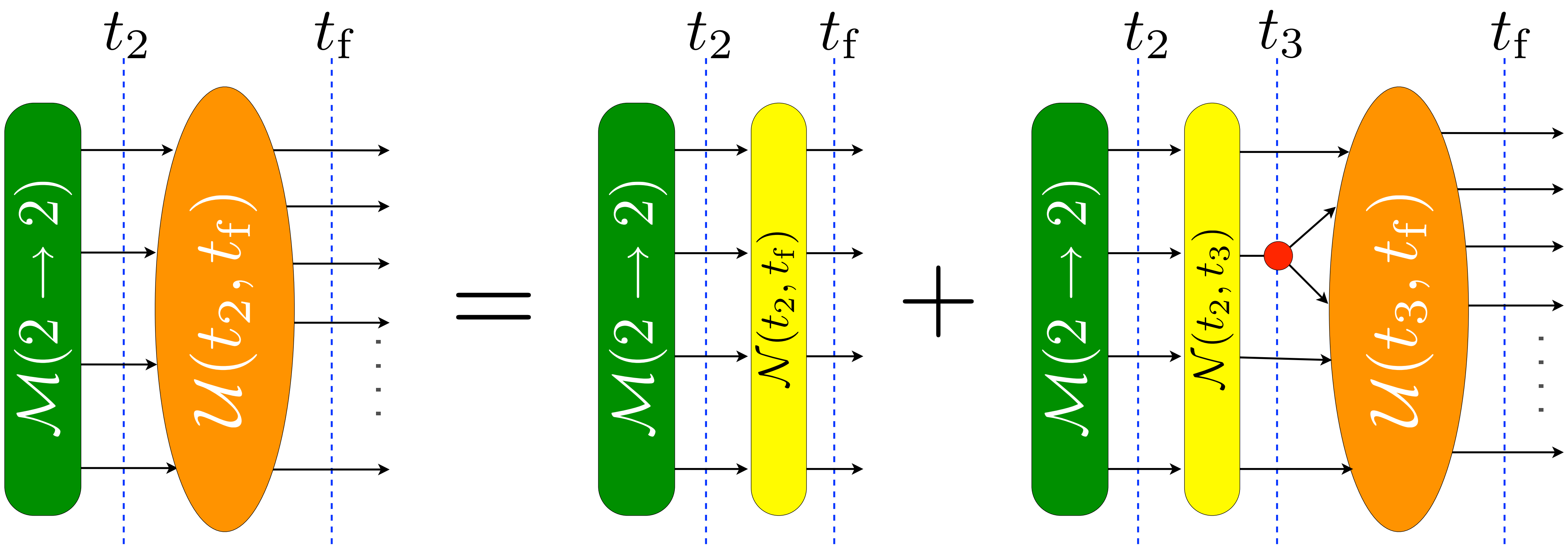
We can write the evolution equation in an integral equation form

$$\mathcal{U}(t_{\rm f}, t_2) = \underbrace{\mathcal{N}(t_{\rm f}, t_2)}_{\text{“Nothing happens”}} + \overbrace{\int_{t_2}^{t_{\rm f}} dt_3\, \mathcal{U}(t_{\rm f}, t_3) \mathcal{H}_I(t_3) \mathcal{N}(t_3, t_2)}^{\text{“Something happens”}}$$

where the non-splitting operator is

$$\mathcal{N}(t', t) = \mathbb{T} \exp\left\{ - \int_t^{t'} d\tau\, \mathcal{V}_I(\tau) \right\}$$

Sudakov operator



Splitting Operator

Very general splitting operator (*no spin correlation*) is

$$
\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{H}(t)\big|\{p,f,c',c\}_m\big) \\
&= \sum_{l=\mathrm{a,b},1,\dots,m} \delta\Big(t - T_l\big(\{\hat{p},\hat{f}\}_{m+1}\big)\Big)\,\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p,f\}_m\big)\,\frac{m+1}{2} \\
&\times \frac{n_\mathrm{c}(a)n_\mathrm{c}(b)\,\eta_\mathrm{a}\eta_\mathrm{b}}{n_\mathrm{c}(\hat{a})n_\mathrm{c}(\hat{b})\,\hat{\eta}_\mathrm{a}\hat{\eta}_\mathrm{b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_\mathrm{a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_\mathrm{b},\mu_F^2)}{f_{a/A}(\eta_\mathrm{a},\mu_F^2)f_{b/B}(\eta_\mathrm{b},\mu_F^2)}\sum_k \Psi_{lk}\big(\{\hat{f},\hat{p}\}_{m+1}\big) \\
&\times \sum_{\beta=L,R}(-1)^{1+\delta_{lk}}\big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}_\beta(l,k)\big|\{c',c\}_m\big)
\end{aligned}
$$

Momentum and flavor mapping

PDF factor

Color dependence

Important: $A_{lk} + A_{kl} = 1$

Arbitrary function, helps to distribute the soft gluon along the collinear directions.

It is only LO!

Splitting kernel is

$$
\Psi_{lk} = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{1}{\hat{p}_l\cdot\hat{p}_{m+1}}\underbrace{\left[A_{lk}\,\frac{2\hat{p}_l\cdot\hat{p}_k}{\hat{p}_k\cdot\hat{p}_{m+1}} + H_{ll}^{\mathrm{coll}}\big(\{\hat{f},\hat{p}\}_{m+1}\big)\right]} + \cdots
$$

Altarelli-Parisi splitting function in a very general form

Finite pieces

Splitting Operator

Very g

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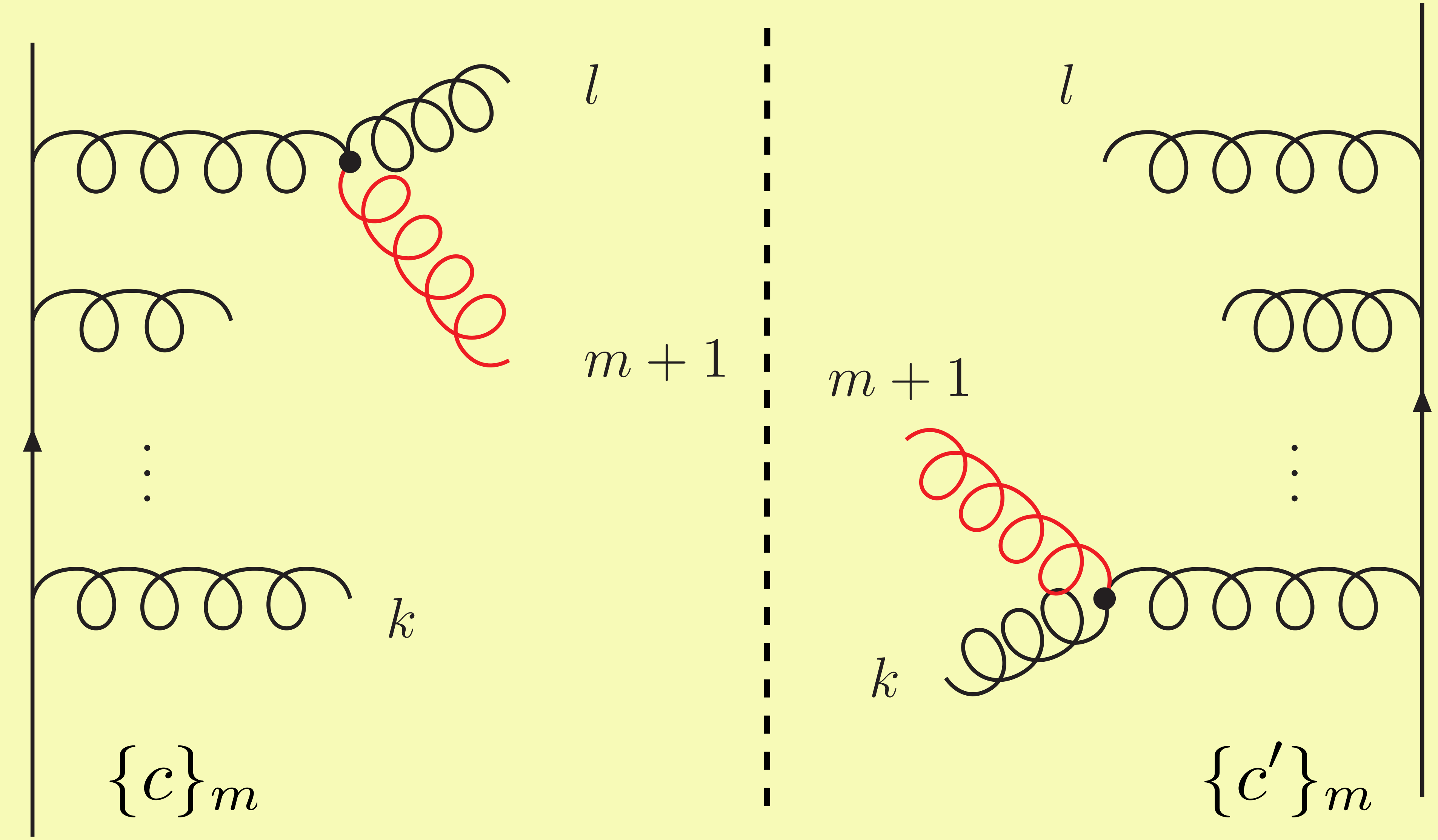
$=$



$$(\{\hat{c}', \hat{c}\}_{m+1} | \mathcal{G}_R(l,k) | \{c', c\}_m) = {}_D\langle \{\hat{c}\}_{m+1} | t_l^\dagger | \{c\}_m \rangle \langle \{c'\}_m | t_k | \{\hat{c}'\}_{m+1} \rangle_D \ .$$

he

Splittin

$\Psi_{lk} =$

Angular Ordered Shower

What would happen if we used angular ordering?

$$t_\angle = T_l\big(\{\hat{p},\hat{f}\}_{m+1}\big) = \log\frac{2\,\hat{Q}^2}{(p_l\cdot\hat{Q})^2} - \log\frac{\hat{p}_l\cdot\hat{p}_{m+1}\,\hat{Q}^2}{\hat{p}_l\cdot\hat{Q}\,\hat{p}_{m+1}\cdot\hat{Q}} = \log\frac{2}{E_l^2(1-\cos\vartheta_{l,m+1})}$$

And let's have a special choice for soft partitioning function:

$$A_{lk} = \theta(\vartheta_{l,m+1} < \vartheta_{l,k})\,\frac{1-\cos\vartheta_{m+1,k}}{1-\cos\vartheta_{l,k}} \quad \Rightarrow \quad A_{lk} + A_{kl} \approx 1$$

$$\Psi_l^{(\mathrm{a.o.})} = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{2}{\hat{p}_l\cdot\hat{p}_{m+1}}\left[\frac{\hat{p}_l\cdot\hat{Q}}{\hat{p}_{m+1}\cdot\hat{Q}} + H_{ll}^{\mathrm{coll}}(\{\hat{f},\hat{p}\}_{m+1})\right]$$

Independent of parton k!!!

(Azimuthal averaging leads to the same result.)

One can perform the sum over the color connected parton analytically

$$-\sum_k \big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}_\beta(l,k)\big|\{c',c\}_m\big) = \big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}_\beta(l,l)\big|\{c',c\}_m\big)$$

$\Rightarrow$ *No complicated color structure.*

Leading Color Approx.

1. Don't have special choice for the evolution variable and the soft partitioning function

Anyway everybody uses transverse momentum and the simplest soft partitioning function :

$$t_\perp = T_l\big(\{\hat{p},\hat{f}\}_{m+1}\big) = \log\frac{\hat{Q}^2}{-k_\perp^2}$$

$$A_{lk} = \frac{\hat{p}_k\cdot\hat{p}_{m+1}}{\hat{p}_k\cdot\hat{p}_{m+1} + \hat{p}_l\cdot\hat{p}_{m+1}}$$

2. But do approximation in the color space by considering only the leading color contributions

$$\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{c}\}_{m+1}\big|\mathcal{H}(t)\big|\{p,f,c\}_m\big) \\
&\quad = \sum_{l=\mathrm{a,b},1,\dots,m} \delta\Big(t - T_l\big(\{\hat{p},\hat{f}\}_{m+1}\big)\Big)\,\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p,f\}_m\big) \\
&\qquad \times \frac{n_\mathrm{c}(a)n_\mathrm{c}(b)\,\eta_\mathrm{a}\eta_\mathrm{b}}{n_\mathrm{c}(\hat{a})n_\mathrm{c}(\hat{b})\,\hat{\eta}_\mathrm{a}\hat{\eta}_\mathrm{b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_\mathrm{a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_\mathrm{b},\mu_F^2)}{f_{a/A}(\eta_\mathrm{a},\mu_F^2)f_{b/B}(\eta_\mathrm{b},\mu_F^2)} \\
&\qquad \times (m+1)\sum_k \Psi_{lk}(\{\hat{f},\hat{p}\}_{m+1})\,\big\langle\{\hat{c}\}_{m+1}\big|a_{lk}^\dagger\big|\{c\}_m\big\rangle \quad .
\end{aligned}$$

Antenna Dipole Shower

The antenna dipole shower is rather a *reorganization of the leading color* partitioned dipole *shower*.

$$\mathcal{H}_{lk}^{\mathrm{part}}(t) \propto [\mathcal{P}_l\, A_{lk} + \mathcal{P}_k\, A_{kl}]\, \frac{\hat{p}_l \cdot \hat{p}_k}{\hat{p}_{m+1} \cdot \hat{p}_l \;\hat{p}_{m+1} \cdot \hat{p}_k}$$

The antenna shower tries to remove the ambiguity of the soft partitioning function A_{lk} by using a new momentum mapping

$$\mathcal{H}_{lk}^{\mathrm{ant}}(t) \propto \mathcal{P}_{lk}\, \frac{\hat{p}_l \cdot \hat{p}_k}{\hat{p}_{m+1} \cdot \hat{p}_l \;\hat{p}_{m+1} \cdot \hat{p}_k}$$

Now the freedom to choose A_{lk} function resides in the freedom to choose P_{lk}. I think the best mapping for antenna shower would be

$$\mathcal{P}_{lk} = \theta(\vartheta_{l,m+1} < \vartheta_{k,m+1})\, \mathcal{P}_l + \theta(\vartheta_{k,m+1} < \vartheta_{l,m+1})\, \mathcal{P}_k$$

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1. Fixes the general structure of the splitting kernels.

2. Fixes the evolution equation.

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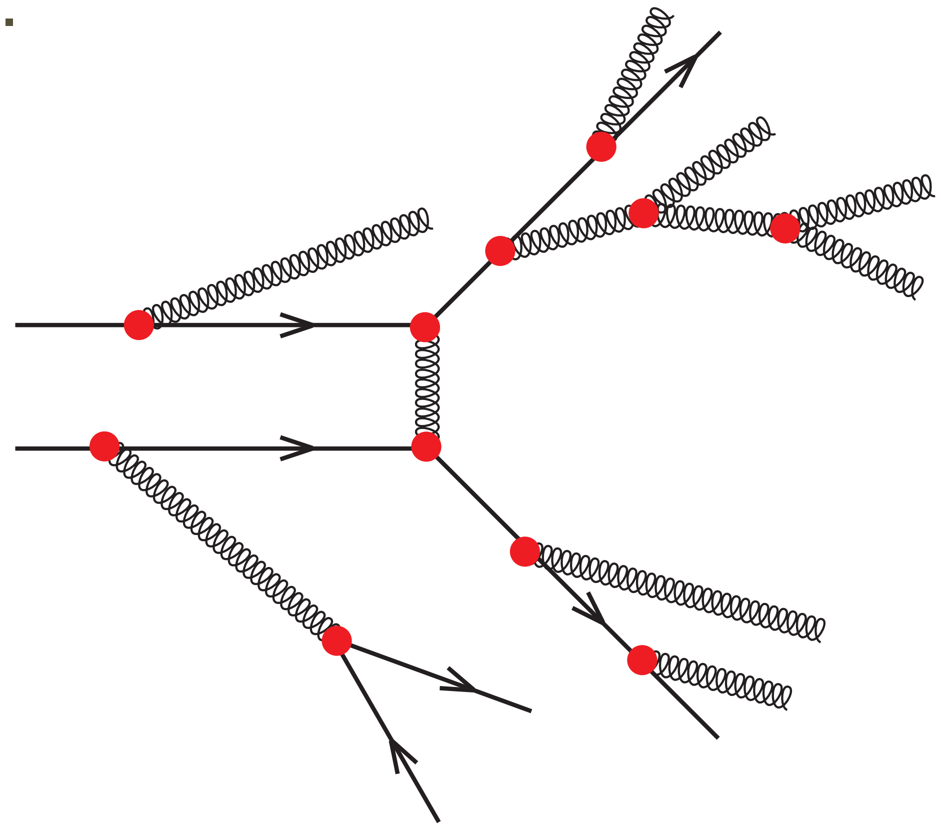
Shower Time

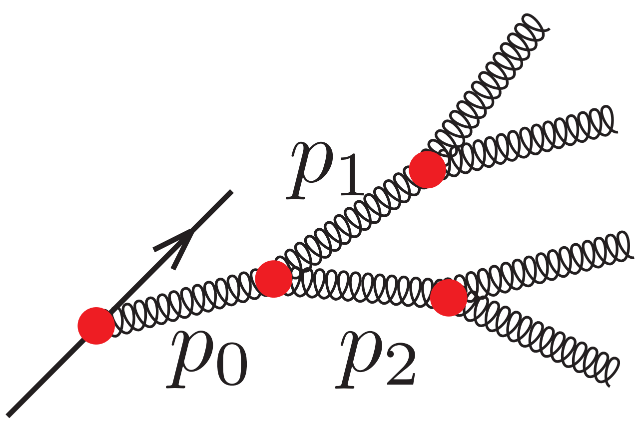
- In a shower history, we need to distinguish which vertices are "harder" and which are "softer."



- Does "harder" means bigger virtuality, $\left|p^2 - m^2\right|$?
- Does "harder" means greater k_T^2 of daughter parton relative to the mother parton axis?

Shower Time

- Examine successive splitting



- Use null-plane momentum components

$$p = (p^+, p_-, \boldsymbol{p}_\perp)$$

- Direction of the jet is *(1,0,0)*

- For mother parton,

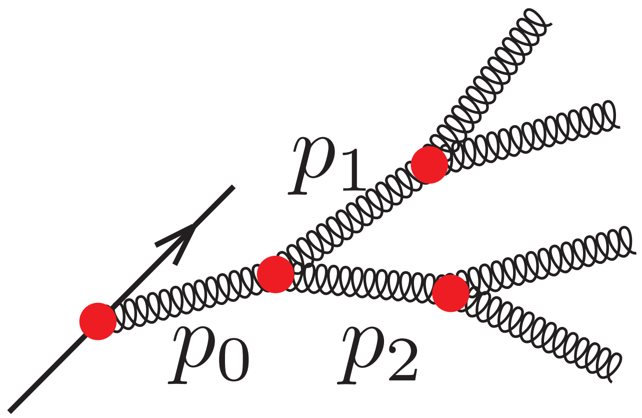
$$p_0 = \left(x_0 P, \frac{\boldsymbol{p}_0^2 + m_0^2 + v_0^2}{2 x_0 P}, \boldsymbol{p}_0 \right)$$

- Here P is the jet momentum, x_0 is the momentum fraction and $v_0^2 = p_0^2 - m_0^2$ is the mother parton virtuality.

Shower Time

- The momentum of the mother parton is

$$p_0 = p_1 + p_2$$



- and the daughters are

$$p_1 = \left(x_1 P, \frac{\boldsymbol{p}_1^2 + m_1^2 + v_1^2}{2x_1 P}, \boldsymbol{p}_1 \right)$$

$$p_2 = \left(x_2 P, \frac{\boldsymbol{p}_2^2 + m_2^2 + v_2^2}{2x_2 P}, \boldsymbol{p}_2 \right)$$

- Now the virtuality of the mother parton is

$$\frac{v_0^2}{x_0} = \frac{(x_2 \boldsymbol{p}_1 - x_1 \boldsymbol{p}_2)^2}{x_0 x_1 x_2} + \frac{m_1^2}{x_1} + \frac{m_2^2}{x_2} - \frac{m_0^2}{x_0} + \frac{v_1^2}{x_1} + \frac{v_2^2}{x_2}$$

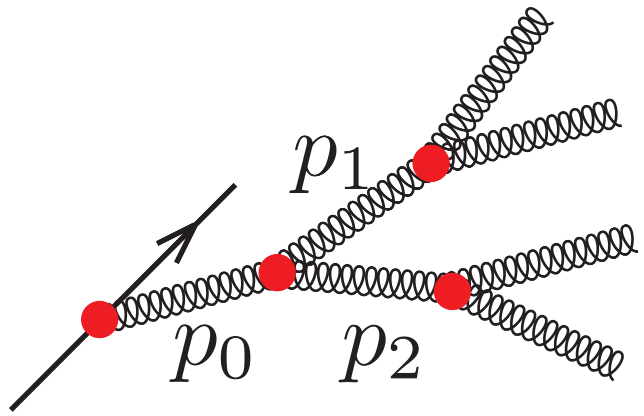
- For factorization graph by graph, it must be a good approximation to neglect v_1^2 and v_2^2 in v_0^2 :

$$\frac{v_0^2}{x_0} > \frac{v_1^2}{x_1} \qquad \frac{v_0^2}{x_0} > \frac{v_2^2}{x_2}$$

Shower Time

- So we demand

$$\Lambda_0^2 > \Lambda_1^2 \quad \text{and} \quad \Lambda_0^2 > \Lambda_2^2$$



where

$$\Lambda_i^2 = \frac{|p_i^2 - m_i^2|}{p_i \cdot Q_0} Q_0^2$$

is the ordering variable and Q_0 is fixed timelike.

- Λ^2 is neither virtuality nor k_T^2

- The transverse momentum and the emission angle are also good ordering variable if the color coherence is preserved, the observable is not sensitive for wide angle soft emission. (*But no graph by graph factorization.*)

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2. *Fixes the evolution equation.*
3. *Fixes the shower time.*

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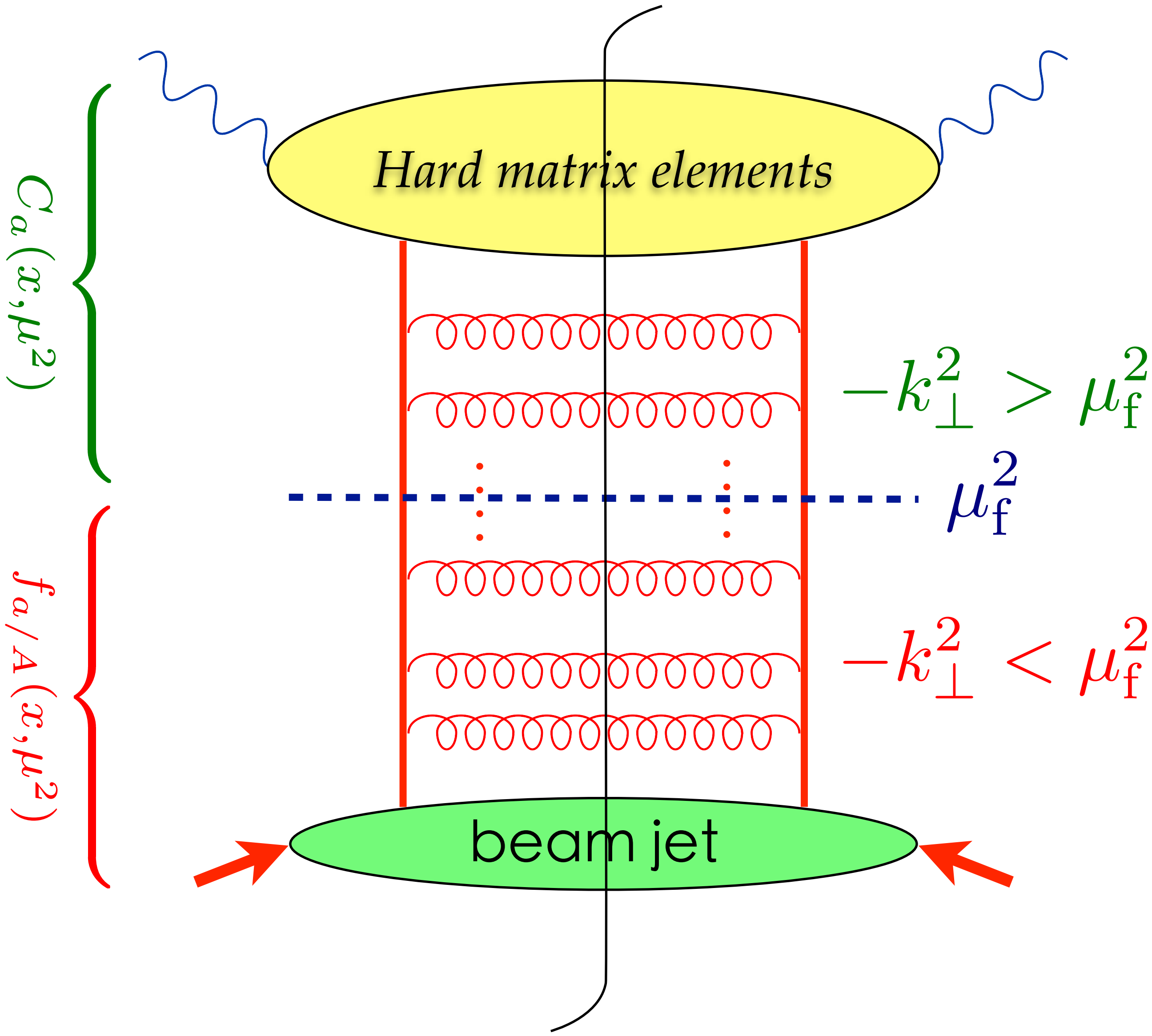
DGLAP Evolution of PDFs



Perturbative part (what we calculate)
Completely independent of the PDFs

$$|\rho(t_\mathrm{f})) = \underbrace{\mathcal{F}(t_\mathrm{f})}\,\overbrace{|\rho_\mathrm{pert}(t_\mathrm{f}))}$$

PDFs: The non-perturbative physics is only here

It MUST BE independent of the PDF, otherwise the perturbative and non-perturbative physics are mixed.

Non-trivial PDF dependence

$$\mathbb{T}\exp\left\{\overbrace{-\int_t^{t'} d\tau\, \mathcal{V}_I(\tau)}\right\} = \mathcal{N}(t',t) = \mathcal{F}(t')\mathcal{N}_\mathrm{pert}(t',t)\mathcal{F}^{-1}(t) = \mathcal{F}(t')\,\mathbb{T}\exp\left\{\overbrace{-\int_t^{t'} d\tau\, \mathcal{V}_I^\mathrm{pert}(\tau)}\right\}\mathcal{F}^{-1}(t)$$

Leads to the evolution equation of the parton distribution functions.

DGLAP Evolution

In general the incoming parton can be massive, this leads to a slightly modified DGLAP evolution. That is

$$\mu^2 \frac{d}{d\mu^2} f_{a/A}(\eta, \mu^2) = \sum_{\hat{a}} \int_0^1 \frac{dz}{z} \, \frac{\alpha_s(\mu^2)}{2\pi} P_{a\hat{a}}\left(z, z\frac{m^2}{\mu^2}\right) f_{\hat{a}/A}(\eta/z, \mu^2)$$

with the modified evolution kernels:

$$P_{qq}(z,\lambda) = C_F \left[\left(\frac{2}{1-z} - (1+z) - 2\lambda\right)\theta\left(\frac{1}{1-z} > 1+\lambda\right)\right]_+ \quad ,$$

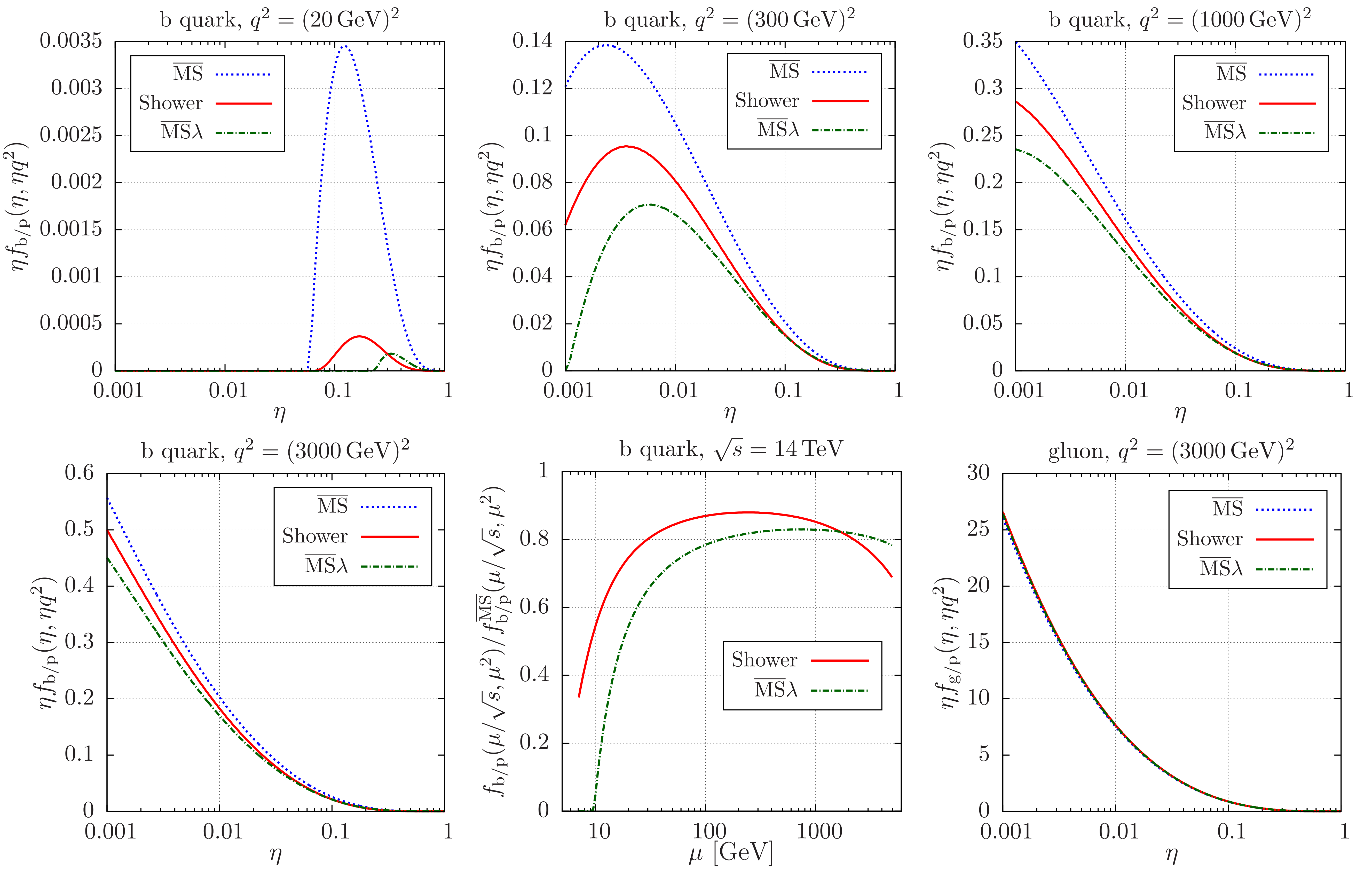
$$P_{gg}(z,\lambda) = 2C_A\left[\frac{1}{(1-z)_+} - 1 + \frac{1-z}{z} + z(1-z)\right] + \gamma_g(\lambda)\,\delta(1-z) \quad ,$$

$$P_{qg}(z,\lambda) = T_R\left[1 - 2\,z\,(1-z) + 2\lambda\right]\theta(z(1-z) > \lambda) \quad ,$$

$$P_{gq}(z,\lambda) = C_F\left[\frac{1+(1-z)^2}{z} - 2\lambda\right]\theta\left(\frac{1}{z} > 1+\lambda\right) \quad .$$

With different shower time the mass depend parts of the DGLAP kernels are different!

Shower PDFs



b quark, $q^2 = (20\,\mathrm{GeV})^2$
b quark, $q^2 = (300\,\mathrm{GeV})^2$
b quark, $q^2 = (1000\,\mathrm{GeV})^2$
b quark, $q^2 = (3000\,\mathrm{GeV})^2$
b quark, $\sqrt{s} = 14\,\mathrm{TeV}$
gluon, $q^2 = (3000\,\mathrm{GeV})^2$
$\overline{\mathrm{MS}}$
Shower
$\overline{\mathrm{MS}}\lambda$
$\eta f_{\mathrm{b/p}}(\eta, \eta q^2)$
$\eta f_{\mathrm{g/p}}(\eta, \eta q^2)$
$f_{\mathrm{b/p}}(\mu/\sqrt{s}, \mu^2) / f^{\overline{\mathrm{MS}}}_{\mathrm{b/p}}(\mu/\sqrt{s}, \mu^2)$
η
μ [GeV]
0.001
0.01
0.1
1
10
100
1000
0
0.0005
0.001
0.0015
0.002
0.0025
0.003
0.0035
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.05
0.15
0.2
0.25
0.3
0.35
0.4
0.5
0.6
0.8
5
15
20
25
30

How to Design Parton Showers?

Mandatory design principles

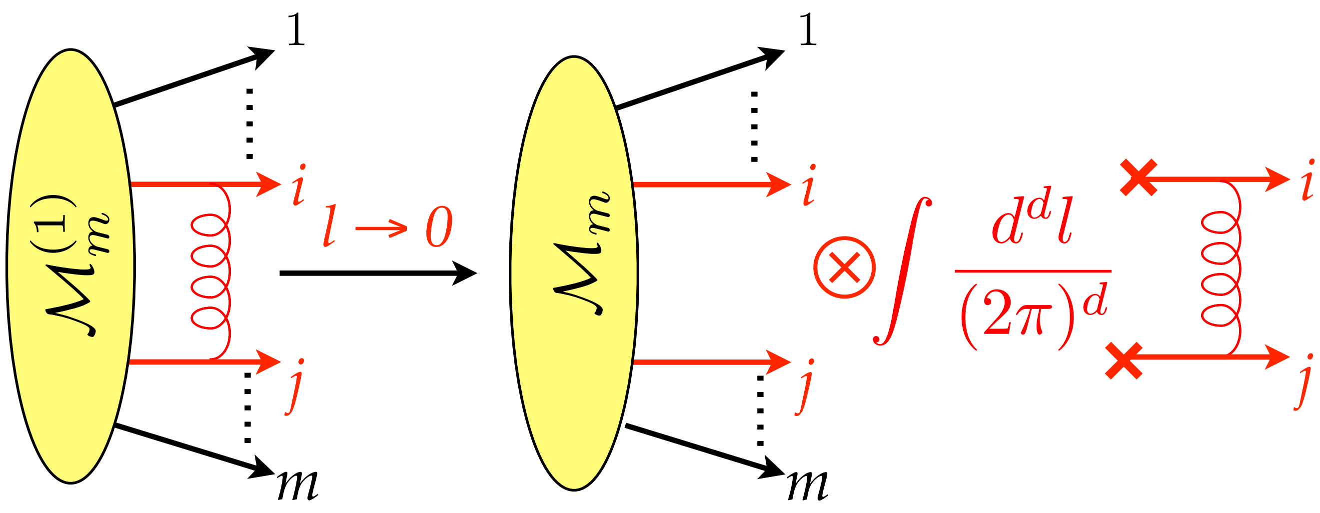
1. Shower calculates cross sections approximately using the soft and collinear factorization of the QCD amplitudes (tree and 1-loop level).
2. The emissions are strongly ordered.
3. The ordering must control the goodness of the soft and collinear approximations.
4. The parton shower must be a perturbative object.

1. *Fixes the general structure of the splitting kernels.*
2. *Fixes the evolution equation.*
3. *Fixes the shower time.*
4. *Fixes the evolution of the PDFs.*

Normalization

5. Shower doesn't change the normalization. This is the unitarity condition.

Unitarity Condition



$$\mathcal{V}_I(t) = \mathcal{V}(t) + \underbrace{i\pi\widetilde{\mathcal{V}}(t)}$$

Coulomb gluon

$$(1|\widetilde{\mathcal{V}}(t) = 0$$

The singularities must be cancelled in the soft and collinear limits between the real and virtual emissions

$$(1|\,[\mathcal{H}_I(t) - \mathcal{V}_I(t)] = \text{Finite(t)} \xrightarrow{t\to\infty} 0$$

In parton shower implementation we always choose

$$\text{Finite(t)} = 0 \quad \text{for every } t$$

Unitarity condition

The shower evolution *doesn't change the normalization.*

Unitarity condition is not God given, not derived from first principles. It is only a convenient choice!!! In some cases it is rather an unpleasant limitation....

How to Design Parton Showers?

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Normalization

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1. *Fixes the general structure of the splitting kernels.*
2. *Fixes the evolution equation.*
3. *Fixes the shower time.*
4. *Fixes the evolution of the PDFs.*
5. *Fixes the virtual operator.*

A general purpose parton shower program must generate partonic final states

- in a *FULLY exclusive way* (momentum, flavor, spin and color are fully resolved)
- as *precisely* as possible (*e.g.*: sums up large logarithms at NLL level).

Splitting Operator

Most of the component of the parton shower have been fixed

Momentum and flavor mapping

$$
\begin{aligned}
&\big(\{\hat p, \hat f, \hat c', \hat c\}_{m+1}\big|\mathcal{H}(t)\big|\{p, f, c', c\}_m\big) \\
&= \sum_{l=\mathrm{a,b},1,\dots,m} \delta\Big(t - T_l\big(\{\hat p, \hat f\}_{m+1}\big)\Big)\,\big(\{\hat p, \hat f\}_{m+1}\big|\mathcal{P}_l\big|\{p, f\}_m\big)\,\frac{m+1}{2} \\
&\quad\times \frac{n_\mathrm{c}(a) n_\mathrm{c}(b)\,\eta_\mathrm{a}\eta_\mathrm{b}}{n_\mathrm{c}(\hat a) n_\mathrm{c}(\hat b)\,\hat\eta_\mathrm{a}\hat\eta_\mathrm{b}}\,\frac{f_{\hat a/A}(\hat\eta_\mathrm{a},\mu_F^2) f_{\hat b/B}(\hat\eta_\mathrm{b},\mu_F^2)}{f_{a/A}(\eta_\mathrm{a},\mu_F^2) f_{b/B}(\eta_\mathrm{b},\mu_F^2)} \sum_k \Psi_{lk}\big(\{\hat f, \hat p\}_{m+1}\big) \\
&\quad\times \sum_{\beta=L,R} (-1)^{1+\delta_{lk}}\big(\{\hat c', \hat c\}_{m+1}\big|\mathcal{G}_\beta(l,k)\big|\{c', c\}_m\big)
\end{aligned}
$$

$$
\Psi_{lk} = \frac{\alpha_\mathrm{s}}{2\pi}\,\frac{1}{\hat p_l\cdot\hat p_{m+1}}\left[A_{lk}\,\frac{2\hat p_l\cdot\hat p_k}{\hat p_k\cdot\hat p_{m+1}} + H_{ll}^{\mathrm{coll}}\big(\{\hat f, \hat p\}_{m+1}\big)\right]
$$

We still have to say something about the

- *momentum mapping*
- *soft partitioning function*
- *color*

Splitting Operator

Most of the component of the parton shower have been fixed

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$$\big(\{\hat{p},\hat{f},\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{H}(t)\big|\{p,f,c',c\}_m\big)$$
$$= \sum_{l=\mathrm{a},\mathrm{b},1,\dots,m} \delta\Big(t - T_l\big(\{\hat{p},\hat{f}\}_{m+1}\big)\Big)\,\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p,f\}_m\big)\,\frac{m+1}{2}$$
$$\times \frac{n_\mathrm{c}(a)n_\mathrm{c}(b)\,\eta_\mathrm{a}\eta_\mathrm{b}}{n_\mathrm{c}(\hat{a})n_\mathrm{c}(\hat{b})\,\hat{\eta}_\mathrm{a}\hat{\eta}_\mathrm{b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_\mathrm{a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_\mathrm{b},\mu_F^2)}{f_{a/A}(\eta_\mathrm{a},\mu_F^2)f_{b/B}(\eta_\mathrm{b},\mu_F^2)}\sum_k \Psi_{lk}(\{\hat{f},\hat{p}\}_{m+1})$$
$$\times \sum_{\beta=L,R}(-1)^{1+\delta_{lk}}\big(\{\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{G}_\beta(l,k)\big|\{c',c\}_m\big)$$

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&\quad\times \frac{n_\mathrm{c}(a)n_\mathrm{c}(b)\,\eta_\mathrm{a}\eta_\mathrm{b}}{n_\mathrm{c}(\hat{a})n_\mathrm{c}(\hat{b})\,\hat{\eta}_\mathrm{a}\hat{\eta}_\mathrm{b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_\mathrm{a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_\mathrm{b},\mu_F^2)}{f_{a/A}(\eta_\mathrm{a},\mu_F^2)f_{b/B}(\eta_\mathrm{b},\mu_F^2)}\sum_k \Psi_{lk}(\{\hat{f},\hat{p}\}_{m+1}) \\
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\end{aligned}
$$

Momentum and flavor mapping

$$
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$$

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Momentum and flavor mapping

$$\left(\{\hat{p},\hat{f},\hat{c}',\hat{c}\}_{m+1}\middle|\mathcal{H}(t)\middle|\{p,f,c',c\}_m\right)$$
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$$\times \frac{n_\mathrm{c}(a)n_\mathrm{c}(b)\,\eta_\mathrm{a}\eta_\mathrm{b}}{n_\mathrm{c}(\hat{a})n_\mathrm{c}(\hat{b})\,\hat{\eta}_\mathrm{a}\hat{\eta}_\mathrm{b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_\mathrm{a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_\mathrm{b},\mu_F^2)}{f_{a/A}(\eta_\mathrm{a},\mu_F^2)f_{b/B}(\eta_\mathrm{b},\mu_F^2)}\sum_k \Psi_{lk}(\{\hat{f},\hat{p}\}_{m+1})$$
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We still have to say something about the

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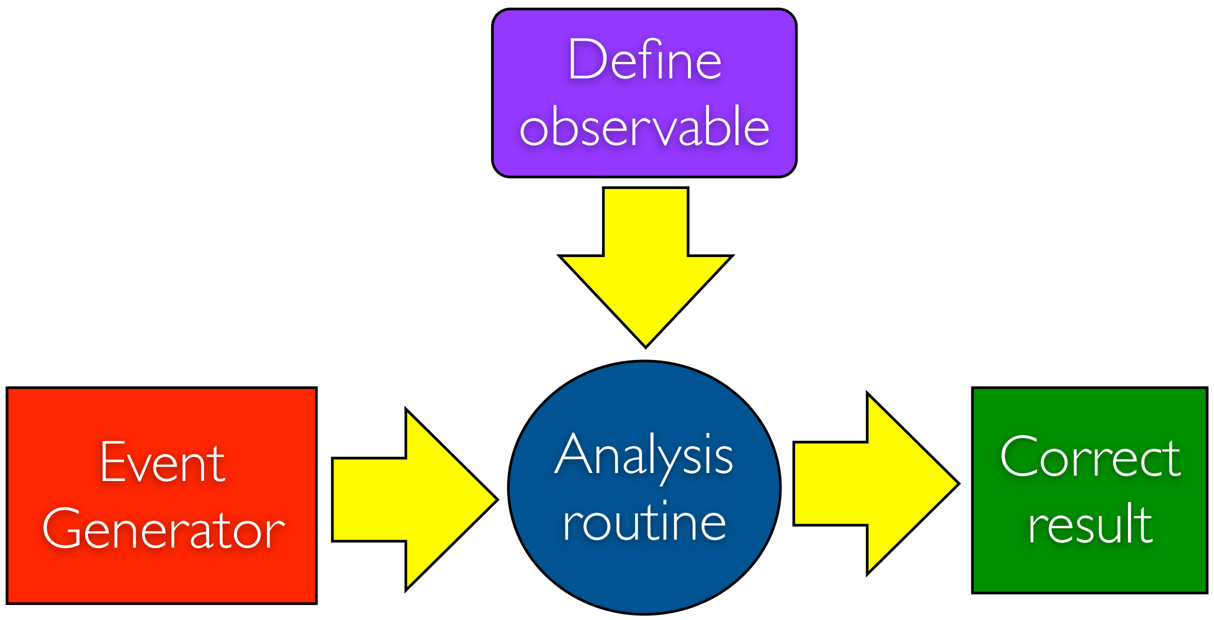
- *color*

Is NLL precision inevitable?

One might imagine that because parton splitting functions are correct in the limits of soft and collinear splittings, all large log summations will come out correctly.



- Eye measure doesn't help to validate parton showers against analytical results.
- One has to solve the shower evolution equation analytically and compare the result at NLL level. (e.g.: Drell-Yan pT-distribution, e+e- event shapes)
- "Minor details" are important. Once they are fixed the resummation works.
- It requires more studies to understand what class of observables can be predicted at (N)LL accuracy from parton showers.
- Recent results gives us only some hints about the *soft partitioning function (A_{lk})* and the *momentum mapping*.

✓ The momentum mappings with *global recoil* are *more preferred.*

✓ The soft partitioning function should depend on *only relative angles*.

(These are only hints, we don't have solid proof, only some counter examples.)

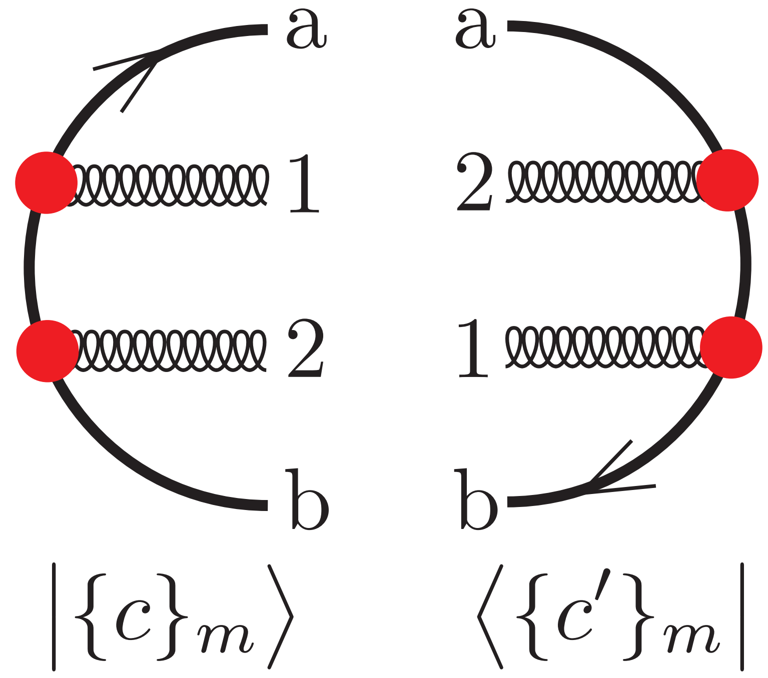
Color

Color

- The fundamental object is the quantum density matrix in color space with basis:

$$\big|\{c\}_m\big\rangle\big\langle\{c'\}_m\big|$$

- A simple but not trivial example for this:

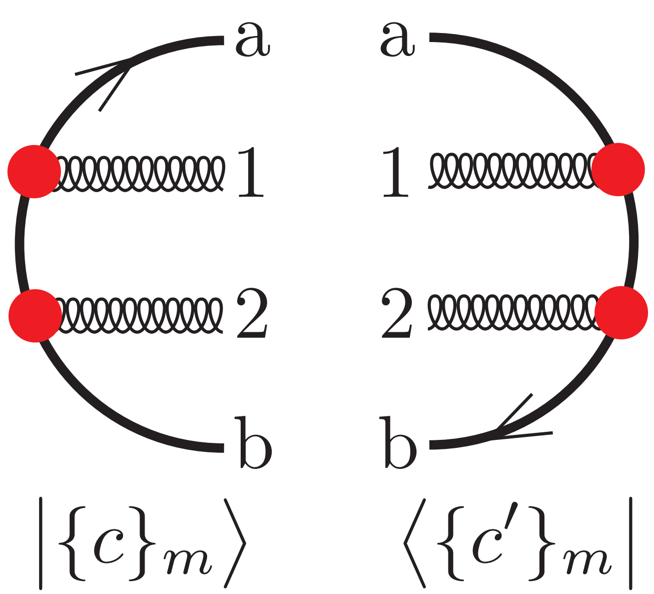


The leading color (LC)approx.

In leading color approximation only states with

$$\{c'\}_m = \{c\}_m$$

are allowed. Thus the shower starts or continues only from diagonal states like this:

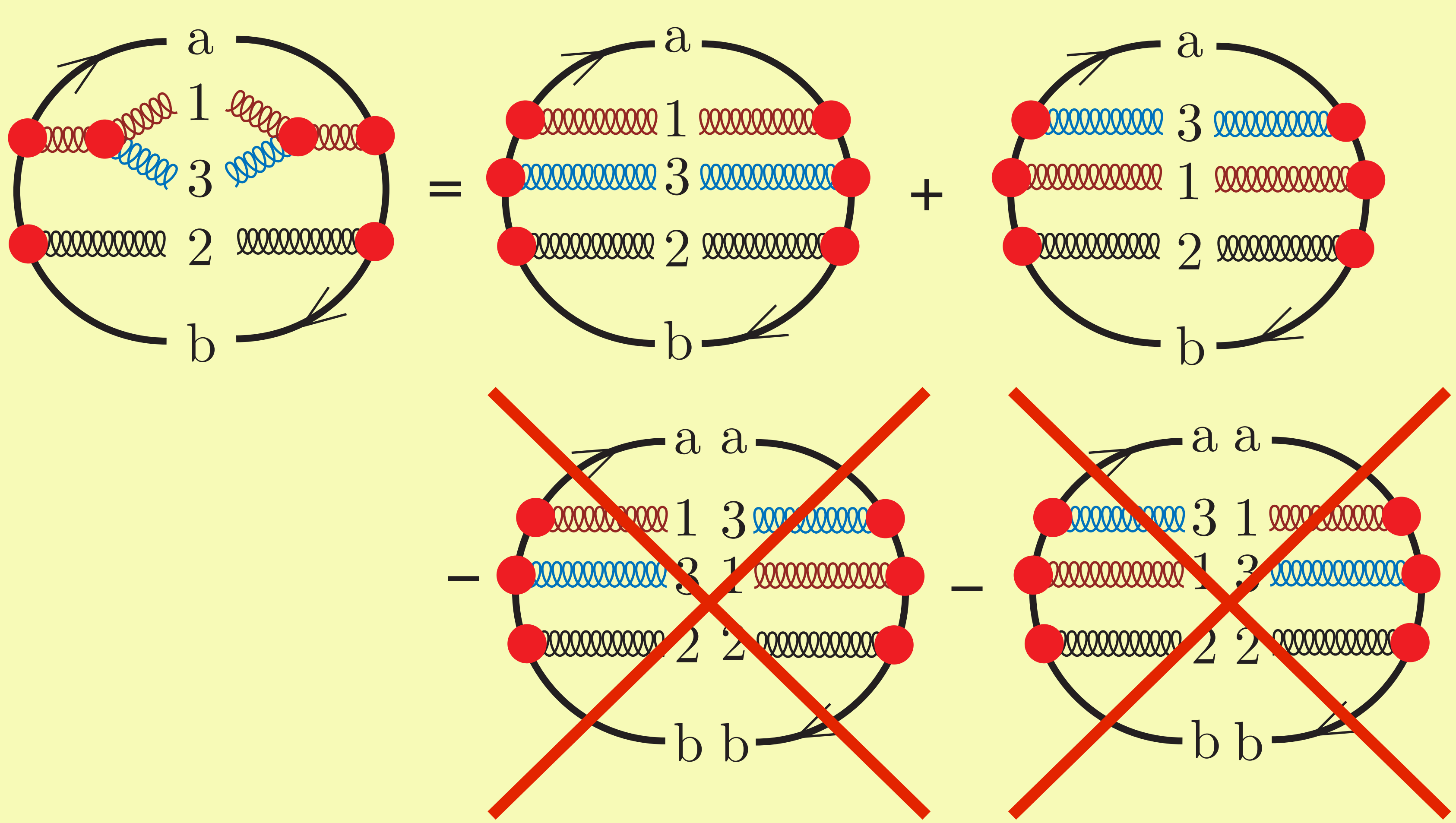


The leading color (LC)approx.

In leading color approximation only states with

are allow

This means we have to throw away terms in every splittings:



Color Suppression Index

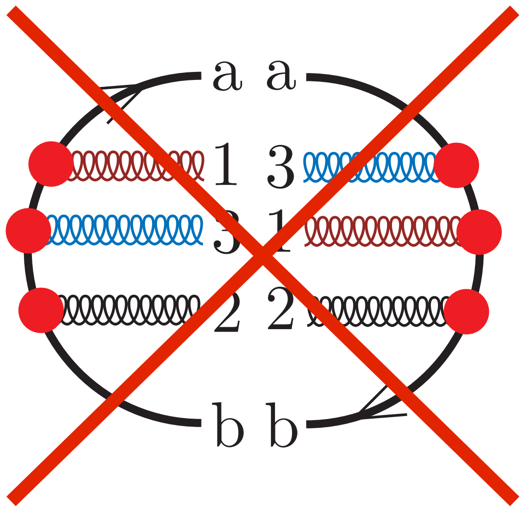
- At each step we calculate the "color suppression index", I
- The $I=0$ corresponds to the leading color approximation.
- At the end of the shower evolution the event is proportional to

$$\frac{1}{N_c^I} \quad \text{and} \quad I \geq 0$$

- At each step of the shower $I_{\mathrm{new}} \geq I_{\mathrm{old}}$
- In leading color approximation at each splitting we neglect terms with

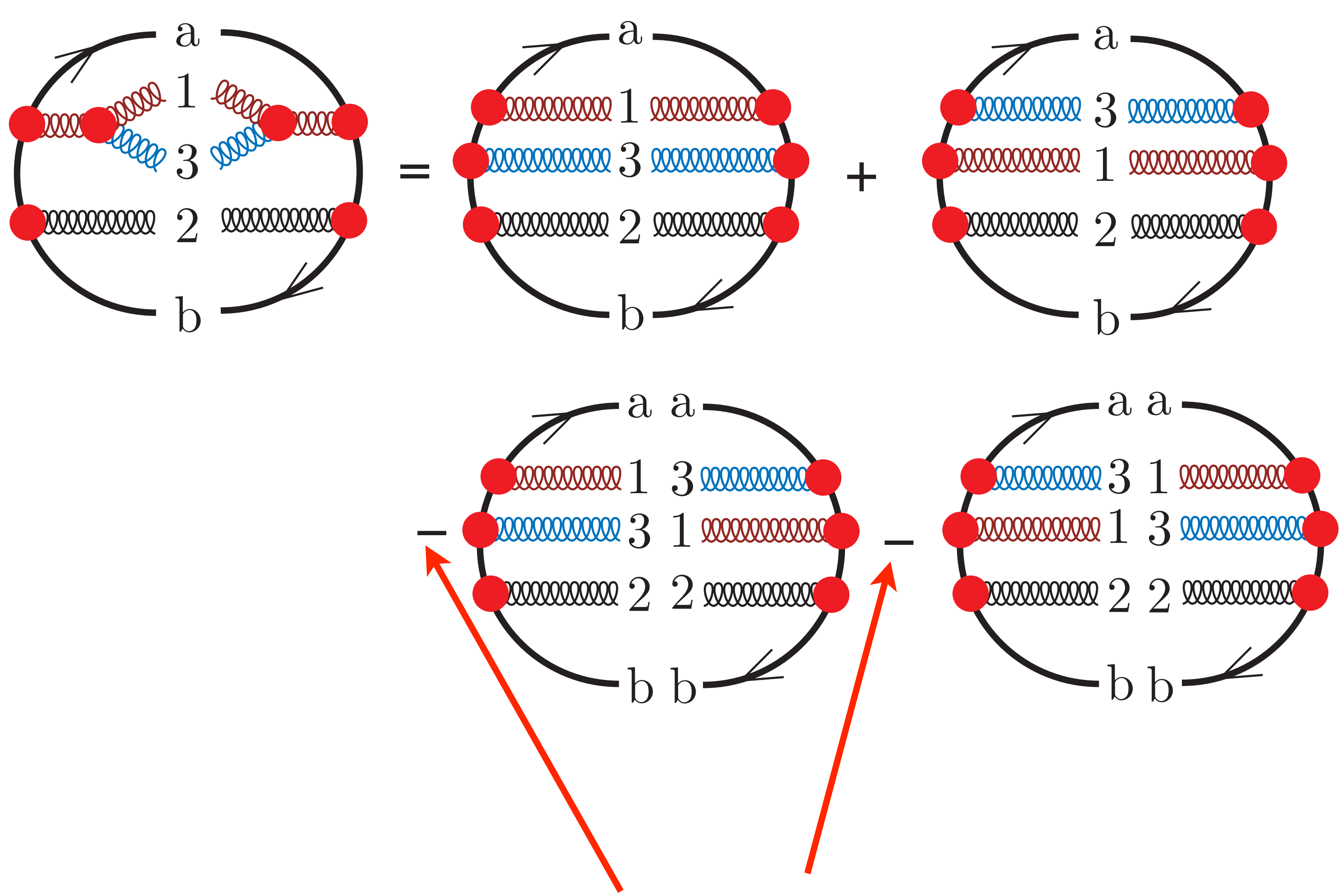
$$I > 0$$

- Thus we neglect $1/N_c^2$ contributions.
- *Are these contributions unimportant?*



LC+ approximation

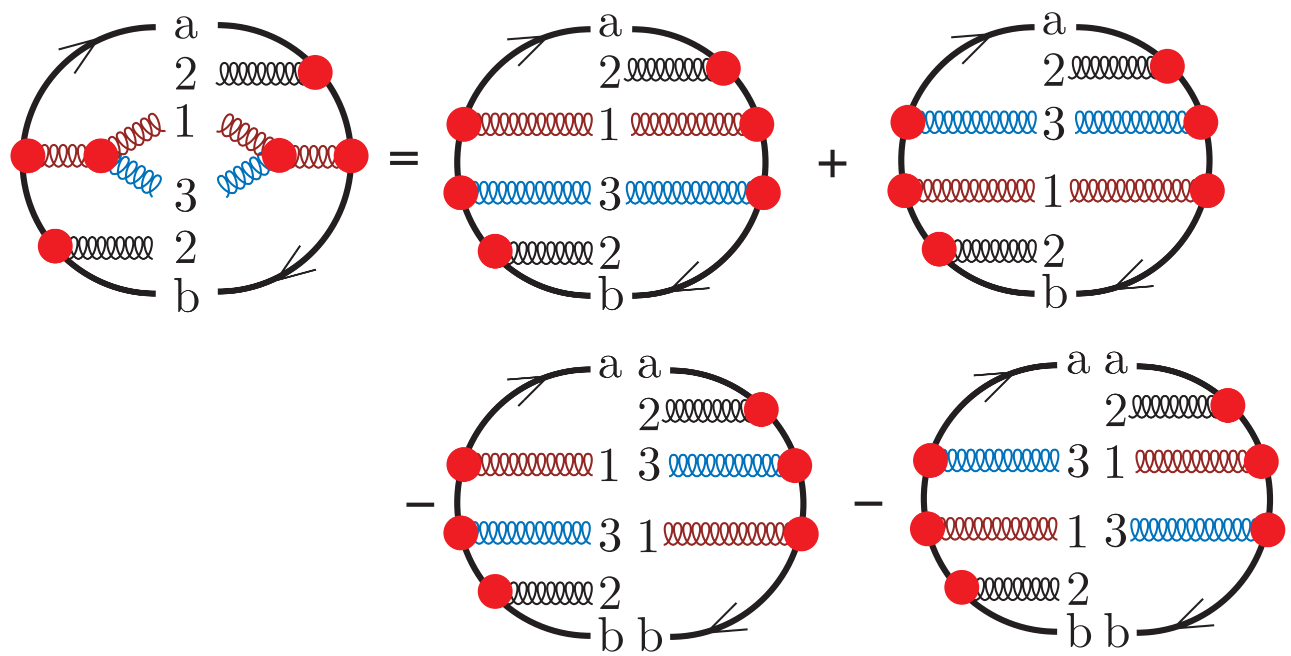
- Start shower from any color configuration and each step of the shower throw away less terms
- Example: Collinear splitting



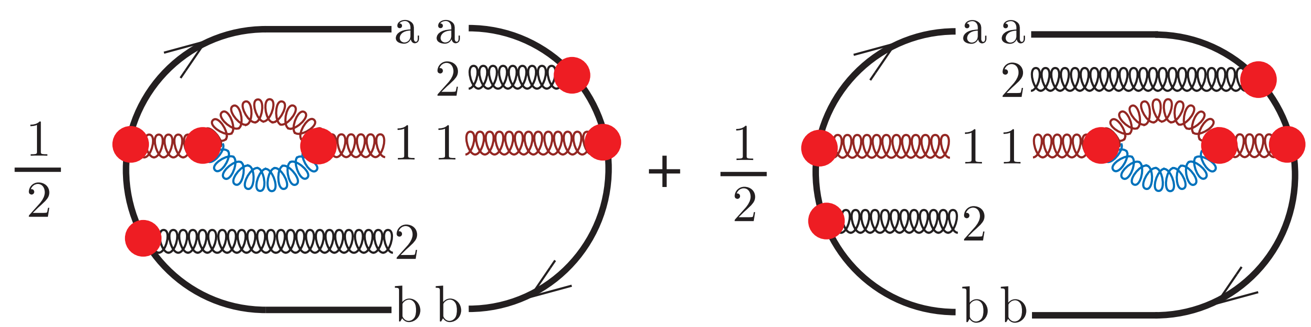
It is not a mistake, we have negative weights

How is this possible?

- For terms kept, the Sudakov exponent needs to be a number not an matrix in the color space.
- For this splitting keep all terms



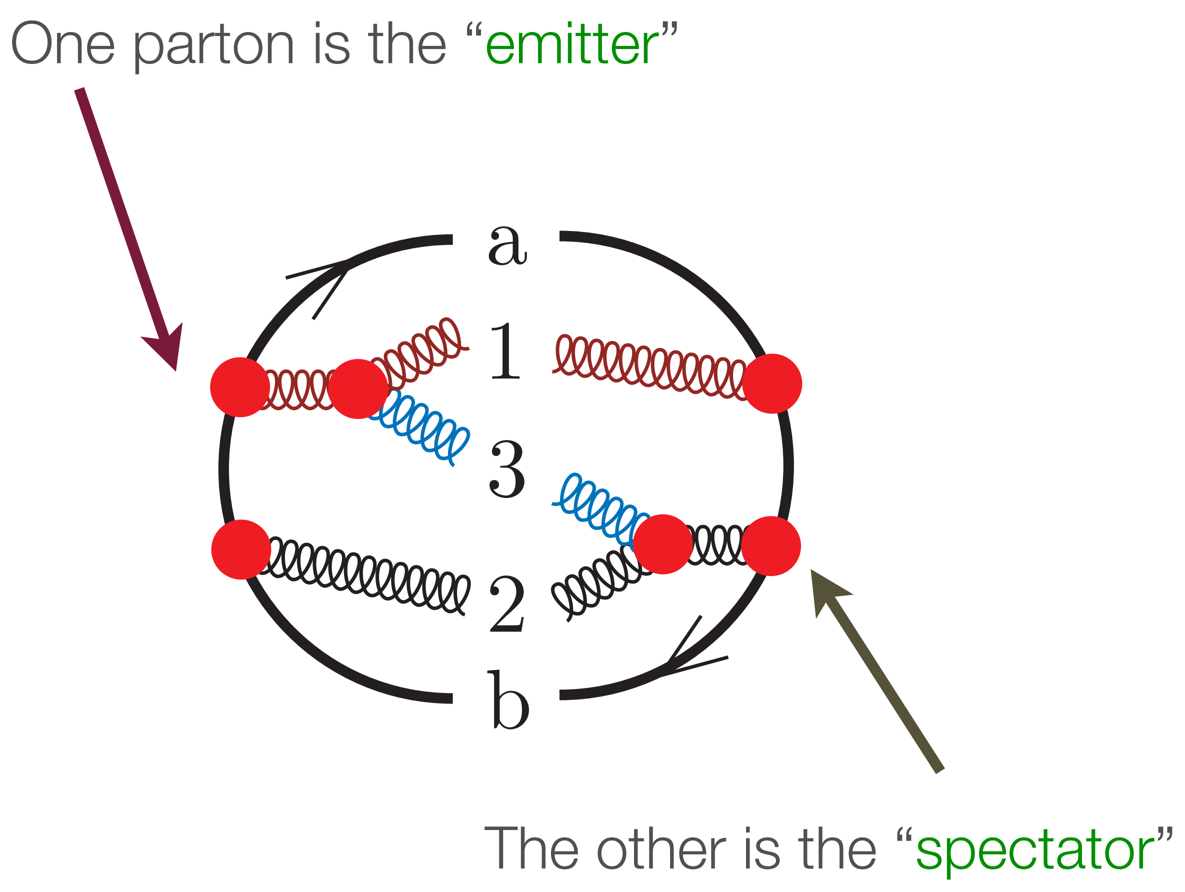
- The corresponding contribution to $\mathcal{V}(t)$ has the color structure:



- The gluon loops simple give a factor of C_A

Interference Graphs

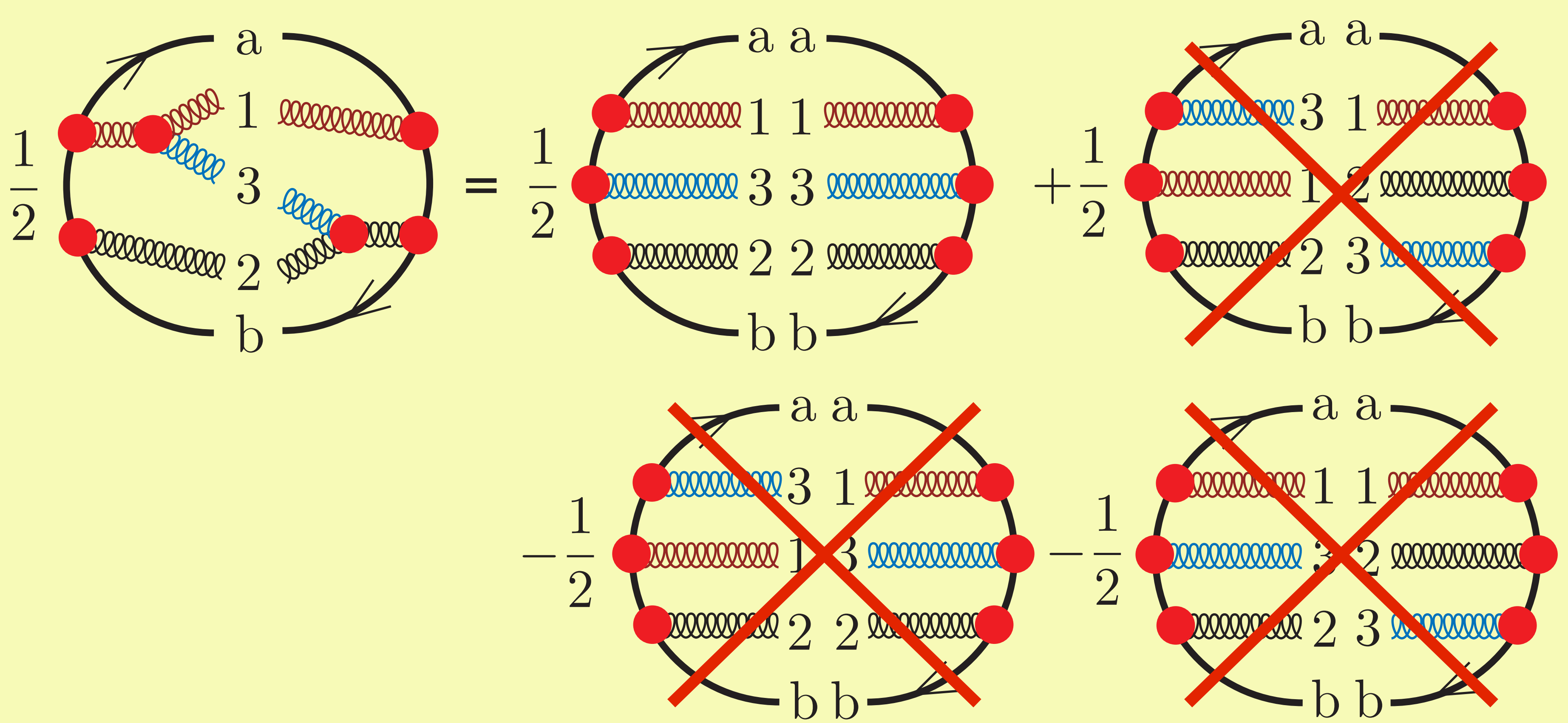
Interference graphs are important for the soft gluon emission



Interference Graphs

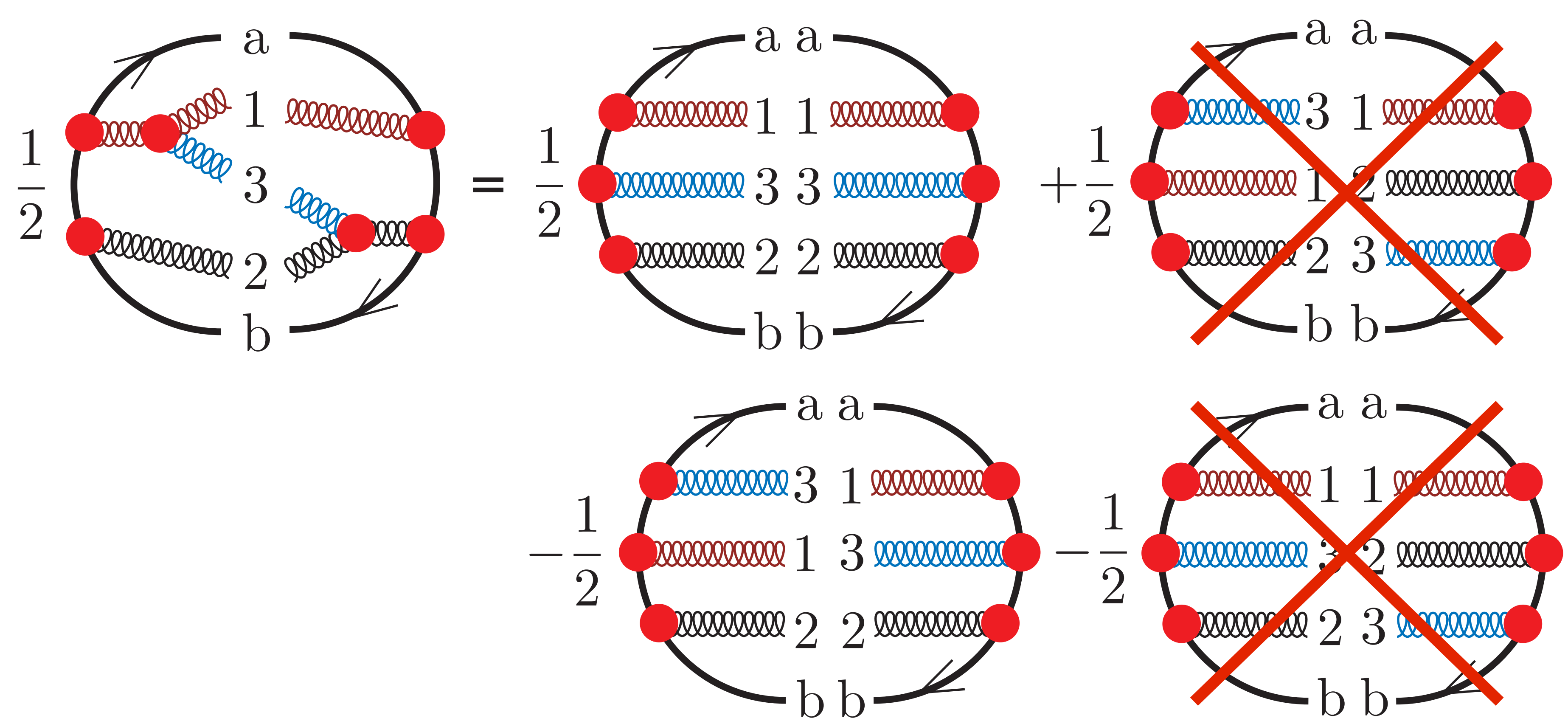
Interference graphs are important for the soft gluon emission

The LC approximation keeps just one contribution

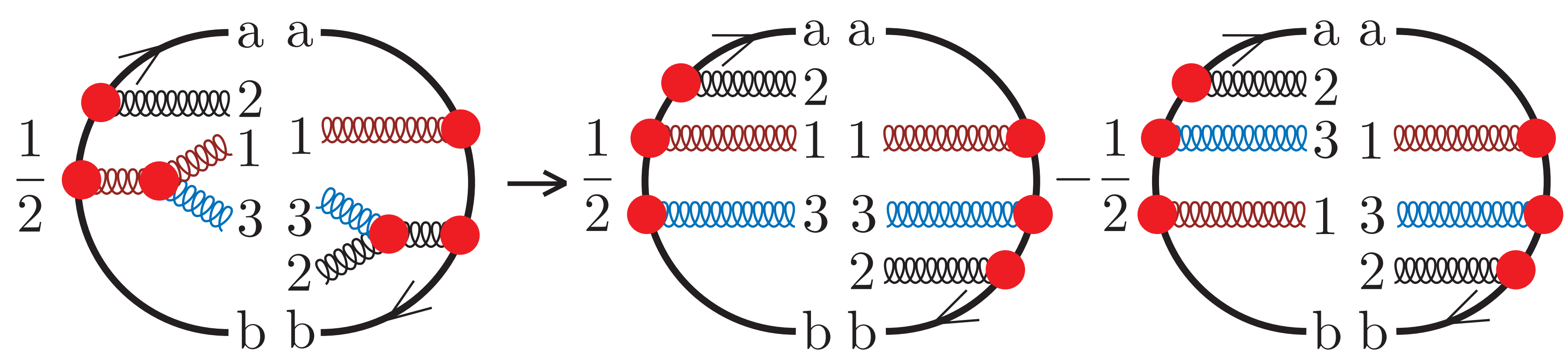


Interference Graphs

The LC+ approximation keeps two terms:

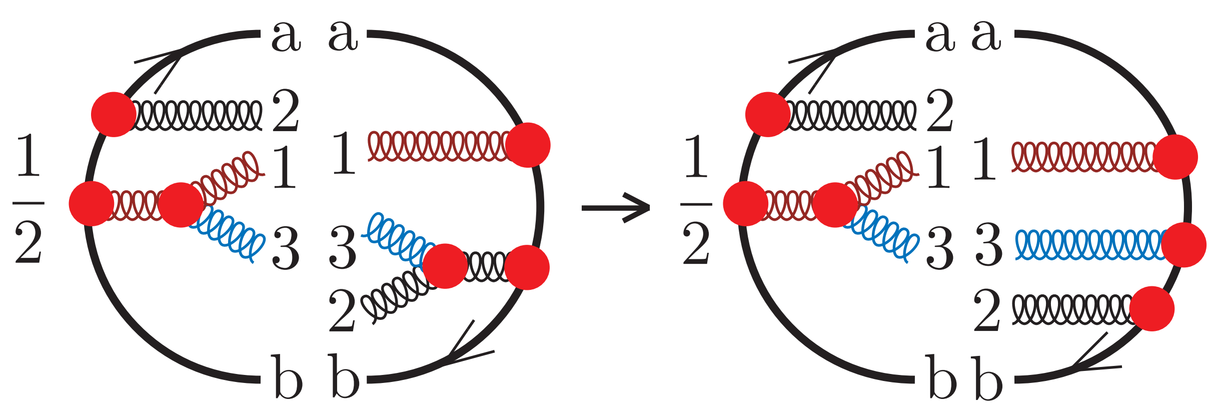


Another example, starting from non-diagonal contribution:

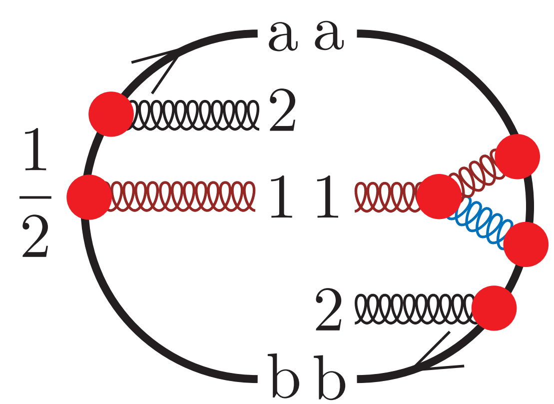


Interference Graphs

This amounts to



The corresponding contribution to $\mathcal{V}(t)$:



This is just a factor of $C_A/4$.

LC+ Approximation

- ✓ LC+ approximation is still an approximation in the color state
- ✓ It can evolve interference contributions.
- ✓ One can start the shower from any non-diagonal color states.
- ✓ The Sudakov exponent is still simple, no need to exponentiate complicated matrix.
- ✗ But we have negative weights.
- ✗ It drops only color suppressed wide angle soft contributions.
- ✓ It is systematically improvable.
- ✓ It can deal with Coulomb gluons.
- ✓ It can be implemented in dipole showers (PYTHIA, SHERPA). *[I think there is a chance to use LC+ approximation antenna shower.]*

Matching at NLO

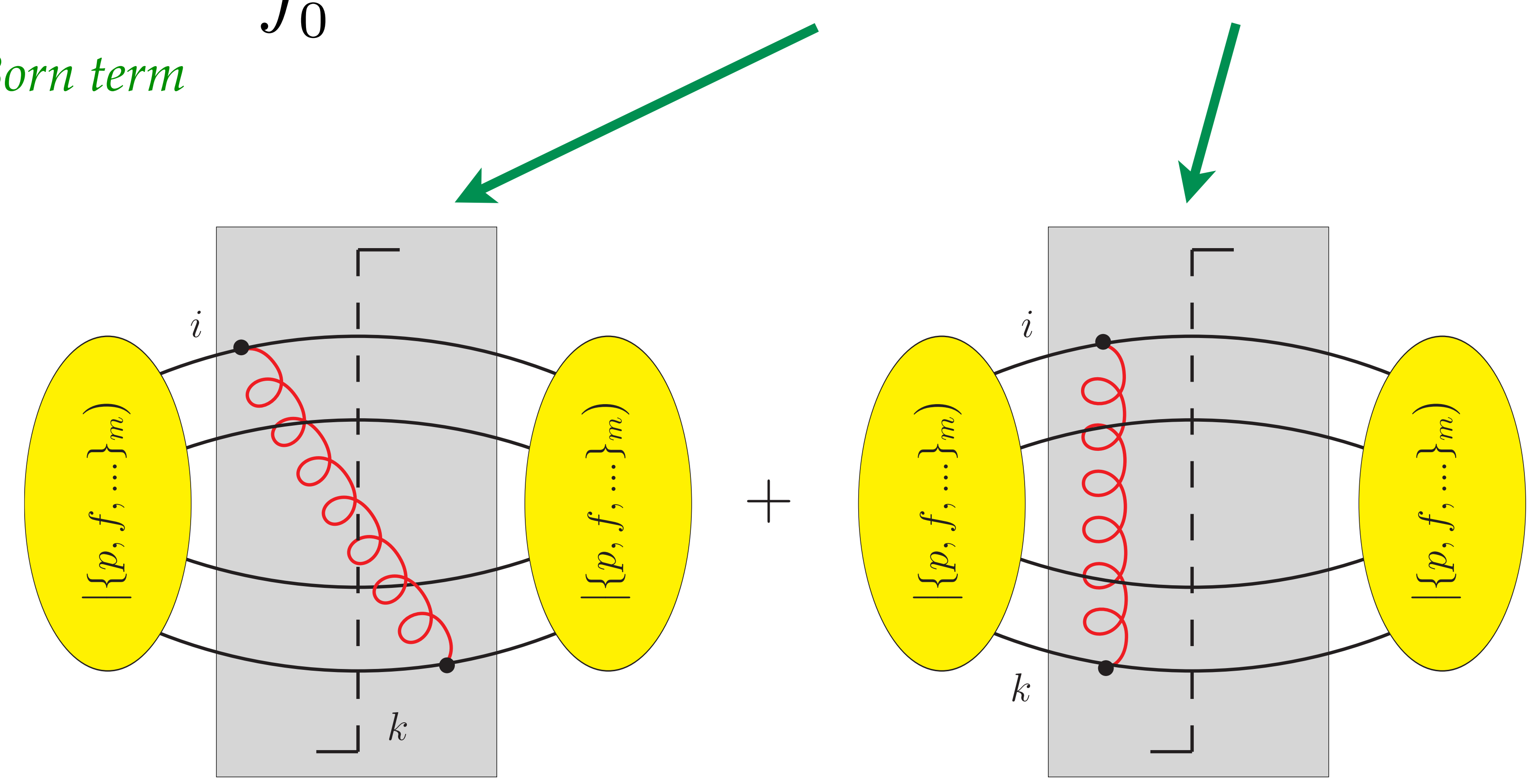
- We want to improve the parton shower with higher multiplicity tree-level and 1-loop level matrix elements.
- At the same time we want to improve the NLO fixed order calculation with parton shower corrections.
- Strictly speaking, it is impossible to do NLO matching with LO partons shower unambiguously. It can be done with NLO level parton shower.
- In the matching procedure we should preserve the “goodness” and the full exclusiveness of the parton shower.
- Expanding the matching formulae in the strong coupling one should obtain the NLO level cross section.
- We should find the general matching/merging formulae based on density operator and make it as *precise* as possible.

Matching

The parton shower starts from the simplest *2→2 like process and* generates the QCD density operator approximately. It would be nice to use *exact tree and 1-loop level amplitudes without double counting and destroying the exclusiveness* of the shower events.

$$\big|\rho(t)\big) = \mathcal{U}(t,0)\big|\rho_0\big) = \big|\rho_0\big) + \int_0^t d\tau\, \mathcal{U}(t,\tau)\left[\mathcal{H}_I(\tau)\big|\rho_0\big) - \mathcal{V}_I(\tau)\big|\rho_0\big)\right]$$

Born term



resolved radiations

unresolved radiation

Matching

The parton shower starts from the simplest *2→2 like process and* generates the QCD density operator approximately. It would be nice to use *exact tree and 1-loop level amplitudes without double counting and destroying the exclusiveness* of the shower events.

$$\big|\rho(t)\big) = \mathcal{U}(t,0)\big|\rho_0\big) + \int_0^t d\tau\, \mathcal{U}(t,\tau)\Big\{ \underbrace{[\mathcal{H}_I(\tau) - \mathcal{V}_I(\tau)]\,\big|\rho_0\big)}_{\approx\, |\rho_R(\tau)) + |\rho_V(\tau))} - [\mathcal{H}_I(\tau) + \mathcal{V}_I(\tau)]\,\big|\rho_0\big)\Big\}$$

$\big|\rho_R(\tau)\big)$: The real contribution is based on the Born level *2→3 amplitudes*

$$\big|\rho_V(\tau)\big) = -\mathcal{V}_I^{(\epsilon)}(\tau)\big|\rho_0\big) + \underbrace{\delta(\tau)\big|\tilde{\rho}_V\big)}$$

Finite part of the 1-loop density operator

$$\lim_{t\to\infty}\int_0^t d\tau \big|\rho_V(\tau)\big) \Leftrightarrow \underbrace{\big|M^{(1)}\big\rangle\big\langle M^{(0)}\big| + c.c.}$$

1-loop density operator with the 1/ε and 1/ε² singularities

Matching

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$$\big|\rho(t)\big) = \mathcal{U}(t,0)\left[\big|\rho_0\big) + \big|\tilde{\rho}_V\big)\right] + \int_0^t d\tau\, \mathcal{U}(t,\tau)\left[\big|\rho_R(\tau)\big) - \mathcal{H}_I(\tau)\big|\rho_0\big)\right]$$

- ✓ This is NLO level matching.
- ✓ Preserves precision and exclusiveness of the shower.
- ✓ This matching is possible because the shower scheme also defines a subtraction scheme to calculate NLO fixed order cross sections.
- ✓ It works only for *2→2 like process.*
- ✓ *No strange Sudakov factor like in POWHEG.*
- ✗ *For higher multiplicity matching we have to work harder.... (and the formalism gets more complicated)*

Naive Matching Formulae

After similar considerations one can derive a matching formulae for higher multiplicities:

Subsequent emissions are softer than any in the hard part

PDF factor

Emissions in the hard part are strongly ordered

$$\big|\psi^{\text{naive}}_{+m}(\tau,\tau_0)\big) = \int_{\tau_0}^{\tau} d\tau_{m+1}\,\mathcal{U}(\tau,\tau_{m+1})\,\mathcal{F}(\tau_{m+1})\int_{\tau_0}^{\tau_{m+1}} d\tau_m \cdots \int_{\tau_0}^{\tau_2} d\tau_1$$
$$\times\Big\{\delta(\tau_{m+1}-\tau_m)\,\Big[\big|\hat\rho^R_m(\tau_m,\ldots,\tau_0)\big)+\big|\hat\rho^F_m(\tau_m,\ldots,\tau_0)\big)\Big]$$
$$+\big|\hat\rho^R_{m+1}(\tau_{m+1},\ldots,\tau_0)\big)-\hat{\mathcal{H}}(\tau_{m+1})\big|\hat\rho^R_m(\tau_m,\ldots,\tau_0)\big)\Big\}$$

Real contribution

Born contribution

Finite part of the 1-loop

Actually this is "quite a good" matching formulae if the measured observable is *m*-jet sensitive and the *m* jets are well separated. This is done with density operators.

Is this compatible with MC@NLO and POWHEG?

Color Averaging

We need an operator to project out a single color from the interference graphs. Thus we define

$$\mathcal{K} = \sum_m \int d\{p,f,c,c\}_{+m} \,\big|\{p,f,c,c\}_{+m}\big)\, p\big(\{p,f,c\}_{+m}\big) \times \sum_{\tilde{c}',\tilde{c}} \big\langle\{c'\}_m\big|\{c\}_m\big\rangle \big(\{p,f,\tilde{c}',\tilde{c}\}_{+m}\big|$$

$$\mathcal{K}\mathcal{K} = \mathcal{K}$$

Probability of choosing a single color flow

The usual choice is based on the tree level color subamplitudes:

$$p\big(\{p,f,c\}_{+m}\big) = \frac{\big|A_0(\{p,f,c\}_{+m})\big|^2}{\sum_{\hat{c}} \big\langle\{\hat{c}\}_m\big|\{\hat{c}\}_m\big\rangle \big|A_0(\{p,f,\hat{c}\}_{+m})\big|^2}$$

This operator washes out all the color correlations:

$$\mathcal{K}\big|\hat{\rho}^R_m(\tau_m,...,\tau_0)\big) = \int d\{p,f,c,c\}_{+m} \,\big|\{p,f,c,c\}_{+m}\big) \times p\big(\{p,f,c\}_{+m}\big)\, \big|M^{(0)}(\{p,f\}_{+m})\big|^2 \times \delta\big(\tau_m,\dots,\tau_0;\{p,f\}_{+m}\big) \;.$$

MC@NLO

As far as I understood MC@NLO is the "color blinded" naive matching formulae. When it was developed the color blinding was essential.

$$\big|\psi_{+m}^{\mathrm{MC@NLO}}(\tau,\tau_0)\big) =$$

$$\mathcal{K}\big|\psi_{+m}^{\mathrm{naive}}(\tau,\tau_0)\big) = \int_{\tau_0}^{\tau} d\tau_{m+1}\,\mathcal{U}^{LC}(\tau,\tau_{m+1})\mathcal{K}\,\mathcal{F}(\tau_{m+1})\int_{\tau_0}^{\tau_{m+1}} d\tau_m \cdots \int_{\tau_0}^{\tau_2} d\tau_1$$
$$\times\Big\{\delta(\tau_{m+1}-\tau_m)\,\big[\big|\hat{\rho}_m^R(\tau_m,...,\tau_0)\big) + \big|\hat{\rho}_m^F(\tau_m,...,\tau_0)\big)\big]$$
$$+\big|\hat{\rho}_{m+1}^R(\tau_{m+1},...,\tau_0)\big) - \hat{\mathcal{H}}(\tau_{m+1})\big|\hat{\rho}_m^R(\tau_m,...,\tau_0)\big)\Big\}$$

The error of this matching formula is estimated by

$$\big|\Delta\psi_{+m}^{\mathrm{MC@NLO}}(\tau,\tau_0)\big) = \int_{\tau_0}^{\tau} d\tau_{m+1}\,\mathcal{F}(\tau_{m+1})\int_{\tau_0}^{\tau_{m+1}} d\tau_m \cdots \int_{\tau_0}^{\tau_2} d\tau_1$$
$$\times\Big[\hat{\mathcal{H}}(\tau_{m+1}) - \hat{\mathcal{V}}(\tau_{m+1}),\,\mathcal{K}\Big]\big|\hat{\rho}_m^R(\tau_m,...,\tau_0)\big)$$

As far as I can see MC@NLO really tried to minimize the error of lacking color evolution.

POWHEG

There are several “variants” of the POWHEG method in the literature, here I discuss the simplified version of the POWHEG, that appears in many SHERPA paper. Starting with the alternative form of the naive matching formulae:

$$\begin{aligned}\big|\psi_{+m}^{\text{naive}}(\tau,\tau_0)\big) = &\int_{\tau_0}^{\tau} d\tau_{m+1}\,\mathcal{U}(\tau,\tau_{m+1})\,\mathcal{F}(\tau_{m+1})\int_{\tau_0}^{\tau_{m+1}} d\tau_m\cdots\int_{\tau_0}^{\tau_2} d\tau_1\\ &\times\Big\{\delta(\tau_{m+1}-\tau_m)\,\Big[\big|\hat{\rho}_m^R(\tau_m,...,\tau_0)\big)+\big|\hat{\rho}_m^F(\tau_m,...,\tau_0)\big)\Big]\\ &\quad+\big|\hat{\rho}_{m+1}^R(\tau_{m+1},...,\tau_0)\big)-\hat{\mathcal{H}}(\tau_{m+1})\big|\hat{\rho}_m^R(\tau_m,...,\tau_0)\big)\Big\}\end{aligned}$$

Let us make some changes (I wouldn’t call them to approximations):

$$\mathcal{H}(\tau)\longrightarrow\mathcal{H}_P(\tau)\;\;,\quad\mathcal{V}(\tau)\longrightarrow\mathcal{V}_P(\tau)\quad\text{and}\quad\mathcal{N}(t,t')\longrightarrow\mathcal{N}_P(t,t')\;\;.$$

POWHEG

There are several "variants" of the POWHEG method in the literature, here I discuss the simplified version of the POWHEG, that appears in many SHERPA paper. Starting with the alternative form of the naive matching formulae:

$$
\begin{aligned}
\big|\psi^{\text{naive}}_{+m}(\tau,\tau_0)\big) = & \int_{\tau_0}^{\tau} d\tau_m \cdots \int_{\tau_0}^{\tau_2} d\tau_1\, \mathcal{N}(\tau,\tau_m)\,\mathcal{F}(\tau_m) \\
& \times \Big[\big|\hat{\rho}^R_m(\tau_m,\ldots,\tau_0)\big) + \big|\hat{\rho}^F_m(\tau_m,\ldots,\tau_0)\big)\Big] \\
& + \int_{\tau_0}^{\tau} d\tau_{m+1} \int_{\tau_0}^{\tau_{m+1}} d\tau_m \cdots \int_{\tau_0}^{\tau_2} d\tau_1\, \mathcal{U}(\tau,\tau_{m+1}) \\
& \quad \times \Big\{\mathcal{F}(\tau_{m+1})\big|\hat{\rho}^R_{m+1}(\tau_{m+1},\ldots,\tau_0)\big) \\
& \qquad - \mathcal{H}(\tau_{m+1})\,\mathcal{F}(\tau_{m+1})\big|\hat{\rho}^R_m(\tau_m,\ldots,\tau_0)\big) \\
& \qquad + \mathcal{H}(\tau_{m+1})\,\mathcal{N}(\tau_{m+1},\tau_m)\,\mathcal{F}(\tau_m) \\
& \qquad\quad \times \Big[\big|\hat{\rho}^R_m(\tau_m,\ldots,\tau_0)\big) + \big|\hat{\rho}^F_m(\tau_m,\ldots,\tau_0)\big)\Big]\Big\} \ .
\end{aligned}
$$

One step of the shower is expaded

Let us make some changes (I wouldn't call them to approximations):

$$
\mathcal{H}(\tau) \longrightarrow \mathcal{H}_P(\tau) \ , \quad \mathcal{V}(\tau) \longrightarrow \mathcal{V}_P(\tau) \quad \text{and} \quad \mathcal{N}(t,t') \longrightarrow \mathcal{N}_P(t,t') \ .
$$

POWHEG

Defining the following function:

$$\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{R}_{MC}(\tau)\big|\{p,f\}_m\big)$$
$$= \sum_{\hat{c}',\hat{c}} \sum_{c',c} \big(\{\hat{p},\hat{f},\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{H}(\tau)\big|\{p,f,c',c\}_m\big)$$
$$\times \int_{-\infty}^{\tau} d\tau_0 \int_{\tau_0}^{\tau} d\tau_m \cdots \int_{\tau_0}^{\tau_2} d\tau_1 \,\big(\{p,f,c',c\}_m\big|\hat{\rho}_m^R(\tau_m,\ldots,\tau_0)\big)$$

Actually this is the subtraction term in the NLO calculation

Real splitting operator:

$$\big(\{\hat{p},\hat{f},\hat{c}',\hat{c}\}_{m+1}\big|\mathcal{H}_P(\tau)\big|\{p,f,c',c\}_m\big)$$
$$= \delta_{\hat{c}',\hat{c}}\, p\big(\{\hat{p},\hat{f},\hat{c}\}_{m+1}\big) \frac{\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{R}_{MC}(\tau)\big|\{p,f\}_m\big)}{B\big(\{p,f\}_m;\tau\big)}$$
$$\times \big(1\big|\{p,f,c',c\}_m\big)$$

Choosing a single m+1 parton color flow .

Summing up m-parton color states

POWHEG

The virtual splitting operator is diagonal:

$$\mathcal{V}_P(\tau)\big|\{p,f,c',c\}_m\big) = \lambda_P\big(\{p,f\}_m;\tau\big)\big|\{p,f,c',c\}_m\big)$$

Where the POWHEG Sudakov exponent is

$$\lambda_M\big(\{p,f\}_m;\tau\big) = \int d\{\hat{p},\hat{f}\}_{m+1}\,\frac{\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{R}_{MC}(\tau)\big|\{p,f\}_m\big)}{B\big(\{p,f\}_m;\tau\big)}$$

- There is no such factorization that is implemented in the splitting operators. It is kind of acceptable only in the strict collinear limit.
- Thus is not an approximation of the “exact” splitting kernel.
- It completely fails for heavy colored objects (*e.g:* top quark), because there is no collinear limit in this case.

Merging

- The idea is to have a "super" shower that has exact high multiplicity matrix element corrections at tree and 1-loop level:

$$\big|\psi^{\rm SUPER}(\tau,\tau_0)\big)$$

- Example: e+e- thrust. It is a 3-jet sensitive observable. The super merging formula gives the proper NLO distribution in the large 1-T region and it is gives the NLL resummation in the small 1-T region

$$\big(1\big|\mathcal{O}(1-T)\big|\psi^{\rm SUPER}(\tau,\tau_0)\big) = \frac{d\sigma}{d(1-T)}$$

- At the same time the "SUPER" shower should be to the NLO total cross section:

$$\int_0^1 d(1-T)\,\big(1\big|\mathcal{O}(1-T)\big|\psi^{\rm SUPER}(\tau,\tau_0)\big) = \sigma_0\left(1+\frac{\alpha_{\rm s}}{\pi}+\cdots\right)$$

Merging

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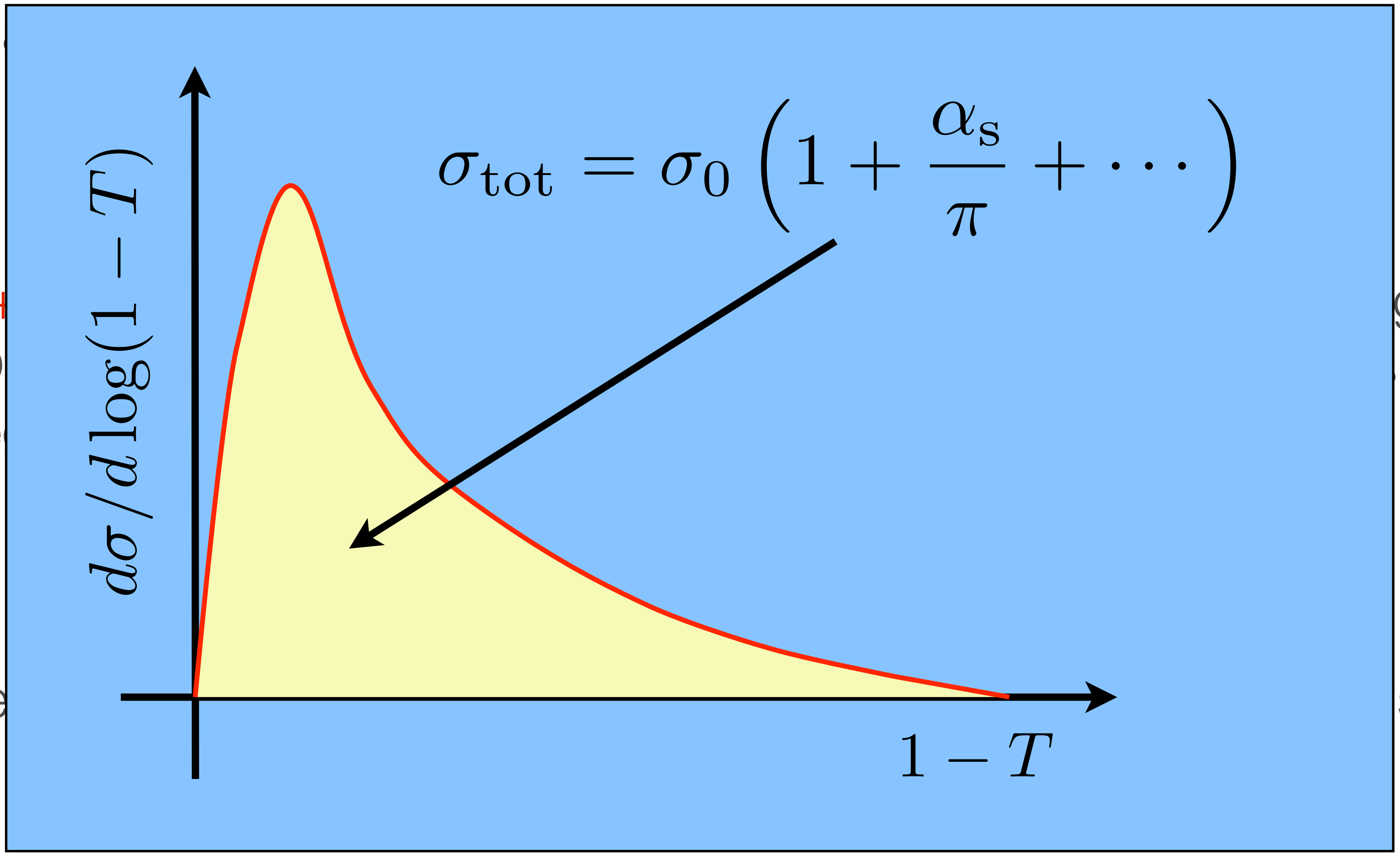

- Example: e+ … g formula gives the proper NLO … ummation in the small 1-T re…

- At the same … section:

$$\int_0^1 d(1-T)\,\big(1\big|\mathcal{O}(1-T)\big|\psi^{\rm SUPER}(\tau,\tau_0)\big) = \sigma_0\left(1+\frac{\alpha_s}{\pi}+\cdots\right)$$

Merging

- The next obvious step is to define a merging formulae. This tries to combine higher multiplicity matrix elements at NLO level (CKKW@NLO)
- Well, it is "easy" just sum up the naive matching formulae. It is OK since the emissions are strongly odered:

$$\big|\psi^{\text{CKKW@NLO}}(\tau,\tau_0)\big) = \sum_{m=0}^{\infty} \big|\psi^{\text{naive}}_{+m}(\tau,\tau_0)\big)$$

$$\big|\psi^{\text{naive}}_{+m}(\tau,\tau_0)\big) = \int_{\tau_0}^{\tau} d\tau_{m+1}\,\mathcal{U}(\tau,\tau_{m+1})\,\mathcal{F}(\tau_{m+1})\int_{\tau_0}^{\tau_{m+1}} d\tau_m \cdots \int_{\tau_0}^{\tau_2} d\tau_1$$
$$\times\Big\{\delta(\tau_{m+1}-\tau_m)\,\Big[\big|\hat\rho^R_m(\tau_m,\dots,\tau_0)\big) + \big|\hat\rho^F_m(\tau_m,\dots,\tau_0)\big)\Big]$$
$$+\big|\hat\rho^R_{m+1}(\tau_{m+1},\dots,\tau_0)\big) - \hat{\mathcal{H}}(\tau_{m+1})\big|\hat\rho^R_m(\tau_m,\dots,\tau_0)\big)\Big\}$$

No Sudakov exponent in the hard matrix elements

- But the hard states should be reweighted by Sudakov factors. This is kind of hard because our Sudakov factors are operator. To do such reweighting requires to recalculate tree and 1-loop amplitudes and nobody wants to do that...

Conclusions

- It is a huge topic....
- ...

Implementation

We calculate Drell-Yan total cross section at 14TeV with $(0.7\text{ GeV})^2 < Q^2 < (1\text{TeV})^2$

$$\sigma_{\text{tot}} = \sigma_{\text{tot}} \left[0.6588 + \underbrace{\frac{1}{N_c^2} 2.097108}_{23\%} + \overbrace{\frac{1}{N_c^4} 6.0241887}^{7.5\%} + \underbrace{\frac{1}{N_c^6} 19.6786989}_{2.7\%} + \cdots \right]$$

The subleading color contributions are not just 10% what we naively expect.

The average transverse momentum of the vector boson is

$$\langle p_T^2 \rangle = 9757.9\text{GeV}^2 \left[0.3958 + \underbrace{\frac{1}{N_c^2} 2.66}_{29.5\%} + \overbrace{\frac{1}{N_c^4} 11.03}^{13.6\%} + \underbrace{\frac{1}{N_c^6} 52.1}_{7.1\%} + \overbrace{\frac{1}{N_c^8} 433.7}^{6.6\%} + \underbrace{\frac{1}{N_c^{10+}} 2042}_{3.4\%} \right]$$

and running a pure leading color shower, the result is

$$\langle p_T^2 \rangle_{\text{LC}} = 9260.641\text{GeV}^2$$