Soft Gluon Effects in Four Parton Hard-Scattering Processes

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Outline

1. Introduction: Sudakov Logs & Threshold Resummation

2. One-particle inclusive cross section:
   a general NLO calculation of the large logarithmic terms

3. Soft-gluon resummation at fixed rapidity

4. Conclusions

Threshold Resummation

- **Perturbative** QCD predictions are at the heart of our understanding of hard-scattering processes at high-energy colliders.

- At the **partonic threshold**, the imbalance between real emission (strongly inhibited) and virtual corrections leads to *enhanced logarithmic terms*, already at NLO.

- The large logarithms spoil the perturbative expansion ($\alpha_s \mathrm{L} \sim 1$). A reliable evaluation of any cross-section in the near-threshold region requires the **all-order resummation** of these logarithms.

[Sterman '87]

[Catani, Trentadue '89]

[Kidonakis, Laenen, Oderda, Sterman '98]

[Bonciani, Catani, Mangano, Nason '98]

Motivation

- Resummation nowadays available for several processes, involving only two partons (DY and H production) or more partons (photoproduction, high-p_T vector and H bosons, heavy quarks, jet and dihadron, single-hadron inclusive production) in QCD and SCET.

- We consider the **single-hadron** inclusive production at **high hadron transverse momentum**. Easily measurable at hadron colliders, it offers both a relevant test of the QCD factorization and quantitative information on the parton fragmentation functions.

- Soft-gluon resummation up to NLL for this process was performed in [De Florian,Vogelsang '05]. The quantitative effect is large, expecially at the typical energies of fixed-target collisions.

Aim

We want to study soft-gluon resummation for the transverse-momentum cross section at **fixed rapidity** of the observed hadron.

- To this aim we compute the NLO QCD corrections close to the partonic threshold, directly factorized in color space.
- Using our general expression of the NLO cross section, we determine the one-loop hard-virtual amplitude that enters into the colour-space factorization structure of the resummation formula.

Related works (threshold resummation):

  - Prompt-photon [Catani, Mangano, Nason '98] [Becher, Schwartz '10]
  - One-particle *integrated over rapidity* [De Florian, Vogelsang '05]
  - Double-particle at large invariant mass [Kelley, Schwartz '11]

One-hadron-inclusive cross-section

$h_1 h_2 \to h_3 X$ $\qquad\qquad$ $a_1 a_2 \to a_3 X$

$$\frac{d\sigma_{h_1 h_2 h_3}}{d^3\boldsymbol{P}_3/E_3}(P_i) = \sum_{a_i} f^{(\mu_F)}_{h_1/a_1} \otimes f^{(\mu_F)}_{h_2/a_2} \otimes d^{(\mu_f)}_{h_3/a_3} \otimes \frac{d\hat{\sigma}_{a_1 a_2 a_3}}{d^3\boldsymbol{p}_3/p_3^0}(p_i, \mu_F, \mu_f)$$

$$d\eta \, d^2\boldsymbol{p}_\perp \longleftarrow$$

- At high p_T: leading contributions at the partonic threshold.

One-hadron-inclusive cross-section

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$$d\eta\, d^2\boldsymbol{p}_\perp \longleftarrow$$

- At high p_T: leading contributions at the partonic threshold.
- If $x = (s+t+u)/s \to 0$, these are:
    - Constant terms $\delta(x) \;\to 1$
    - Large Logs $\left(\dfrac{\ln^\ell x}{x}\right)_+ \to (\ln N)^{\ell+1}$
- While regular terms are suppressed: $\mathcal{O}(1/N)$

NLO result near threshold

Already in the literature [Aversa,Chiappetta,Greco,Guillet '89]

In terms of the kinematical variables $v = 1 + t/s$ and $w = -u/(s+t)$, the partonic cross section can be written as

$$\frac{d\hat{\sigma}_{a_1 a_2 a_3}}{d^3\boldsymbol{p}_3/p_3^0}(p_i;\mu_F,\mu_f) = \frac{\alpha_{\rm S}^2(\mu_R^2)}{\pi\, s}\Big[\frac{1}{v}\frac{d\hat{\sigma}^{(0)}(s,v)}{dv}\;\delta(1-w) + \frac{\alpha_{\rm S}(\mu_R^2)}{2\,\pi}\;\frac{1}{v\,s}\,\mathcal{C}^{(1)}(s,v,w;\mu_R,\mu_F,\mu_f) + \mathcal{O}(\alpha_{\rm S}^2)\Big]$$

where the NLO term $\mathcal{C}^{(1)}$ has the structure

$$\mathcal{C}^{(1)}(s,v,w;\mu_R,\mu_F,\mu_f) = \mathcal{C}_3(v)\left(\frac{\ln(1-w)}{1-w}\right)_+ + \mathcal{C}_2(v;s,\mu_F,\mu_f)\left(\frac{1}{1-w}\right)_+ + \mathcal{C}_1(v;s,\mu_R,\mu_F,\mu_f)\,\delta(1-w) + \mathcal{C}_0(1-w,v;s,\mu_R,\mu_F,\mu_f)\,.$$

We have performed an independent calculation, by using soft and collinear approximations.

- $a_1 a_2 \to a_3 a_4$ @ 1-loop: [Kunszt,Signer,Trocsanyi '94]

$$|\mathcal{M}^{(1)}\rangle = \boldsymbol{I}^{(1)}_{\rm sing} \, |\mathcal{M}^{(0)}\rangle + |\mathcal{M}^{(1)\,{\rm fin}}\rangle \, .$$

The color operator $\boldsymbol{I}^{(1)}_{\rm sing}$ embodies the one-loop IR divergence, while $|\mathcal{M}^{(1)\,{\rm fin}}\rangle$ is finite as $\epsilon \to 0$. We use the expression

$$\boldsymbol{I}^{(1)}_{\rm sing} = \frac{1}{2} \frac{1}{\Gamma(1-\epsilon)} \left[\frac{1}{\epsilon^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \boldsymbol{T}_i \boldsymbol{T}_j \left(\frac{4\pi \mu_R^2 \, e^{-i\lambda_{ij}\pi}}{2 p_i \cdot p_j} \right)^{\epsilon} - \frac{1}{\epsilon} \sum_{i=1}^{4} \gamma_{a_i} \left(\frac{4\pi \mu_R^2 \, s}{u\, t} \right)^{\epsilon} \right]$$

Full color structure: $\boldsymbol{T}_i$ color operators.

- $a_1 a_2 \to a_3 X$, with real emission of $X = \{2 \text{ partons}\}$.

  We use soft and collinear factorization formulae [Catani,Seymour '97]

  - The soft configuration is treated via eikonal approximation.
  - $X = \{$*hard-collinear* pair$\}$? Matching to the *full* AP behaviour.
  - Collinear-divergent counterterms (NLO PDFs).

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  - Collinear-divergent counterterms (NLO PDFs).

- Our result has a rather compact form: one simple formula valid for **all** the flavour and colour channels.

→ It is factorized on the colour space: the coefficients of $(1/(1-x))_+$ and $\delta(1-x)$ depend on colour-correlation operators $\boldsymbol{T}_i \boldsymbol{T}_j$.

→ It is consistent with (the dominant contribution of) known results:
  - Photoproduction from qg and $q\overline{q}$ channels [Gordon and Vogelsang '93]
  - qq and qg scattering [Aversa,Chiappetta,Greco,Guillet '89]

$$16\pi\, N^{(in)}\, \mathcal{C}^{(1)} = \langle \mathcal{M}^{(0)} |\, \boldsymbol{C}^{(1)} | \mathcal{M}^{(0)} \rangle + \left(\langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)\,\mathrm{fin}} \rangle + \mathrm{c.c.} \right) \delta(1-w) + \mathcal{O}\left((1-w)^0 \right)$$

$$\begin{aligned}
\boldsymbol{C}^{(1)}_{a_1 a_2 a_3 a_4}(s, v, w; \mu_R, \mu_F, \mu_f) &= 2 \left(\frac{\ln(1-w)}{1-w} \right)_+ \left[2 \sum_{i=1}^{3} \boldsymbol{T}_i^2 - \boldsymbol{T}_4^2 \right] \\
&- \left(\frac{1}{1-w} \right)_+ \Bigg[2 \sum_{i=1}^{3} \boldsymbol{T}_i^2 \left(\ln \frac{1-v}{v} + \ln \frac{\mu_{Fi}^2}{s} \right) - 2\, \boldsymbol{T}_4^2 \ln(1-v) \\
&\qquad + \gamma_{a_4} + 8 \Big(\boldsymbol{T}_1 \cdot \boldsymbol{T}_3 \ln(1-v) + \boldsymbol{T}_2 \cdot \boldsymbol{T}_3 \, \ln v \Big) \Bigg] \\
&+ \delta(1-w) \Bigg\{ \frac{\pi^2}{2} \left(\boldsymbol{T}_1^2 + \boldsymbol{T}_2^2 + 3 \boldsymbol{T}_3^2 - \frac{4}{3} \boldsymbol{T}_4^2 \right) - \sum_{i=1}^{3} \gamma_{a_i} \ln \frac{\mu_{Fi}^2}{s\, v(1-v)} + \gamma_{a_4} \ln(1-v) \\
&\qquad - 2 \boldsymbol{T}_3^2 \ln v \, \ln \frac{\mu_f^2}{s} + 2 \boldsymbol{T}_2^2 \ln \frac{1-v}{v} \ln \frac{\mu_F^2}{s} + \ln v \ln(1-v) \left(\boldsymbol{T}_4^2 - \boldsymbol{T}_1^2 - \boldsymbol{T}_2^2 - \boldsymbol{T}_3^2 \right) \\
&\qquad + \boldsymbol{T}_2 \cdot \boldsymbol{T}_3 \Big(2\pi^2 + 2 \ln v \left(2 \ln(1-v) - 3 \ln v \right) \Big) + \ln^2(1-v) \left(\boldsymbol{T}_1^2 + \boldsymbol{T}_3^2 - \boldsymbol{T}_4^2 \right) \\
&\qquad + \boldsymbol{T}_1 \cdot \boldsymbol{T}_3 \Big(2\pi^2 + 2 \ln(1-v) \left(\ln(1-v) - 2 \ln v \right) \Big) + \ln^2 v \left(\boldsymbol{T}_2^2 + \boldsymbol{T}_3^2 \right) + K_{a_4} \Bigg\}
\end{aligned}$$

All-order soft-gluon resummation

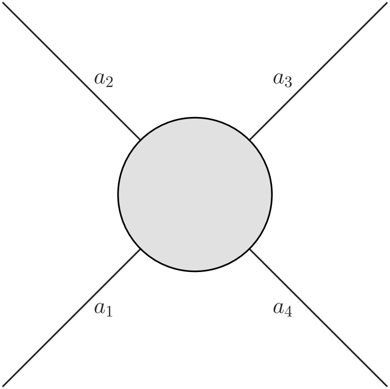
- We introduce $x_\omega = -\dfrac{t+u}{s}, \quad r = \dfrac{u}{t}, \quad p_T^2 = \dfrac{t\,u}{s},$
  - The threshold region is $x_\omega \to 1$.
  - The threshold variable x_ω is **symmetric** w.r.t. $t \leftrightarrow u$. Otherwise the truncation of the resummed series would produce unphysical asymmetries in the angular distribution.
- In terms of these variables

$$\frac{d\hat{\sigma}_{a_1 a_2 a_3}}{d^3\boldsymbol{p}_3/p_3^0} = \frac{\overline{|\mathcal{M}^{(0)}_{a_1 a_2 a_3 a_4}(r, p_T^2)|^2}}{(4\pi s)^2}\, \Sigma_{a_1 a_2 \to a_3}(x_\omega, r; p_T^2, \mu_F, \mu_f).$$

- We perform resummation in the Mellin space of N conjugated to x_ω, neglecting $\mathcal{O}(1/N)$ contributions. The all-order expression of $\Sigma_{a_1 a_2 \to a_3, N}$ is obtained by using the BCMN resummation formalism.

- Resumed Multiparton Cross Section [Bonciani, Catani, Mangano, Nason '03]

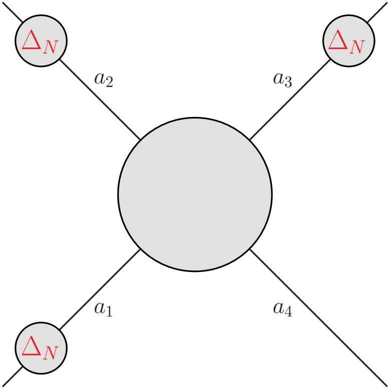
$$\Sigma^{\mathrm{res}}_{a_1 a_2 a_3, N} =$$



- Resummed Multiparton Cross Section [Bonciani, Catani, Mangano, Nason '03]

$$\Sigma^{\rm res}_{a_1 a_2 a_3, N} = \prod_{i=1,2,3} \Delta_{a_i, N_i}(Q_i^2, \mu_i^2)$$

Δ_{a_i,N_i}: IS-like radiation (soft-collinear)



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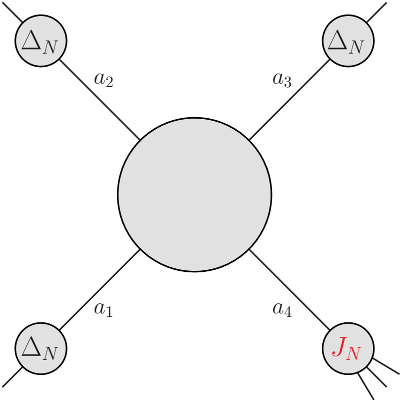
$$\Sigma^{\text{res}}_{a_1 a_2 a_3, N} = \prod_{i=1,2,3} \Delta_{a_i, N_i}(Q_i^2, \mu_i^2)\, J_{a_4, N_4}(Q_4^2)$$



Δ_{a_i, N_i}: IS-like radiation (soft-collinear)

J_{a_4, N_4}: Jet function (collinear, soft and hard, radiation)

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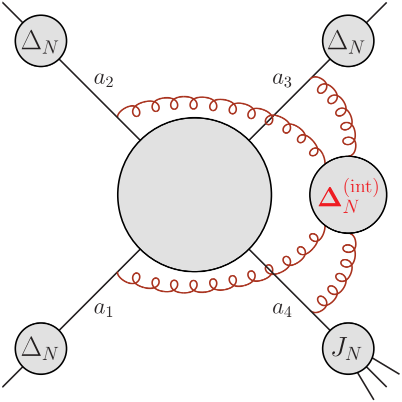

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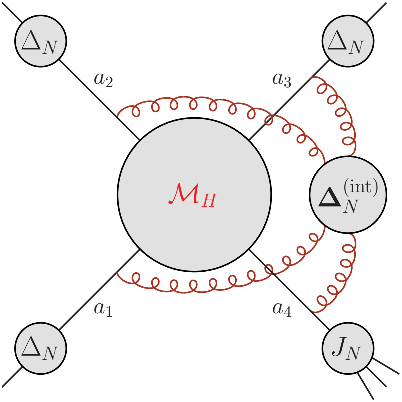
$\boldsymbol{\Delta}_N^{(\rm int)}$: Color-correlated large-angle soft emission

- Resumed Multiparton Cross Section [Bonciani, Catani, Mangano, Nason '03]

$$\Sigma^{\mathrm{res}}_{a_1 a_2 a_3, N} = \prod_{i=1,2,3} \Delta_{a_i,N_i}(Q_i^2, \mu_i^2)\, J_{a_4,N_4}(Q_4^2)\, \frac{\langle \mathcal{M}_{\mathrm{H}} | \mathbf{\Delta}_N^{(\mathrm{int})} | \mathcal{M}_{\mathrm{H}} \rangle}{\left| \mathcal{M}^{(0)} \right|^2}$$



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$\mathcal{M}_{\mathrm{H}}$: Process-dependent constant terms ∼ hard virtual corrections

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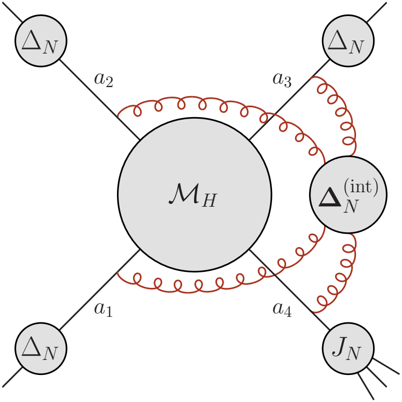

Δ_{a_i, N_i}: IS-like radiation (soft-collinear)

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$\boldsymbol{\Delta}_N^{(\mathrm{int})}$: Color-correlated large-angle soft emission

$\mathcal{M}_{\mathrm{H}}$: Process-dependent constant terms $\sim$ hard virtual corrections

$$N_1 = \frac{N\, r}{1+r}, \; N_2 = \frac{N}{1+r},$$

$$N_3 = N, \; N_4 = \frac{N\, r}{(1+r)^2}, \quad Q_i^2 = p_T^2$$

Collinear radiation

- The collinear radiation is **diagonal** in colour-space:

$$\ln \Delta_{a,N}(Q^2;\mu^2) = \int_0^1 \frac{z^{N-1}-1}{1-z} \int_{\mu^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A_a(\alpha_{\mathrm{S}}(q^2))$$

$$\ln J_{a,N}(Q^2) = \int_0^1 \frac{z^{N-1}-1}{1-z} \Big[\int_{(1-z)^2 Q^2}^{(1-z)Q^2} \frac{dq^2}{q^2} A_a(\alpha_{\mathrm{S}}(q^2)) + \frac{1}{2} B_a\left(\alpha_{\mathrm{S}}\left((1-z)Q^2\right)\right) \Big]$$

- The coefficients A_a and B_a have perturbative expansions:

$$A_a(\alpha_{\mathrm{S}}) = \sum_{k=1}^{\infty} \left(\frac{\alpha_{\mathrm{S}}}{\pi}\right)^n A_a^{(n)} \qquad B_a(\alpha_{\mathrm{S}}) = \sum_{k=1}^{\infty} \left(\frac{\alpha_{\mathrm{S}}}{\pi}\right)^n B_a^{(n)}$$

$$A_a^{(1)} = C_a \qquad A_a^{(2)} = \frac{1}{2} C_a \left[C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{10}{9} n_F T_R \right]$$

$$B_a^{(1)} = -\gamma_a$$

Soft large-angle radiation

- The colour-space radiative factor embodies **all** the quantum-interference effects induced by soft-gluons radiated at **large angles**:

$$\boldsymbol{\Delta}_N^{(\mathrm{int})} = \boldsymbol{V}_N^{\dagger}\boldsymbol{V}_N$$

$$\boldsymbol{V}_N = P_z \exp\left\{\sum_{i\neq j}\int_0^1 \frac{z^{N-1}-1}{1-z}\,\boldsymbol{\Gamma}\left(\alpha_{\mathrm{S}}\left((1-z)^2 p_T^2\right), r\right)\right\}$$

P_z denotes z-ordering in the expansion of the exponential matrix

- The anomalous-dimension matrix $\boldsymbol{\Gamma}(\alpha_{\mathrm{S}}, r)$ has the perturbative expansion

$$\boldsymbol{\Gamma}(\alpha_{\mathrm{S}}, r) = \frac{\alpha_{\mathrm{S}}}{\pi}\,\boldsymbol{\Gamma}^{(1)}(r) + \mathcal{O}(\alpha_{\mathrm{S}}^2),$$

$$\boldsymbol{\Gamma}^{(1)}(r) = \boldsymbol{T}_t^2\,\ln(1+r) + \boldsymbol{T}_u^2\,\ln\frac{1+r}{r} + i\pi\,\boldsymbol{T}_s^2,$$

$$\boldsymbol{T}_s^2 = (\boldsymbol{T}_1 + \boldsymbol{T}_2)^2, \quad \boldsymbol{T}_t^2 = (\boldsymbol{T}_1 + \boldsymbol{T}_3)^2, \quad \boldsymbol{T}_u^2 = (\boldsymbol{T}_2 + \boldsymbol{T}_3)^2.$$

Hard components

- $|\mathcal{M}_{\rm H}\rangle$ embodies the residual terms of Σ_N that are constant and it is perturbatively computable as a power series in $\alpha_{\rm S}$.

- $|\mathcal{M}_{\rm H}^{(1)}\rangle$ can be obtained from the result of our NLO calculation:
  - Expand & truncate the resummed formula, compare with Mellin transformed fixed order result;
  - All the logs must match $\rightarrow$ the constant terms give $\mathcal{M}_{\rm H}$.

- The colour-space factorization of our NLO is essential to obtain $|\mathcal{M}_{\rm H}^{(1)}\rangle$ as an **amplitude**!

Full Color Structure

@NLO: $|\mathcal{M}_{\mathrm{H}}^{(1)}\rangle = |\mathcal{M}^{(1)}\rangle - \boldsymbol{I}_{\mathrm{H}}^{(1)} |\mathcal{M}^{(0)}\rangle$

$$\begin{aligned}
\boldsymbol{I}_{\mathrm{H}}^{(1)} = {} & \boldsymbol{I}_{\mathrm{sing}}^{(1)} + \frac{\pi^2}{4}\left(\boldsymbol{T}_1^2 + \boldsymbol{T}_2^2 + \boldsymbol{T}_3^2 + \frac{4}{3}\boldsymbol{T}_4^2\right) + \frac{1}{2}\sum_{i=1}^{3}\gamma_{a_i} \ln\frac{\mu_{Fi}^2}{p_T^2} \\
& - \frac{1}{2}\ln(1+r)\ln\frac{1+r}{r}\left(\boldsymbol{T}_1^2 + \boldsymbol{T}_2^2 - 3\boldsymbol{T}_3^2 + \boldsymbol{T}_4^2\right) \\
& - \boldsymbol{T}_t^2\left(\frac{\pi^2}{2} + \frac{1}{2}\ln^2(1+r) + \ln(1+r)\ln\frac{1+r}{r}\right) \\
& - \boldsymbol{T}_u^2\left(\frac{\pi^2}{2} + \frac{1}{2}\ln^2\frac{1+r}{r} + \ln(1+r)\ln\frac{1+r}{r}\right) - \frac{1}{2}K_{a_4}
\end{aligned}$$

$$K_q = K_{\bar{q}} = \left(\frac{7}{2} - \frac{\pi^2}{6}\right) C_F, \qquad K_g = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} n_F T_R$$

NNLL

The extension to NNLL resummation still requires:

- $A_a^{(3)}$ in $\Delta_{a,N}$ and $J_{a,N}$ [Moch,Vermaseren,Vogt '04]
- $B_a^{(2)}$ in $J_{a,N}$
- $|\mathcal{M}_{\mathrm{H}}^{(2)}\rangle$ in $|\mathcal{M}_{\mathrm{H}}\rangle$
- $\boldsymbol{\Gamma}^{(2)} \sim (K/2\,\boldsymbol{\Gamma}^{(1)})$ in $\boldsymbol{\Delta}_N^{(\mathrm{int})}$

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Note that $\langle\mathcal{M}_{\mathrm{H}}|\mathbf{\Delta}_N^{(\mathrm{int})}|\mathcal{M}_{\mathrm{H}}\rangle$ leads to the second-order contribution

$$\alpha_s^2 \ln N \left(\langle\mathcal{M}^{(0)}| \left(\mathbf{\Gamma}^{(1)} + \mathbf{\Gamma}^{(1)\dagger}\right) |\mathcal{M}_{\mathrm{H}}^{(1)}\rangle + \mathrm{c.c.}\right)$$

The colour interferences between $\mathbf{\Delta}_N^{(\mathrm{int})}$ and $|\mathcal{M}_{\mathrm{H}}\rangle$ are relevant, starting from $\mathcal{O}(\alpha_s(\alpha_s \ln N)^n)$.

Conclusions

- We have considered the one-particle-inclusive cross section at high transverse energies in hadron collisions.
- We have presented the general structure of the logarithmically enhanced terms at NLO.
  - ✓ Agreement with previous specific results in the literature.
- We have presented the all-order resummation formula of the logarithmically enhanced terms at fixed rapidity and extracted the colour structure of the hard coefficient at $\mathcal{O}(\alpha_{\mathrm{S}})$.
  - → These resummation results are valid for both spin-unpolarized and spin-polarized hard scattering.
  - → The same technique can be applied to other multiparton hard scattering processes, such as jet and heavy-quark production.

Thank you!

Backup: Rapidity-integrated cross section

Scaling variable: $x_T = 2p_T/\sqrt{s}$ s.t. $x_\omega = x_T \cosh\eta$.

$$\frac{d\hat{\sigma}_{a_1 a_2 \to a_3}}{d^2\boldsymbol{p}_T} = \frac{1}{(4\pi s)^2}\,\widetilde{\Sigma}_{a_1 a_2 \to a_3}(x_T; p_T^2, \mu_F, \mu_f)$$

$$\widetilde{\Sigma}(x_T) = \int d\eta\ \overline{|\mathcal{M}^{(0)}_{a_1 a_2 a_3 a_4}(r = e^{2\eta})|}^2\ \Theta(1 - x_T \cosh\eta)\ \Sigma(x_T \cosh\eta, r = e^{2\eta})$$

The threshold limit $x_T \to 1$ kinematically forces $\eta \to 0, r \to 1$.
Since $\Sigma(x_\omega, r)$ is smooth in this limit, we can use $\Sigma(x_\omega, r = 1)$.
The resulting convolution is diagonalized in Mellin space:

$$\widetilde{\Sigma}_{a_1 a_2 \to a_3, N}(p_T^2, \mu_{f_i}) = \widetilde{\Sigma}^{(0)}_{a_1 a_2 \to a_3, N}\left[\Sigma^{\mathrm{res}}_{a_1 a_2 \to a_3 a_4, N}(r = 1; p_T^2, \mu_{f_i}) + \mathcal{O}\left(1/N\right)\right]$$

✓ Consistent with the NLL resummed result of [De Florian,Vogelsang '05]

NB: $\boldsymbol{\Gamma}^{(1)}(r = 1) = \boldsymbol{T}_s^2(-\ln 2 + i\pi) + \sum_{i=1}^{4} C_{a_i} \ln 2\,.$