

Replacing 1-loop tensor reduction by contractions

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<http://www.ippp.dur.ac.uk/Workshops/13/RADCOR2013/index.html>



Introduction

Recent more detailed overviews of our approach to tensor reduction:

J. Fleischer, at:

"Frontiers in Perturbative Quantum Field Theory"

10-12 September 2012, Bielefeld University, Germany

http://www2.physik.uni-bielefeld.de/fileadmin/user_upload/workshops/fleischer.pdf

T. Riemann, at:

4th Workshop "HP2 – High Precision for Hard Processes"

4-7 September 2012, Max Planck Institute for Physics, Munich, Germany

<https://indico.mpp.mpg.de/contributionDisplay.py?contribId=27&confId=1369>

T. Riemann, at:

5th Helmholtz International Summer School - Workshop

Dubna International Advanced School of Theoretical Physics - DIAS TH

Calculations for Modern and Future Colliders

July 23 - August 2, 2012, Dubna, Russia

<http://theor.jinr.ru/calc2012/>

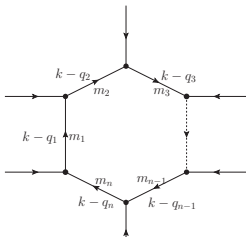
Definitions

n -point tensor integrals of rank R : (n,R) -integrals

$$I_n^{\mu_1 \dots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}},$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\epsilon$$



tensor integrals due to, e.g.:

- fermion propagators
- three-gauge boson couplings

A simple example

1-loop self-energy:

$$I_2^\mu = \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2][(k+p)^2 - M_2^2]}$$

$$\text{Ansatz : } I_2^\mu = p_\mu \cdot B_1(p, M_1, M_2)$$

Solve:

$$\begin{aligned} p_\mu \cdot I_2^\mu &= p^2 \cdot B_1(p, M_1, M_2) \\ &= \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2][(k+p)^2 - M_2^2]} = \int \frac{d^d k}{i\pi^{d/2}} \frac{pk}{D_1 D_2} \\ &= \int \frac{d^d k}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1 D_2} \right], \end{aligned}$$

$$B_1(p, M_1, M_2) = \frac{1}{2p^2} \left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2) \right]$$

A **tensor** Feynman integral may be expressed in terms of **scalar** Feynman integrals.

Passarino-Veltman algorithm

- 1 Contract n -point and R -rank Feynman integral with *external momenta* p_i^μ and with $g^{\mu\nu}$, and cancel propagators
- 2 Invert the resulting system of linear equations
- 3 The result consists of $(n - 1)$ -point and $(R - 1)$ -rank functions

Reducing tensor rank introduces inverse Gram determinant:

$$I_5^{\mu_1 \dots \mu_{R-1} \mu_R} = \sum_{i=1}^5 \frac{q_i^{\mu_R}}{\det(G_5)} \left[A_{0i} I_5^{\mu_1 \dots \mu_{R-1}} - \sum_{s=1}^5 A_{si} I_4^{\mu_1 \dots \mu_{R-1}, s} \right]$$

Gram determinant G_n :

$$G_n = |2q_i q_j|, \quad i, j = 1, \dots, n-1 \quad (1)$$

and A_{0i} , A_{si} are kinematic coefficients. The q_i are **internal** momenta.

Systematic approach to tensor reductions:

1,2,3,4-point functions:

- Passarino, Veltman 1978 [1]

Open-source Source-open programs for 5,6-point reductions:

- LoopTools/FF ($n \leq 5, rank \leq 4$), T. Hahn [2, 3] 1998,1990.
- Golem95 T. Binoth et al. [4] 2008
- PJFry ($n \leq 5, rank \leq 5$), V. Yundin et al. [5, 6] July 2011

Need in addition a library of scalar functions:

- 't Hooft, Veltman 1979 [7]
- LoopTools/FF T. Hahn [2, 3] 1998,1990
- QCDloop/FF K. Ellis and G. Zanderighi [8, 3] 2007,1990
- OneLooP (complex masses) van Hameren [9] 2010

This talk: Efficient contraction (not: reduction) formulae in the algebraic Davydychev-Tarasov-Fleischer-Jegerlehner-TR approach

- Get $n > 4$ tensor reduction with \dots :
- \dots arbitrary masses
- \dots inverse pentagon Gram determinants killed – if wanted
- \dots full kinematics treated, also with small inverse sub-diagram Gram determinants

- **new:** \dots multiple sums over tensor coefficients made efficient by **contracting with external momenta**

Fleischer, TR [10] PLB 701(2011)646 + further simplifications by A. Almasy

Program: **OLEC** (C++), I. Dubovik, J. Fleischer, J. Gluza, M. Gluza, TR [11]

will include **CONTRACTIONS** (F95), Andrea Almasy, J. Fleischer, TR [12]

- \dots higher n point functions, $n \geq 7$ Fleischer, TR [13] PLB 707(2012)375
Programs: to be done

History of the Approach - not a complete list of references

- [14] Melrose 1965: Reduction of Feynman diagrams and Cayley determinants
- [15] Davydychev 1991: Integrals in different space-time dimension.
- [16] See also Bern et al. 1993
- [17] Tarasov 1996: Dimensional recurrence relations
- [18] Fleischer, Jegerlehner, Tarasov 2000: 1-loop reductions and signed minors.
 - [4] Binoth, Guillet, Heinrich, Pilon, Schubert, 2005: Algebraic/numerical formalism for one-loop multi-leg amplitudes
- [19] Fleischer and T. Riemann (since 2007) 2011: Complete reduction of 1-loop tensors.
 - See also Diakonidis et al. [20] 2009 and [21] 2009
- [22] Yundin's package PJFry 2011; <https://github.com/Vayu/PJFry>.
 - See also Fleischer, TR, Yundin [5, 6]
- [10] Fleischer and T. Riemann 2011: Contracted tensor Feynman integrals.
- [23] Fleischer and T. Riemann 2012: A solution for tensor reduction of one-loop n -point functions with $n \geq 6$

Tensor integrals expressed in terms of scalar integrals in higher dimensions

$D = d + 2l = 4 - 2\epsilon, 6 - 2\epsilon, \dots$ [Davydychev:1991], also [Fleischer et al.:2000]

$$n_{ij} = \nu_{ij} = 1 + \delta_{ij}, n_{ijk} = \nu_{ij}\nu_{ijk}, \nu_{ijk} = 1 + \delta_{ik} + \delta_{jk}$$

$$I_n^\mu = \int^d k^\mu \prod_{r=1}^n c_r^{-1} = - \sum_{i=1}^n q_i^\mu I_{n,i}^{[d+]}$$

$$I_n^{\mu\nu} = \int^d k^\mu k^\nu \prod_{r=1}^n c_r^{-1} = \sum_{i,j=1}^n q_i^\mu q_j^\nu n_{ij} I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} I_n^{[d+]}$$

$$I_n^{\mu\nu\lambda} = \int^d k^\mu k^\nu k^\lambda \prod_{r=1}^n c_r^{-1} = - \sum_{i,j,k=1}^n q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^n g^{\mu\nu} q_i^\lambda I_{n,i}^{[d+]^2}$$

$$\begin{aligned}
 I_n^{\mu\nu\lambda\rho} &= \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu k^\nu k^\lambda k^\rho}{\prod_{r=1}^n c_r} = \sum_{i,j,k,l=1}^n q_i^\mu q_j^\nu q_k^\lambda q_l^\rho n_{ijkl} I_{n,ijkl}^{[d+]^4} \\
 &\quad - \frac{1}{2} \sum_{i,j=1}^n g^{\mu\nu} q_i^\lambda q_j^\rho n_{ij} I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{\mu\nu} g^{\lambda\rho} I_n^{[d+]^2}
 \end{aligned} \tag{2}$$

$$\begin{aligned}
I_n^{\mu\nu\lambda\rho\sigma} &= \int \frac{d^d k}{i\pi^{d/2}} \frac{k^\mu k^\nu k^\lambda k^\rho k^\sigma}{\prod_{j=1}^n c_j} \\
&= - \sum_{i,j,k,l,m=1}^n q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma n_{ijklm} I_{n,ijklm}^{[d+]}{}^5 \\
&\quad + \frac{1}{2} \sum_{i,j,k=1}^n g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^\sigma] n_{ijk} I_{n,ijk}^{[d+]}{}^4 - \frac{1}{4} \sum_{i=1}^n g^{[\mu\nu} g^{\lambda\rho} q_i^\sigma] I_{n,i}^{[d+]}{}^3.
\end{aligned}$$

The integrals are defined in $[d+]^l = 4 - 2\epsilon + 2l$ dimensions.

$I_{n-1,ab}^{\{\mu_1, \dots\}, s}$, $a, b \neq s$ is obtained from $I_n^{\{\mu_1, \dots\}}$

by

- shrinking line s
- raising the powers of inverse propagators a, b .

Notations: Gram and modified Cayley determinant, signed minors

[Melrose:1965]

Gram determinant G_n :

$$G_n = |2q_i q_j|, i, j = 1, \dots, n-1 \quad (3)$$

Modified Cayley determinant $(\)_N$ of a diagram with N internal lines and chords q_j :

$$(\)_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix} \quad (4)$$

with the matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \dots N) \quad (5)$$

The propagators are: $c_i = (k - q_i)^2 - m_i^2$ For the choice $q_n = 0$, both determinants are related:

$$(\)_N = -G_N$$

 \Rightarrow The modified Cayley determinant $(\)_N$ does not depend on masses.

Notations: signed minors [Melrose:1965]

signed minors of $(\)_N$ are constructed by deleting m rows and m columns from $(\)_N$, and multiplying with a sign factor:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N &\equiv \\ &\equiv (-1)^{\sum_i (j_i + k_i)} \operatorname{sgn}_{\{j\}} \operatorname{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \end{aligned} \quad (6)$$

where $\operatorname{sgn}_{\{j\}}$ and $\operatorname{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N \equiv \begin{vmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{12} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \cdots & Y_{NN} \end{vmatrix}, \quad (7)$$

Tarasov's dimensional recurrences for scalars

Following [Tarasov:1996 [17], Fleischer:1999 [18]]
 apply **recurrence relations**, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

shift dimension + index:

$$\nu_j(j^+ l_5^{[d+]}) = \frac{1}{\binom{0}{5}} \left[-\binom{j}{0}_5 + \sum_{k=1}^5 \binom{j}{k}_5 \mathbf{k}^- \right] l_5 \quad (8)$$

shift dimension:

$$\left(d - \sum_{i=1}^5 \nu_i + 1 \right) l_5^{[d+]} = \frac{1}{\binom{0}{5}} \left[\binom{0}{0}_5 - \sum_{k=1}^5 \binom{0}{k}_5 \mathbf{k}^- \right] l_5, \quad (9)$$

shift index:

$$\nu_j j^+ l_5 = \frac{1}{\binom{0}{0}_5} \sum_{k=1}^5 \binom{0j}{0k}_5 \left[d - \sum_{i=1}^5 \nu_i (\mathbf{k}^- i^+ + 1) \right] l_5 \quad (10)$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1 .

Recursions for tensors – Alternative to dimensional recurrences of scalars

Express any $(5, R)$ pentagon by a $(5, R - 1)$ pentagon plus $(4, R - 1)$ boxes

[Diakonidis, Fleischer, T. Riemann, Tausk: Phys. Lett. **B683** (2010) [21]]

5-point tensor recursion:

$$I_5^{\mu_1 \dots \mu_{R-1} \mu} = I_5^{\mu_1 \dots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1 \dots \mu_{R-1}, s} Q_s^\mu,$$

For $n = 6, 7, 8, \dots$ things are close but differ a bit.

auxiliary vectors **with inverse Gram determinants**

$$Q_s^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{s}{i}_5}{\binom{s}{i}_5}, \quad s = 0, \dots, 5$$

For e.g. $R = 3$, again $[1/\binom{s}{i}_5]^3$ will occur.

Contractions

One may combine now *Tarasov's dimensional recurrence relations* and the *tensor rank recursions* in order to derive especially useful representations.

After that, we will perform contractions with external momenta.

So, the following equations are symbolic:

$$q_{i_1 \mu_1} \cdots q_{i_R \mu_R} I_5^{\mu_1 \cdots \mu_R} = \int \frac{d^d k}{i\pi^{d/2}} \frac{\prod_{r=1}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j},$$

$$g_{\mu_1, \mu_2} q_{i_1 \mu_3} \cdots q_{i_R \mu_R} I_5^{\mu_1 \cdots \mu_R} \neq \int \frac{k^2 d^d k}{i\pi^{d/2}} \frac{\prod_{r=3}^R (q_{i_r} \cdot k)}{\prod_{j=1}^5 c_j}$$

One may arrange a one-loop calculation such that all the one-loop integrals appear **only** in such contractions.

Important:

The contraction with g_{μ_1, μ_2} is shown here in a symbolic form; in practice we work strictly 4-dimensional with g_{μ_1, μ_2} .

One option was to avoid the appearance of inverse Gram determinants $1/(\)_5$.

For rank $R = 5$, e.g.:

$$I_5^{\mu\nu\lambda\rho\sigma} = \sum_{s=1}^5 \left[\sum_{i,j,k,l,m=1}^5 q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma E_{ijklm}^s + \sum_{i,j,k=1}^5 g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} E_{00ijk}^s + \sum_{i=1}^5 g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E_{0000i}^s \right] \quad (11)$$

The tensor coefficients are expressed in terms of integrals $I_{4,i\dots}^{[d+],s}$, e.g.:

$$E_{ijklm}^s = -\frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0l}{sm}_5 n_{ijk} I_{4,ijk}^{[d+],s} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l) \right] + \binom{0s}{0m}_5 n_{ijkl} I_{4,ijkl}^{[d+],s} \right\}.$$

Now, in a next step, one may avoid the appearance of inverse sub-Gram determinants $(\)_4$.

Further, the complete dependence on the indices i of the tensor coefficients is contained now in the pre-factors with signed minors.

One can say that **the indices decouple from the integrals**.

As an example, we reproduce the 4-point part of $I_{4,ijkl}^{[d+]}$:

$$\begin{aligned}
 n_{ijkl} I_{4,ijkl}^{[d+]} &= \frac{\binom{0}{i} \binom{0}{j} \binom{0}{k} \binom{0}{l}}{\binom{0}{0} \binom{0}{0} \binom{0}{0} \binom{0}{0}} d(d+1)(d+2)(d+3) I_4^{[d+]} \\
 &+ \frac{\binom{0i}{0j} \binom{0}{k} \binom{0}{l} + \binom{0i}{0k} \binom{0}{j} \binom{0}{l} + \binom{0j}{0k} \binom{0}{i} \binom{0}{l} + \binom{0i}{0l} \binom{0}{j} \binom{0}{k} + \binom{0j}{0l} \binom{0}{i} \binom{0}{k} + \binom{0k}{0l} \binom{0}{i} \binom{0}{j}}{\binom{0}{0}^3} \\
 &\times d(d+1) I_4^{[d+]} \\
 &+ \frac{\binom{0i}{0l} \binom{0j}{0k} + \binom{0j}{0l} \binom{0i}{0k} + \binom{0k}{0l} \binom{0i}{0j}}{\binom{0}{0}^2} I_4^{[d+]} + \dots
 \end{aligned} \tag{12}$$

In (12), one has to understand the 4-point integrals to carry the corresponding index s and the signed minors are $\binom{0}{k} \rightarrow \binom{0s}{ks}_5$ etc.

✓ no scalar 5-point integrals in higher dimensions

✓ no inverse Gram det. $\binom{0}{5}$

✓ **4-point integrals without indices**

† scalar 4-point integrals in higher dimensions: $I_4^{[d+],s}$ etc.

† inverse Gram det. $\binom{0}{5} \equiv \binom{0}{4}$

Contractions for the 5-point functions with rank $R = 1$

A chord is the momentum shift of an internal line due to external momenta, $D_i = (k - q_i)^2 - m_i^2 + i\epsilon$, and $q_i = (p_1 + p_2 + \dots + p_i)$, with $q_n = 0$.

The tensor 5-point integral of rank $R = 1$ is ([19], eq. (4.6)):

$$\begin{aligned}
 I_5^\mu &= - \sum_{i=1}^5 q_i^\mu I_{5,i}^{[d+]} \\
 &= - \sum_{i=1}^4 q_i^\mu \sum_{s=1}^5 \frac{\binom{0i}{0s}_5}{\binom{0}{0}_5} I_4^s
 \end{aligned} \tag{13}$$

This yields, when contracted with a chord,

$$q_{a\mu} I_5^\mu = - \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \left[\sum_{i=1}^4 (q_a \cdot q_i) \binom{0i}{0s}_5 \right] I_4^s. \tag{14}$$

In fact, the sum over i may be performed explicitly:

$$\Sigma_a^{1,s} \equiv \sum_{i=1}^4 (q_a \cdot q_i) \begin{pmatrix} 0s \\ 0i \end{pmatrix}_5 = +\frac{1}{2} \left\{ \begin{pmatrix} s \\ 0 \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\},$$

We get immediately

$$q_{a\mu} l_5^\mu = -\frac{1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5} \sum_{s=1}^5 \Sigma_a^{1,s} l_4^s. \quad (15)$$

Contractions for the 5-point functions with rank $R = 2$

$$I_5^{\mu\nu} = \sum_{i,j=1}^4 q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \quad (16)$$

has the following tensor coefficients free of $1/()_5$:

$$E_{00} = - \sum_{s=1}^5 \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, \quad (17)$$

$$E_{ij} = \sum_{s=1}^5 \frac{1}{\binom{0}{0}_5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. \quad (18)$$

Equation (16) yields for the contractions with chords:

$$q_{a\mu} q_{b\nu} l_5^{\mu\nu} = \sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) E_{ij} + (q_a \cdot q_b) E_{00}. \quad (19)$$

and finally (19) simply reads

$$\begin{aligned} q_{a\mu} q_{b\nu} l_5^{\mu\nu} &= \frac{1}{4} \sum_{s=1}^5 \left\{ \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} (\delta_{ab} \delta_{as} + \delta_{5s}) + \frac{\binom{s}{0}_5}{\binom{0s}{0s}_5} [(\delta_{as} - \delta_{5s})(Y_{b5} - Y_{55}) \right. \\ &\quad \left. + (\delta_{bs} - \delta_{5s})(Y_{a5} - Y_{55}) + \frac{\binom{s}{0}_5}{\binom{0}{0}_5} (Y_{a5} - Y_{55})(Y_{b5} - Y_{55})] \right\} l_4^{[d+],s} \\ &\quad + \frac{1}{\binom{0}{0}_5} \sum_{s=1}^5 \frac{\sum_b^{1,s}}{\binom{0s}{0s}_5} \sum_{t=1}^5 \sum_a^{2,st} l_3^{st}, \end{aligned}$$

with

$$\begin{aligned}\Sigma_a^{2,st} &\equiv \sum_{i=1}^4 (q_a \cdot q_i) \begin{pmatrix} 0st \\ 0si \end{pmatrix}_5 \\ &= \frac{1}{2} (1 - \delta_{st}) \left\{ \begin{pmatrix} ts \\ 0s \end{pmatrix}_5 (Y_{a5} - Y_{55}) + \begin{pmatrix} 0s \\ 0s \end{pmatrix}_5 (\delta_{at} - \delta_{5t}) - \begin{pmatrix} 0s \\ 0t \end{pmatrix}_5 (\delta_{as} - \delta_{5s}) \right\}\end{aligned}$$

This has been extended also to higher ranks.

We need at most double sums, e.g.:

$$\begin{aligned}\Sigma_{ab}^{2,s} &\equiv \sum_{i,j=1}^4 (q_a \cdot q_i)(q_b \cdot q_j) \begin{pmatrix} si \\ sj \end{pmatrix}_5 \\ &= \frac{1}{2} (q_a \cdot q_b) \begin{pmatrix} s \\ s \end{pmatrix}_5 - \frac{1}{4} ()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \quad (20)\end{aligned}$$

The sums over signed minors, weighted with scalar products of chords are given in J. Fleischer, T.R., PLB 2011 [10].

Contractions for the 5-point functions with rank $R = 4$

$$I_5^{\mu\nu\lambda\rho} = I_5^{\mu\nu\lambda} \cdot Q_0^\rho - \sum_{s=1}^5 I_4^{\mu\nu\lambda,s} \cdot Q_s^\rho. \quad (21)$$

Contracted with chords (differences of external momenta):

$$q_a^\mu q_b^\nu q_c^\lambda q_d^\rho I_5^{\mu\nu\lambda\rho} = q_a^\mu q_b^\nu q_c^\lambda I_5^{\mu\nu\lambda} (q_d^\rho Q_0^\rho) - C_{5,abcd} \quad (22)$$

Here:

$$Q_s^\rho = \sum_{i=1}^5 q_i^\rho \frac{\binom{s}{i}_5}{\binom{0}{5}}, \quad s = 0 \dots 5 \quad (23)$$

The first term $q_a^\mu q_b^\nu q_c^\lambda I_5^{\mu\nu\lambda}$ is known, the second term has to be determined:

$$C_{5,abcd} = - \sum_{s=1}^5 q_{a\mu} q_{b\nu} q_{c\lambda} I_4^{\mu\nu\lambda,s} \frac{1}{2} (\delta_{ds} - \delta_{5s}), \quad (24)$$

becomes:

$$\begin{aligned}
C_{5,abcd} = & \frac{1}{16} \left\{ G^5 + \delta_{ab}\delta_{ac}\delta_{ad}G^d - I_1^{5abc} - I_1^{5abd} - I_1^{5acd} - I_1^{5bcd} + I_1^{abcd} - J_3^{a5} - J_3^{b5} - J_3^{c5} - J_3^{d5} \right. \\
& + R^{5ab} + R^{5ac} + R^{5bc} + R^{5da} + R^{5db} + R^{5dc} + \delta_{bc}\delta_{bd} \left(J_3^{ad} - J_3^{5d} \right) + \delta_{ac}\delta_{ad} \left(J_3^{bd} - J_3^{5d} \right) \\
& + \delta_{ab}\delta_{ad} \left(J_3^{cd} - J_3^{5d} \right) + \delta_{ab}\delta_{ac} \left(J_3^{dc} - J_3^{5c} \right) + \delta_{ab}\delta_{cd}\tilde{J}_3^{db} + \delta_{ad}\delta_{bc}\tilde{J}_3^{dc} + \delta_{ac}\delta_{bd}\tilde{J}_3^{dc} \\
& + \delta_{ab} \left(\tilde{J}_3^{5b} - R^{b5c} - R^{bd5} + R^{bdc} \right) + \delta_{ac} \left(\tilde{J}_3^{5c} - R^{c5b} - R^{cd5} + R^{cdb} \right) \\
& + \delta_{ad} \left(\tilde{J}_3^{d5} - R^{d5b} - R^{d5c} + R^{dbc} \right) + \delta_{bc} \left(\tilde{J}_3^{5c} - R^{c5a} - R^{cd5} + R^{cda} \right) \\
& + \delta_{bd} \left(\tilde{J}_3^{d5} - R^{d5a} - R^{d5c} + R^{dac} \right) + \delta_{cd} \left(\tilde{J}_3^{d5} - R^{d5a} - R^{d5b} + R^{dab} \right) \\
& + \delta_{ab}\delta_{ad}J_4^d Y_c + \delta_{ac}\delta_{ad}J_4^d Y_b + \delta_{bc}\delta_{bd}J_4^d Y_a + \delta_{ab} \left(R^{bd} - R^{b5} \right) Y_c + \delta_{ac} \left(R^{cd} - R^{c5} \right) Y_b \\
& + \delta_{ad} \left(R^{dc} - R^{d5} \right) Y_b + \delta_{ad} \left(R^{db} - R^{d5} \right) Y_c + \delta_{bc} \left(R^{cd} - R^{c5} \right) Y_a + \delta_{bd} \left(R^{dc} - R^{d5} \right) Y_a \\
& + \delta_{bd} \left(R^{da} - R^{d5} \right) Y_c + \delta_{cd} \left(R^{da} - R^{d5} \right) Y_b + \delta_{cd} \left(R^{db} - R^{d5} \right) Y_a + \left(I_4^d - I_4^5 \right) Y_a Y_b Y_c \\
& + \left(I_3^{cd} - I_3^{5c} - I_3^{5d} + R^5 + \delta_{cd}R^d \right) Y_a Y_b + \left(I_3^{bd} - I_3^{5b} - I_3^{5d} + R^5 + \delta_{bd}R^d \right) Y_a Y_c \\
& + \left(I_3^{ad} - I_3^{5a} - I_3^{5d} + R^5 + \delta_{ad}R^d \right) Y_b Y_c + \left(I_2^{bcd} - I_2^{5bc} - I_2^{5bd} - I_2^{5cd} - J_4^5 + R^{5b} + R^{5c} + R^{5d} \right) Y_a \\
& + \left(I_2^{acd} - I_2^{5ac} - I_2^{5ad} - I_2^{5cd} - J_4^5 + R^{5a} + R^{5c} + R^{5d} \right) Y_b \\
& \left. + \left(I_2^{abd} - I_2^{5ab} - I_2^{5ad} - I_2^{5bd} - J_4^5 + R^{5a} + R^{5b} + R^{5d} \right) Y_c \right\} \tag{25}
\end{aligned}$$

where we have introduced:

$$J_3^{st} \equiv \frac{1}{\binom{st}{st}_5} \left\{ -\binom{S}{S}_5 I_3^{[d+],st} + \binom{ts}{0S}_5 R^{ts} - \sum_{u=1}^5 \binom{ts}{uS}_5 R^{tsu} \right\}, \quad (26)$$

$$\tilde{J}_3^{st} \equiv \frac{1}{\binom{st}{st}_5} \left\{ \binom{S}{t}_5 I_3^{[d+],st} + \binom{St}{0t}_5 R^{ts} - \sum_{u=1}^5 \binom{St}{ut}_5 R^{tsu} \right\}, \quad (27)$$

$$G^s \equiv \frac{1}{\binom{s}{s}_5} \left\{ -2\binom{}{}_5 R^{[d+],s} + \binom{S}{0}_5 J_4^s - \sum_{t=1}^5 \binom{S}{t}_5 J_3^{ts} \right\}. \quad (28)$$

J_4^s and $R^{[d+],s}$ are given in Eqs. (2.24) and (2.44) of [19] respectively.

We assume throughout that $q_5 = 0$, where q_1, \dots, q_5 are chords – differences of external momenta.

Further abbreviations (see (2.24, 2.49, 2.9, 2.17, 2.34, 2.41) of [19]) :

$$J_4^s \equiv \frac{1}{\binom{s}{s}_5} \left\{ -()_5 I_4^{[d+],s} + \binom{s}{0}_5 R^s - \sum_{t=1}^5 \binom{s}{t}_5 R^{st} \right\} \quad (29)$$

$$= \frac{-1}{\binom{0s}{0s}_5} \left\{ ()_5 (d-2)(d-1) I_4^{[d+],2,s} - \binom{0}{0}_5 I_4^{[d+],s} + \sum_{t=1}^5 \binom{t}{0}_5 (d-2) I_3^{[d+],st} + \sum_{t=1}^5 \binom{0s}{0t}_5 R^{st} \right\}$$

$$R^{[d+],s} = \frac{1}{\binom{s}{s}_5} \left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^5 \binom{s}{t}_5 I_3^{[d+],st} \right] = \frac{1}{\binom{0s}{0s}_5} \left[\binom{s}{0}_5 (d-1) I_4^{[d+],2,s} - \sum_{t=1}^5 \binom{0s}{0t}_5 I_3^{[d+],st} \right] \quad (30)$$

$$R^s \equiv \frac{1}{\binom{s}{s}_5} \left[\binom{s}{0}_5 I_4^s - \sum_{t=1}^5 \binom{s}{t}_5 I_3^t \right] = \frac{1}{\binom{0s}{0s}_5} \left[\binom{s}{0}_5 I_4^{[d+],s} - \sum_{t=1}^5 \binom{0s}{0t}_5 I_3^{st} \right]. \quad (31)$$

$$R^{st} \equiv \frac{1}{\binom{st}{st}_5} \left[\binom{st}{0t}_5 I_3^{st} - \sum_{u=1}^5 \binom{st}{ut}_5 I_2^{stu} \right] = \frac{1}{\binom{0st}{0st}_5} \left[\binom{st}{0t}_5 (d-2) I_3^{[d+],st} - \sum_{u=1}^5 \binom{0st}{0ut}_5 I_2^{stu} \right]. \quad (32)$$

$$R^{tsu} = \frac{1}{\binom{tsu}{tsu}_5} \left[\binom{tsu}{0su}_5 I_2^{tsu} - \sum_{v=1}^5 \binom{tsu}{vsu}_5 I_1^{stuv} \right]$$

$$= \frac{1}{\binom{0tsu}{0tsu}_5} \left[\binom{tsu}{0su}_5 (d-1) I_2^{[d+],tsu} - \sum_{v=1}^5 \binom{0tsu}{0vsu}_5 I_1^{stuv} \right]. \quad (33)$$

$$Y_a = Y_{a5} - Y_{55}, \quad Y_{ab} = -(q_a - q_b)^2 + m_a^2 + m_b^2 \quad (34)$$

Contractions for the 5-point functions with rank $R = 5$ – NEW, to be tested yet – I

For the rank $R = 5$ case $I_5^{\mu\nu\lambda\rho\sigma}$, we have three types of contractions:

- contraction with momenta, $q_{a\mu} q_{b\nu} q_{c\lambda} q_{d\rho} q_{e\sigma}$
- contractions with momenta and **one metric tensor**, $q_{a\mu} q_{b\nu} q_{c\lambda} g_{\rho\sigma}$
- contractions with momenta and **two metric tensors**, $q_{a\mu} g_{\nu\lambda} g_{\rho\sigma}$

The contraction with momenta yields the following final expression:

$$\begin{aligned} CE5(a, b, c, d, e) &\equiv q_{a\mu} q_{b\nu} q_{c\lambda} q_{d\rho} q_{e\sigma} I_5^{\mu\nu\lambda\rho\sigma} \\ &= -\frac{1}{2} CE4(a, b, c, d) Y_e + C_{5,abcde} \end{aligned} \quad (35)$$

where

$$C_{5,abcde} = -\sum_{s=1}^5 q_{a\mu} q_{b\nu} q_{c\lambda} q_{d\rho} I_4^{\mu\nu\lambda\rho,s} \frac{1}{2} (\delta_{es} - \delta_{5s}),$$

is explicitly evaluated to be:

$$\begin{aligned}
C_{5,abcde} = & \frac{1}{32} \left\{ \delta_{ab}\delta_{ac}\delta_{ad}\delta_{ae}F^e - F^1 + J_2^{5ea} + J_2^{5eb} + J_2^{5ec} + J_2^{ab5} + J_2^{ac5} + J_2^{ad5} + J_2^{bc5} + J_2^{bd5} \right. \\
& + J_2^{cd5} + J_2^{de5} - G^{a5} - G^{b5} - G^{c5} - G^{d5} - G^{e5} - R^{5abc} - R^{5abd} - R^{5abe} - R^{5acd} - R^{5ace} - R^{5ade} \\
& - R^{5bcd} - R^{5bce} - R^{5bde} - R^{5cde} - \delta_{ae}\delta_{bc}\delta_{bd}G_1^{de} - \delta_{ac}\delta_{ad}\delta_{be}G_1^{de} - \delta_{ab}\delta_{ad}\delta_{ce}G_1^{de} \\
& - \delta_{ab}\delta_{cd}\delta_{ce}G_1^{eb} - \delta_{ad}\delta_{ae}\delta_{bc}G_1^{ec} - \delta_{ac}\delta_{bd}\delta_{be}G_1^{ec} + \delta_{bc}\delta_{bd}\delta_{be} \left(G^{5e} - G^{ae} - G^e Y_a \right) \\
& + \delta_{ac}\delta_{ad}\delta_{ae} \left(G^{5e} - G^{be} - G^e Y_b \right) + \delta_{ab}\delta_{ad}\delta_{ae} \left(G^{5e} - G^{ce} - G^e Y_c \right) + \delta_{ab}\delta_{ac}\delta_{ae} \left(G^{5e} - G^{de} - G^e Y_d \right) \\
& + \delta_{ab}\delta_{ac}\delta_{ad} \left(G^{5d} - G^{ed} \right) - \delta_{ac}\delta_{ae}\delta_{bd}\tilde{G}^{de} - \delta_{ad}\delta_{bc}\delta_{be}\tilde{G}^{de} - \delta_{ab}\delta_{ae}\delta_{cd}\tilde{G}^{de} - \delta_{ab}\delta_{ac}\delta_{de}\tilde{G}^{ec} \\
& + \delta_{bc}\delta_{bd} \left[J_2^{5ad} + J_2^{5ed} - J_2^{aed} - G_1^{d5} + \left(J_3^{5d} - J_3^{ed} \right) Y_a \right] \\
& + \delta_{ac}\delta_{ad} \left[J_2^{5bd} + J_2^{5ed} - J_2^{bed} - G_1^{d5} + \left(J_3^{5d} - J_3^{ed} \right) Y_b \right] \\
& + \delta_{ab}\delta_{ad} \left[J_2^{5cd} + J_2^{5ed} - J_2^{ced} - G_1^{d5} + \left(J_3^{5d} - J_3^{ed} \right) Y_c \right] \\
& + \delta_{cd}\delta_{ce} \left[J_2^{5ae} + J_2^{5be} - J_2^{abe} - G_1^{e5} + \left(J_3^{5e} - J_3^{be} \right) Y_a + \left(J_3^{5e} - J_3^{ae} \right) Y_b - J_4^e Y_a Y_b \right] \\
& + \delta_{bd}\delta_{be} \left[J_2^{5ae} + J_2^{5ce} - J_2^{ace} - G_1^{e5} + \left(J_3^{5e} - J_3^{ce} \right) Y_a + \left(J_3^{5e} - J_3^{ae} \right) Y_c - J_4^e Y_a Y_c \right] \\
& + \delta_{ad}\delta_{ae} \left[J_2^{5be} + J_2^{5ce} - J_2^{bce} - G_1^{e5} + \left(J_3^{5e} - J_3^{ce} \right) Y_b + \left(J_3^{5e} - J_3^{be} \right) Y_c - J_4^e Y_b Y_c \right] \\
& + \delta_{bc}\delta_{de} \left(\tilde{J}_2^{5ce} - \tilde{J}_2^{ace} - \tilde{J}_3^{ce} Y_a \right) + \delta_{be}\delta_{cd} \left(\tilde{J}_2^{5de} - \tilde{J}_2^{ade} - \tilde{J}_3^{de} Y_a \right) + \delta_{bd}\delta_{ce} \left(\tilde{J}_2^{5de} - \tilde{J}_2^{ade} - \tilde{J}_3^{de} Y_a \right) \\
& + \delta_{ac}\delta_{de} \left(\tilde{J}_2^{5ce} - \tilde{J}_2^{bce} - \tilde{J}_3^{ce} Y_b \right) + \delta_{ae}\delta_{cd} \left(\tilde{J}_2^{5de} - \tilde{J}_2^{bde} - \tilde{J}_3^{de} Y_b \right) + \delta_{ad}\delta_{ce} \left(\tilde{J}_2^{5de} - \tilde{J}_2^{bde} - \tilde{J}_3^{de} Y_b \right) \\
& + \delta_{ab}\delta_{de} \left(\tilde{J}_2^{5be} - \tilde{J}_2^{cbe} - \tilde{J}_3^{be} Y_c \right) + \delta_{ae}\delta_{bd} \left(\tilde{J}_2^{5de} - \tilde{J}_2^{cde} - \tilde{J}_3^{de} Y_c \right) + \delta_{ad}\delta_{be} \left(\tilde{J}_2^{5de} - \tilde{J}_2^{cde} - \tilde{J}_3^{de} Y_c \right) \\
& + \delta_{ab}\delta_{ce} \left(\tilde{J}_2^{5be} - \tilde{J}_2^{dbe} - \tilde{J}_3^{be} Y_d \right) + \delta_{ae}\delta_{bc} \left(\tilde{J}_2^{5ce} - \tilde{J}_2^{dce} - \tilde{J}_3^{ce} Y_d \right) + \delta_{ac}\delta_{be} \left(\tilde{J}_2^{5ce} - \tilde{J}_2^{dce} - \tilde{J}_3^{ce} Y_d \right) \dots
\end{aligned}$$

$$\begin{aligned}
& \dots + \delta_{ab}\delta_{cd} (\tilde{J}_2^{5bd} - \tilde{J}_2^{ebd}) + \delta_{ad}\delta_{bc} (\tilde{J}_2^{5cd} - \tilde{J}_2^{ecd}) + \delta_{ac}\delta_{bd} (\tilde{J}_2^{5cd} - \tilde{J}_2^{ecd}) \\
& + \delta_{ab}\delta_{ac} \left[J_2^{5dc} + J_2^{5ec} - J_2^{dec} - \tilde{G}^{5c} + (J_3^{5c} - J_3^{ec}) Y_d \right] \\
& + \delta_{bc}\delta_{be} \left[J_2^{5ae} + J_2^{5de} - J_2^{ade} - \tilde{G}^{5e} + (J_3^{5e} - J_3^{de}) Y_a + (J_3^{5e} - J_3^{ae}) Y_d - J_4^e Y_a Y_d \right] \\
& + \delta_{ac}\delta_{ae} \left[J_2^{5be} + J_2^{5de} - J_2^{bde} - \tilde{G}^{5e} + (J_3^{5e} - J_3^{de}) Y_b + (J_3^{5e} - J_3^{be}) Y_d - J_4^e Y_b Y_d \right] \\
& + \delta_{ab}\delta_{ae} \left[J_2^{5ce} + J_2^{5de} - J_2^{cde} - \tilde{G}^{5e} + (J_3^{5e} - J_3^{de}) Y_c + (J_3^{5e} - J_3^{ce}) Y_d - J_4^e Y_c Y_d \right] \\
& + \delta_{ab} \left[G_1^{5b} - \tilde{J}_2^{5cb} - \tilde{J}_2^{d5b} - \tilde{J}_2^{e5b} + R^{b5cd} + R^{b5ce} + R^{b5de} - R^{bcde} + (-\tilde{J}_3^{5b} + R^{b5d} + R^{b5e} - R^{bde}) Y_c \right. \\
& \left. + (-\tilde{J}_3^{5b} + R^{b5c} + R^{b5e} - R^{bce}) Y_d + (R^{b5} - R^{be}) Y_c Y_d \right] + \delta_{bc} \left[G_1^{5c} - \tilde{J}_2^{a5c} - \tilde{J}_2^{d5c} - \tilde{J}_2^{e5c} + R^{c5ad} + R^{c5de} \right. \\
& \left. + R^{c5ae} - R^{cade} - R^{bcde} + (-\tilde{J}_3^{5c} + R^{c5d} + R^{c5e} - R^{cde}) Y_a + (-\tilde{J}_3^{5c} + R^{c5a} + R^{c5e} - R^{cae}) Y_d \right. \\
& \left. + (R^{c5} - R^{ce}) Y_a Y_d \right] + \delta_{ac} \left[G_1^{5c} - \tilde{J}_2^{b5c} - \tilde{J}_2^{d5c} - \tilde{J}_2^{e5c} + R^{c5bd} + R^{c5be} + R^{c5de} - R^{cbde} \right. \\
& \left. + (-\tilde{J}_3^{5c} + R^{c5d} + R^{c5e} - R^{cde}) Y_b + (-\tilde{J}_3^{5c} + R^{c5b} + R^{c5e} - R^{cbe}) Y_d + (R^{c5} - R^{ce}) Y_b Y_d \right] \\
& + \delta_{ce} \left[G_1^{5e} - \tilde{J}_2^{a5e} - \tilde{J}_2^{b5e} - \tilde{J}_2^{d5e} + R^{e5ab} + R^{e5ad} + R^{e5bd} - R^{eabd} + (-\tilde{J}_3^{5e} + R^{e5b} + R^{e5d} - R^{ebd}) Y_a \right. \\
& \left. + (-\tilde{J}_3^{5e} + R^{e5a} + R^{e5d} - R^{ead}) Y_b + (-\tilde{J}_3^{5e} + R^{e5a} + R^{e5b} - R^{eab}) Y_d + (R^{e5} - R^{ed}) Y_a Y_b \right. \\
& \left. + (R^{e5} - R^{eb}) Y_a Y_d + (R^{e5} - R^{ea}) Y_b Y_d - R^e Y_a Y_b Y_d \right] + \delta_{be} \left[G_1^{5e} - \tilde{J}_2^{a5e} - \tilde{J}_2^{c5e} - \tilde{J}_2^{d5e} + R^{e5ac} + R^{e5ad} \right. \\
& \left. + R^{e5cd} - R^{eacd} + (-\tilde{J}_3^{5e} + R^{e5c} + R^{e5d} - R^{ecd}) Y_a + (-\tilde{J}_3^{5e} + R^{e5a} + R^{e5d} - R^{ead}) Y_c \right. \\
& \left. + (-\tilde{J}_3^{5e} + R^{e5a} + R^{e5c} - R^{eac}) Y_d + (R^{e5} - R^{ed}) Y_a Y_c + (R^{e5} - R^{ec}) Y_a Y_d + (R^{e5} - R^{ea}) Y_c Y_d - R^e Y_a Y_c Y_d \right] \\
& + \delta_{ae} \left[G_1^{5e} - \tilde{J}_2^{b5e} - \tilde{J}_2^{c5e} - \tilde{J}_2^{d5e} + R^{e5bc} + R^{e5bd} + R^{e5cd} - R^{ebcd} + (-\tilde{J}_3^{5e} + R^{e5c} + R^{e5d} - R^{ecd}) Y_b \dots \right]
\end{aligned}$$

$$\begin{aligned}
& \dots + (-\tilde{J}_3^{5e} + R^{e5b} + R^{e5d} - R^{ebd}) Y_c + (-\tilde{J}_3^{5e} + R^{e5b} + R^{e5c} - R^{ebc}) Y_d + (R^{e5} - R^{ed}) Y_b Y_c \\
& + (R^{e5} - R^{ec}) Y_b Y_d + (R^{e5} - R^{eb}) Y_c Y_d - R^e Y_b Y_c Y_d + \delta_{cd} [-\tilde{J}_2^{a5d} - \tilde{J}_2^{b5d} - \tilde{J}_2^{e5d} + R^{d5ab} + R^{d5ae} + R^{d5be} \\
& - R^{dabe} + \tilde{G}^{d5} + (-\tilde{J}_3^{d5} + R^{d5b} + R^{d5e} - R^{dbe}) Y_a + (-\tilde{J}_3^{d5} + R^{d5a} + R^{d5e} - R^{dae}) Y_b + (R^{d5} - R^{de}) Y_a Y_b] \\
& + \delta_{bd} [-\tilde{J}_2^{a5d} - \tilde{J}_2^{c5d} - \tilde{J}_2^{e5d} + R^{d5ac} + R^{d5ae} + R^{d5ce} - R^{dace} + \tilde{G}^{d5} + (-\tilde{J}_3^{d5} + R^{d5c} + R^{d5e} - R^{dce}) Y_a \\
& + (-\tilde{J}_3^{d5} + R^{d5a} + R^{d5e} - R^{dae}) Y_c + (R^{d5} - R^{de}) Y_a Y_c] + \delta_{ad} [-\tilde{J}_2^{b5d} - \tilde{J}_2^{c5d} - \tilde{J}_2^{e5d} + R^{d5bc} + R^{d5be} \\
& + R^{d5ce} - R^{dbce} + \tilde{G}^{d5} + (-\tilde{J}_3^{d5} + R^{d5c} + R^{d5e} - R^{dce}) Y_b + (-\tilde{J}_3^{d5} + R^{d5b} + R^{d5e} - R^{dbe}) Y_c \\
& + (R^{d5} - R^{de}) Y_b Y_c] + \delta_{de} [-\tilde{J}_2^{a5e} - \tilde{J}_2^{b5e} - \tilde{J}_2^{c5e} + R^{e5ab} + R^{e5ac} + R^{e5bc} - R^{eabc} + \tilde{G}^{e5} \\
& + (-\tilde{J}_3^{5e} + R^{e5b} + R^{e5c} - R^{ebc}) Y_a + (-\tilde{J}_3^{5e} + R^{e5a} + R^{e5c} - R^{eac}) Y_b + (-\tilde{J}_3^{5e} + R^{e5a} + R^{e5b} - R^{eab}) Y_c \\
& + (R^{e5} - R^{ec}) Y_a Y_b + (R^{e5} - R^{eb}) Y_a Y_c + (R^{e5} - R^{ea}) Y_b Y_c - R^e Y_a Y_b Y_c] + (I_1^{5bcd} + I_1^{5bce} + I_1^{5bde} + I_1^{5cde} \\
& - I_1^{bcde} + J_3^{b5} + J_3^{c5} + J_3^{d5} + J_3^{e5} - G^5 - R^{5bc} - R^{5bd} - R^{5be} - R^{5cd} - R^{5ce} - R^{5de}) Y_a + (I_1^{5acd} + I_1^{5ace} \\
& + I_1^{5ade} + I_1^{5cde} - I_1^{acde} + J_3^{a5} + J_3^{c5} + J_3^{d5} + J_3^{e5} - G^5 - R^{5ac} - R^{5ad} - R^{5ae} - R^{5cd} - R^{5ce} - R^{5de}) Y_b \\
& + (I_1^{5abd} + I_1^{5abe} + I_1^{5ade} + I_1^{5bde} - I_1^{abde} + J_3^{a5} + J_3^{b5} + J_3^{d5} + J_3^{e5} - G^5 - R^{5ab} - R^{5ad} - R^{5ae} - R^{5bd} \\
& - R^{5be} - R^{5de}) Y_c + (I_1^{5abc} + I_1^{5abe} + I_1^{5ace} + I_1^{5bce} - I_1^{abce} + J_3^{a5} + J_3^{b5} + J_3^{c5} + J_3^{e5} - G^5 - R^{5ab} \\
& - R^{5ac} - R^{5ae} - R^{5bc} - R^{5be} - R^{5ce}) Y_d + (I_2^{5cd} + I_2^{5ce} + I_2^{5de} - I_2^{cde} + J_4^5 - R^{5c} - R^{5d} - R^{5e}) Y_a Y_b \\
& + (I_2^{5bd} + I_2^{5be} + I_2^{5de} - I_2^{bde} + J_4^5 - R^{5b} - R^{5d} - R^{5e}) Y_a Y_c + (I_2^{5bc} + I_2^{5be} + I_2^{5ce} - I_2^{bce} + J_4^5 - R^{5b} \\
& - R^{5c} - R^{5e}) Y_a Y_d + (I_2^{5ad} + I_2^{5ae} + I_2^{5de} - I_2^{ade} + J_4^5 - R^{5a} - R^{5d} - R^{5e}) Y_b Y_c + (I_2^{5ac} + I_2^{5ae} + I_2^{5ce} \dots
\end{aligned}$$

$$\begin{aligned}
& \dots - I_2^{ace} + J_4^5 - R^{5a} - R^{5c} - R^{5e}) Y_b Y_d + (I_2^{5ab} + I_2^{5ae} + I_2^{5be} - I_2^{abe} + J_4^5 - R^{5a} - R^{5b} - R^{5e}) Y_c Y_d \\
& + (I_3^{5d} + I_3^{5e} - I_3^{de} - R^5) Y_a Y_b Y_c + (I_3^{5c} + I_3^{5e} - I_3^{ce} - R^5) Y_a Y_b Y_d + (I_3^{5b} + I_3^{5e} - I_3^{be} - R^5) Y_a Y_c Y_d \\
& + (I_3^{5a} + I_3^{5e} - I_3^{ae} - R^5) Y_b Y_c Y_d + (I_4^5 - I_4^e) Y_a Y_b Y_c Y_d \}. \tag{36}
\end{aligned}$$

with some additional definitions not shown here

Some numerics for 5-point function with rank $R = 4$

We use OneLOop for scalar functions and compare with LoopTools/FF

the kinematics:

pls = 1.1163688400000000E-002	p2s = 2.6109999999999998E-007	p3s = 0.0000000000000000
p4s = 2.6109999999999998E-007	p5s = 1.1163688400000000E-002	
s12 = -0.708582781900000001	s23 = -1.5343299000000002E-003	s34 = -0.128518604299999998
s45 = -0.610239379499999996	s15 = 0.926689424200000000	
m1s = 1.1163688361676107E-002	m2s = 0.0000000000000000	m3s = 2.6112003932088364E-007
m4s = 2.6112003932088364E-007	m5s = 0.0000000000000000	

OneLOop-3.3.1

is used for the evaluation of 1-loop scalar 1-, 2-, 3- and 4-point functions

[van Hameren arXiv:1007.4716] and [van Hameren, Papadopoulos, Pittau arXiv:0903.4665]

the R=4 contractions, a,b,c,d=3,3,3,3	(-48094.1074 54542318	, -47802.08746 5035322)
LoopTools	(-48094.1074 65	, -47802.08746 05)

the R=4 contractions, a,b,c,d=3,3,3,4	(-18463.1204 24842149	, -23446.4704 12257226)
LoopTools	(-18463.1204 31	, -23446.4704 09)

the R=4 contractions, a,b,c,d=3,3,3,5	(0.0000000000000000	, 0.0000000000000000)
LoopTools	(0.0000000000000000	, 0.0000000000000000)

compared to: LoopTools/FF [Hahn arXiv:hep-ph/9807565] and [van Oldenborgh CPC66(1991)]

Summary

- Explicit **Analytical, recursive treatment** of **heptagon, hexagon and pentagon tensor integrals** of rank R in terms of pentagons and boxes of rank $R - 1$
- Systematic derivation of expressions which are explicitly **free of inverse Gram determinants** $(\)_5$ until pentagons of rank $R = 5$
- Numerical **tensor reduction package PJFry** for C, C++, Mathematica, Fortran
- Numerical **packages for contracted tensor integrals OLEC and CONTRACTIONS** for C++ and Fortran under development

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