3-loop gauge β-functions in non-simple gauge groups

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L.M., J. Salomon, M. Steinhauser: *PRL 108 (2012), PRD 86 (2012)*

L. Di Luzio, L.M.: *PRD 87 (2013)*

Outline

- Motivation
- Introduction and Framework
- Calculation
- Results

Motivation

β functions are fundamental quantities for any gauge theory

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- Precision tests of the SM (EWPO)
- Indirect Search for New Physics (through RGEs)

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- Indirect Search for New Physics (through RGEs)
  - gauge coupling unification
  - stability bounds for the EW vacuum

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- SM Gauge sector :

$$\beta_i(\{\alpha_j\}) \quad = \quad -\left(\frac{\alpha_i}{\pi}\right)^2 \left[\beta_0 + \sum_j \frac{\alpha_j}{\pi}\beta_{1,j} + \sum_{j,k} \frac{\alpha_j}{\pi}\frac{\alpha_k}{\pi}\beta_{2,jk} + \cdots\right]$$

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  - 1 loop: [Gross, Wilczek’73; Politzer ’73]

$$\beta_0 = \frac{11}{3} C_2(G_i) - \sum_F \frac{2}{3} T_2(F_i) - \sum_S \frac{4}{3} T_2(S_i)$$

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    & Yukawa [Fischler, Hill'81; Machacek, Vaughn'83; Jack, Osborn'84]

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  - 3 loops: QCD [Tarasov, Vladimirov, Zharkov'80, Larin, Vermaseren'93]
    QCD & top Yukawa [Steinhauser'99]
    simple gauge group [Pickering, Gracey, Jones '01];[Curtright'80; Jones'80]
    SM [L.M., Salomon, Steinhauser '12], [Bednyakov, Pikelner, Velizhanin '12]

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    SM [L.M., Salomon, Steinhauser '12], [Bednyakov, Pikelner, Velizhanin '12]
  - 4 loops: QCD [Ritbergen, Vermaseren, Larin'97; Czakon '05]
  - 5 loops: Chetyrkin's talk

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- Higgs quartic couplings:
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this talk: generalisation of the SM results for gauge sector

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$$F_R = P_R F = \left(\frac{1+\gamma_5}{2}\right) F$$

SM: $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \; u_R, \, d_R; \quad L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \; e_R;$

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  - Complex scalars: S
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- Interactions:
  - gauge: α
  - Yukawa: Y, $Y^\dagger$
  - scalar quartic interactions: λ

Framework(2)

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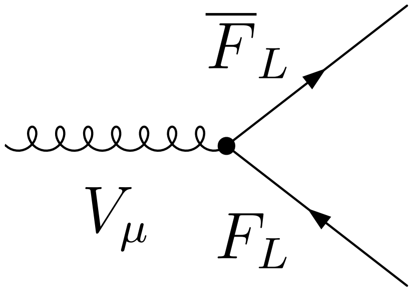
  - all vertices acquire P_L or P_R



$$\sim\ i g_i \gamma_\mu P_L$$

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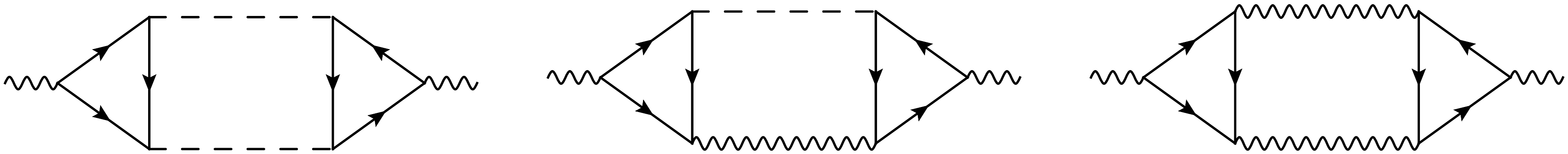
- Group invariants: $[R^A, R^B] = if^{ABC} R^C$

$$\begin{aligned} \mathrm{Tr}\left(R^A R^B\right) &= \delta^{AB} T_2(R) \\ R^A_{ac} R^A_{cb} &= \delta_{ab} C_2(R) \\ \delta^{AA} &= d(G) \\ \delta_{aa} &= d(R) \end{aligned}$$

R^A generators of gauge group G in irr. rep. $\mathcal{R}$

Treatment of γ_5 in $d \neq 4$

- γ_5 present in fermion loops



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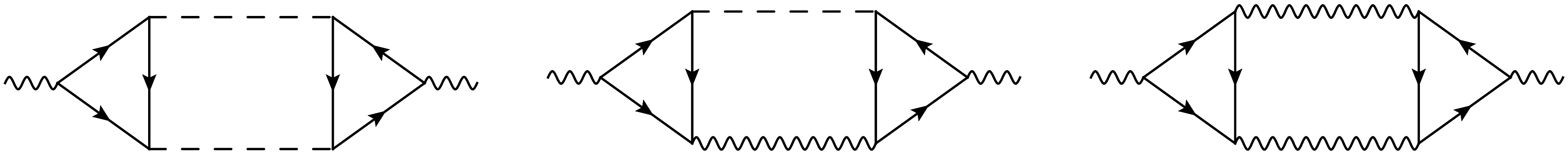

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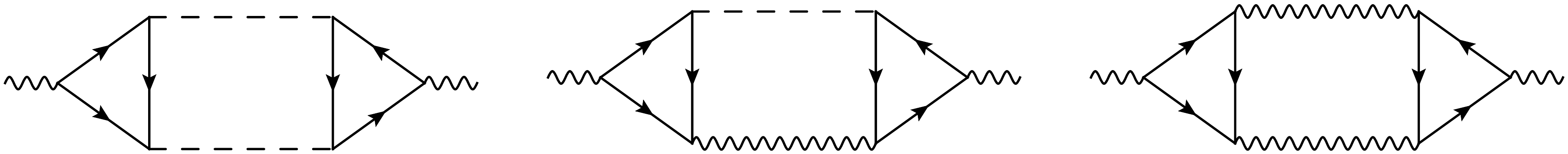

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- "Semi-naive" scheme:

$$\{\gamma_5, \gamma_\mu\} = 0 \quad , \quad \gamma_5^2 = 1$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5) = -4\mathrm{i}\, \tilde{\varepsilon}^{\mu\nu\rho\sigma} + \mathcal{O}(\epsilon)$$

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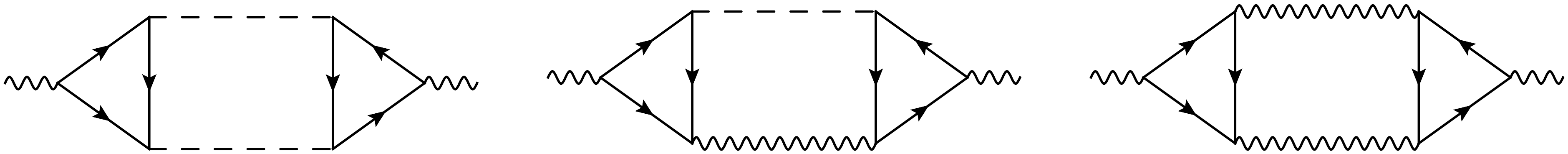

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- $\tilde{\varepsilon}^{\mu\nu\rho\sigma}$ is a formal object
  - completely anti-symmetric

$$\tilde{\varepsilon}^{\mu\nu\rho\sigma}\tilde{\varepsilon}_{\mu'\nu'\rho'\sigma'} = g^{[\mu}_{[\mu'}\, g^{\ \nu}_{\ \nu'}\, g^{\ \rho}_{\ \rho'}\, g^{\ \sigma]}_{\ \sigma']}$$

Calculation

$$\beta_i = -\left[\epsilon\frac{\alpha_i}{\pi} + 2\frac{\alpha_i}{Z_{g_i}} \sum_{j=1, j\neq i}^{7} \frac{\partial Z_{g_i}}{\partial \alpha_j}\beta_j\right]\left(1 + 2\frac{\alpha_i}{Z_{g_i}}\frac{\partial Z_{g_i}}{\partial \alpha_i}\right)^{-1}$$

Z_{g_i} charge renormalization $\quad i = 1, 2, 3, \ldots$

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Z_{g_i} charge renormalization $\quad i = 1, 2, 3, \ldots$

- Calculation of β_i at 3 loops requires:
  - Z_{g_i} to 3 loops
  - β_{Yuk} up to 2 loops
  - β_λ up to 1 loop

Calculation

Calculation of Z_{g_i} to 3 loops:

$$Z_{g_i} \quad = \quad \frac{Z_{\text{V}}}{\prod_k \sqrt{Z_{k,\text{WF}}}} = \frac{Z_{1,g_i c_i \bar{c}_i}}{Z_{2,c_i}\sqrt{Z_{3,g_i}}} = \frac{Z_{1,g_i g_i g_i}}{(\sqrt{Z_{3,g_i}})^3} = \cdots$$

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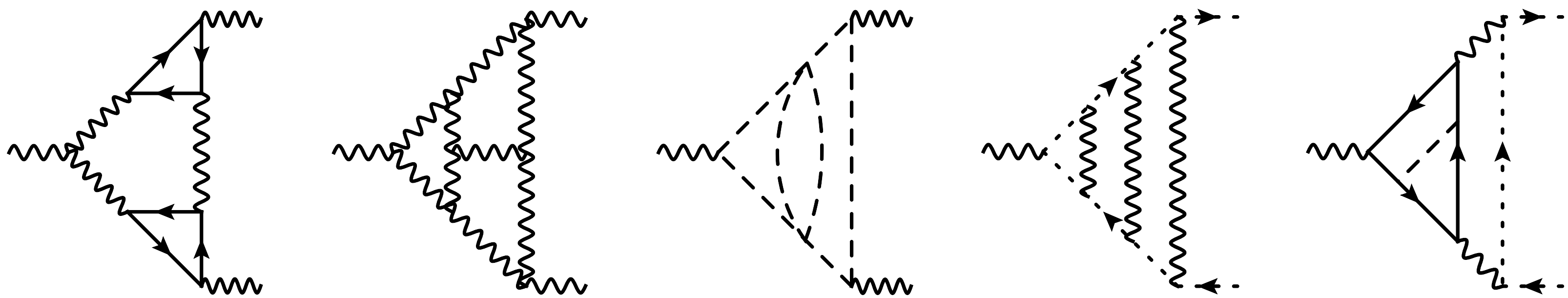
- $\mathcal{O}(5 \times 10^4)$ Feynman diagrams



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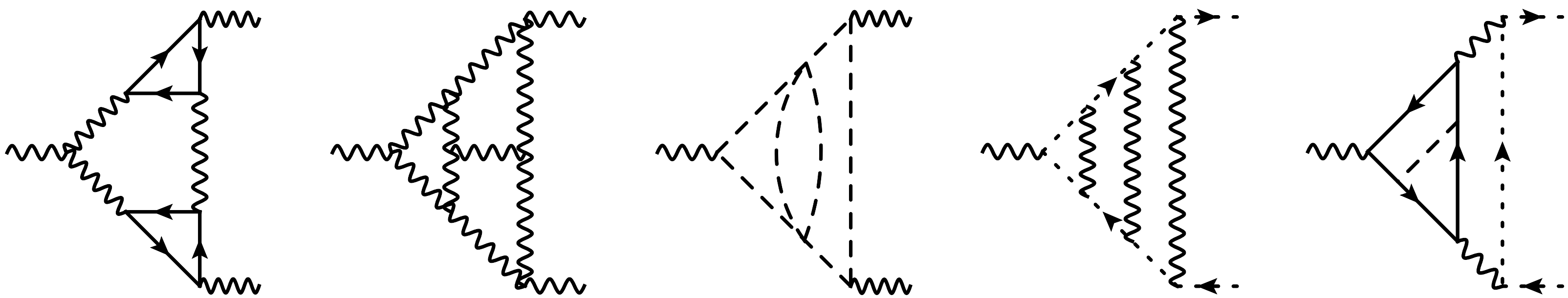

- $\overline{\rm MS}$ scheme

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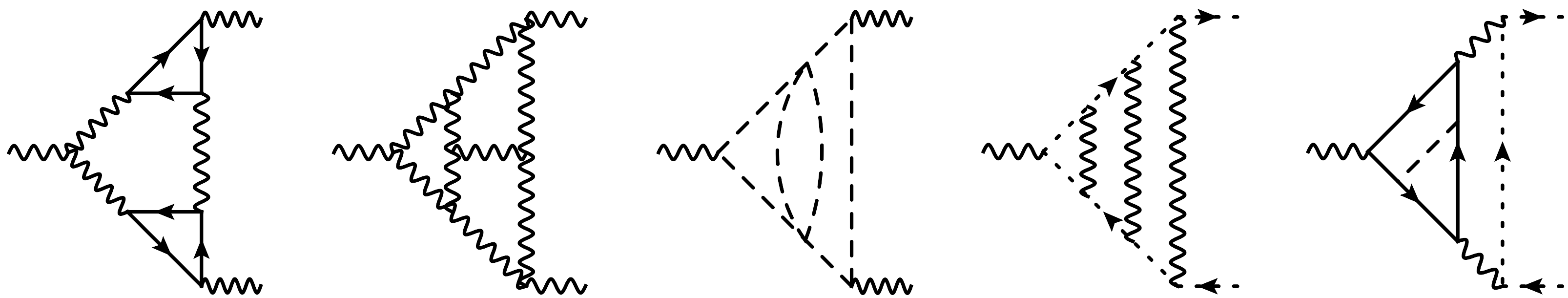
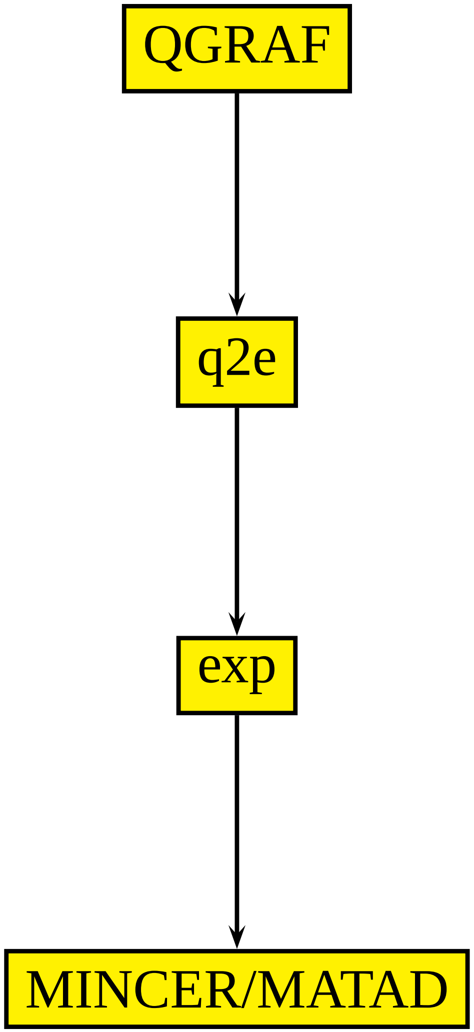


- $\overline{\rm MS}$ scheme
- 1 non zero external momentum & all masses set to zero
  $\Rightarrow$ MINCER [Larin, Tkachov, Vermaseren'91]

Automation



QGRAF [Nogueira'91]

q2e/exp [Harlander, Seidensticker, Steinhauser'97]

MINCER [Larin, Tkachov, Vermaseren'91]

MATAD [Steinhauser'00]

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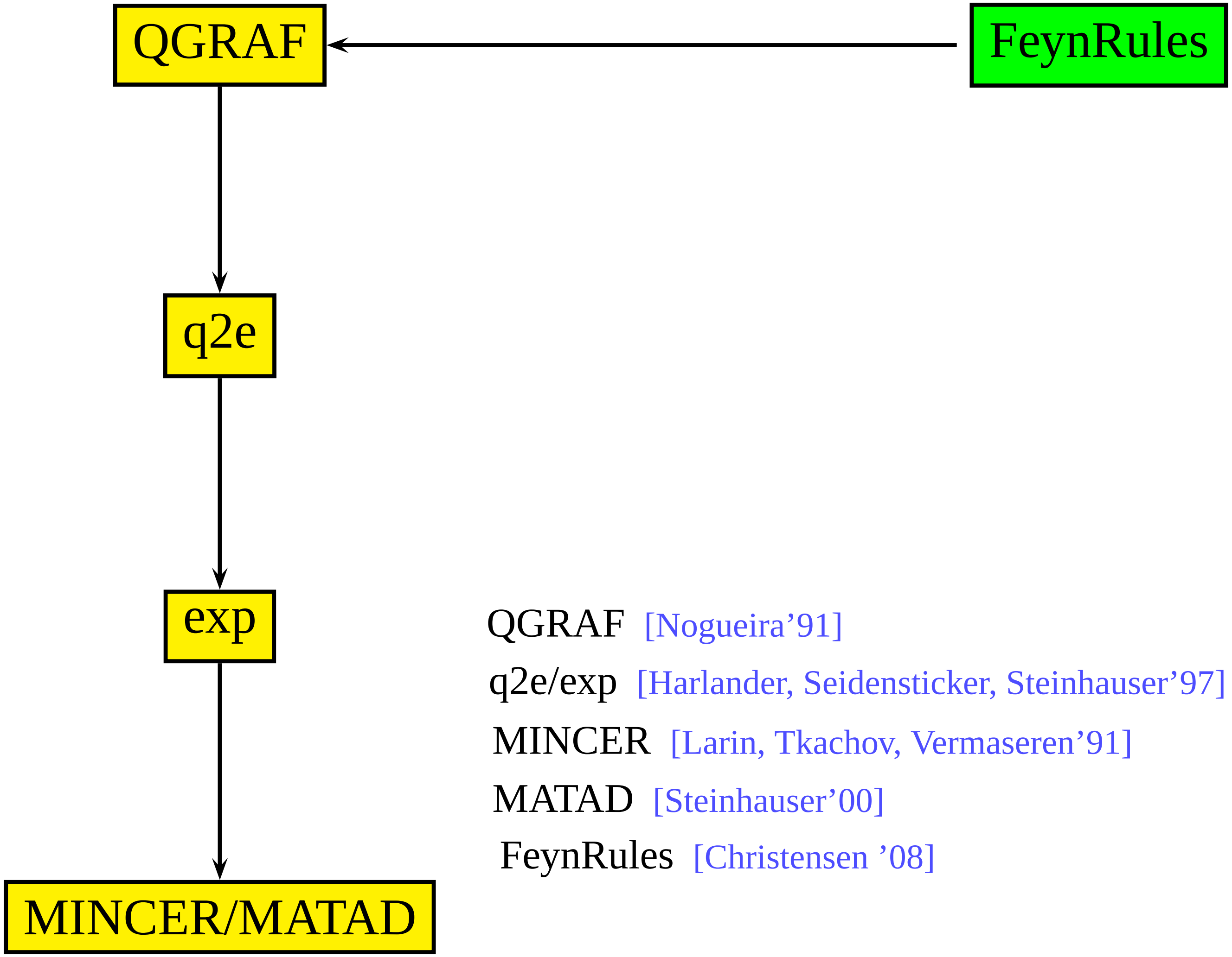

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FeynRules [Christensen '08]

Checks

Internal checks

- $Z_{g_i} = \frac{Z_{\rm V}}{\prod_k \sqrt{Z_{k,\rm WF}}} = \frac{Z_{1,g_i c_i \bar{c}_i}}{Z_{2,c_i}\sqrt{Z_{3,g_i}}} = \frac{Z_{1,g_i g_i g_i}}{(\sqrt{Z_{3,g_i}})^3} = \ldots$
- calculation for arbitrary R_ξ gauge
- complicated gauge invariants cancel out
- G_i=abelian : $3V$ vertex zero to 3 loops (sum of $10\,000$ diagrams)

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- calculation for arbitrary R_ξ gauge
- complicated gauge invariants cancel out
- G_i=abelian : $3V$ vertex zero to 3 loops (sum of $10\,000$ diagrams)
- IR safe for SM case:

  mass $M \neq 0$ for all particles

  asymptotic expansion for $q^2 \gg M^2$

  $\Rightarrow$ NO $\ln(M^2/q^2)$ (up to 35 sub-diagrams/diagram)

Checks

Internal checks

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Comparison with the literature

- 2-loop $\beta_{\rm Yukawa}$ [Fischler, Hill'81; Machacek, Vaughn'83; Jack, Osborn'84]
- 3-loop $\beta_{\rm gauge}$ for simple gauge group [Pickering, Gracey, Jones '01]
- 3-loop $\beta_{\rm gauge}$ in SM [L.M., Salomon, Steinhauser '12], [Bednyakov et al '12]
- 3-loop $\beta_{\rm gauge}$ in SQCD [Jack, Jones, North '96], [Harlander, L.M., Steinhauser '09]

Results for a gauge: $G_1 \times G_2 \times G_3 \ldots$

$$\beta^{(1l)}_{\alpha_i} = \left(\frac{\alpha_i}{\pi}\right)^2 \frac{1}{4}\left[-\frac{11}{3}C_2(G_i) + \sum_F \frac{2}{3}T(F_i)d(F_j) + \sum_S \frac{1}{3}T(S_i)d(S_j)\right]$$

Results for a gauge: $G_1 \times G_2 \times G_3 \ldots$

$$
\begin{aligned}
\beta_{\alpha_i}^{(2l)} &= \left(\frac{\alpha_i}{\pi}\right)^2 \frac{1}{16}\left\{\frac{\alpha_i}{\pi}\left[-\frac{34}{3}C_2(G_i)^2\right.\right.\\
&+ \sum_F [\frac{10}{3}C_2(G_i) + 2C_2(F_i)]T(F_i)d(F_j) + \sum_S [\frac{2}{3}C_2(G_i) + 4C_2(S_i)]T(S_i)d(S_j)\Big]\\
&+ \sum_{j\neq i}\frac{\alpha_j}{\pi}\left[\sum_F 2C_2(F_j)d(F_j)T(F_i)d(F_k) + \sum_S 4C_2(S_j)d(S_j)T(S_i)d(S_k)\right]\\
&+ \left.\sum_F\left[-T(F_i)\frac{1}{\pi}\mathrm{Tr}(\mathrm{Y}_{\mathrm{F_R}}\mathrm{Y}^\dagger_{\mathrm{F_R}})\right]\right\}
\end{aligned}
$$

Results for a gauge: $G_1 \times G_2 \times G_3 \ldots$

$$
\begin{aligned}
\beta^{(3l)}_{\alpha_i} &= \left(\frac{\alpha_i}{\pi}\right)^3 \frac{1}{64}\Bigg\{\frac{\alpha_i}{\pi}\Bigg[-\frac{2857}{54}C_2(G_i)^3 \\
&+ \sum_F [\frac{1415}{54}C_2(G_i)^2 + \frac{205}{18}C_2(G_i)C_2(F_i) - C_2(R_i)^2]T(F_i)d(F_j) \\
&+ \sum_S [\frac{545}{108}C_2(G_i)^2 + \frac{1129}{36}C_2(G_i)C_2(S_i) + \frac{29}{2}C_2(S_i)^2]T(S_i)d(S_j) \\
&- \sum_F \sum_S [\frac{29}{27}C_2(G_i) + \frac{23}{18}C_2(F_i) + \frac{25}{9}C_2(S_i)]T(R_i)T(S_i)d(F_j)d(S_j) \\
&- \sum_{F_m,F_n} [\frac{79}{54}C_2(G_i) + \frac{11}{9}C_2(F_{m,i})]T(F_{m,i})T(F_{n,i})d(F_{m,j})d(F_{n,j}) \\
&+ \sum_{S_m,S_n} [\frac{1}{27}C_2(G_i) - \frac{49}{18}C_2(S_{m,i})]T(S_{m,i})T(S_{n,i})d(S_{m,j})d(S_{n,j})\Bigg] \\
&+ \sum_{j\neq i} \frac{\alpha_j}{\pi}\Bigg[\sum_F 2[2C_2(G_i) - C_2(F_i)]T(R_i)C_2(F_j)d(F_k) \\
&\qquad + \sum_S [\frac{25}{2}C_2(G_i) + 29C_2(S_i)]T(S_i)C_2(S_j)d(S_k)\Bigg]
\end{aligned}
$$

Results for a gauge: $G_1 \times G_2 \times G_3 \ldots$

$$
\begin{aligned}
+ \quad & \left(\frac{\alpha_i}{\pi}\right)^2 \frac{1}{64} \Bigg\{ \sum_{j \neq k} \frac{\alpha_j}{\pi} \frac{\alpha_k}{\pi} \Big[- \sum_F C_2(F_j) C_2(F_k) T(F_i) d(F_l) \Big] \\
+ \quad & \sum_j \left(\frac{\alpha_j}{\pi}\right)^2 \Big[\sum_F [\frac{133}{18} C_2(G_j) - C_2(F_j)] C_2(F_j) T(F_i) d(F_k) \\
+ \quad & \sum_S [\frac{679}{36} C_2(G_j) + \frac{29}{2} C_2(S_j)] C_2(S_j) T(S_i) d(S_k) \\
- \quad & \sum_{F_m, F_n} \frac{11}{9} C_2(F_{m,j}) T(F_{n,j}) T(F_{m,i}) d(F_{m,k}) d(F_{n,l}) \\
- \quad & \sum_{S_m, S_n} \frac{49}{18} C_2(S_{m,j}) T(S_{n,j}) T(S_{m,i}) d(S_{m,k}) d(S_{n,l}) \\
- \quad & \sum_S \sum_F [\frac{25}{9} C_2(S_j) T(S_i) T(F_j) + \frac{23}{18} C_2(F_j) T(F_i) T(S_j)] d(F) d(S) \Big] \\
+ \quad & \ldots \Bigg\}
\end{aligned}
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\begin{aligned}
+ \quad & \left(\frac{\alpha_i}{\pi}\right)^2 \frac{1}{64} \Bigg\{ \sum_{j \neq k} \frac{\alpha_j}{\pi} \frac{\alpha_k}{\pi} \Bigg[- \sum_F C_2(F_j) C_2(F_k) T(F_i) d(F_l) \Bigg] \\
+ \quad & \sum_j \left(\frac{\alpha_j}{\pi}\right)^2 \Bigg[\sum_F [\frac{133}{18} C_2(G_j) - C_2(F_j)] C_2(F_j) T(F_i) d(F_k) \\
+ \quad & \sum_S [\frac{679}{36} C_2(G_j) + \frac{29}{2} C_2(S_j)] C_2(S_j) T(S_i) d(S_k) \\
- \quad & \sum_{F_m, F_n} \frac{11}{9} C_2(F_{m,j}) T(F_{n,j}) T(F_{m,i}) d(F_{m,k}) d(F_{n,l}) \\
- \quad & \sum_{S_m, S_n} \frac{49}{18} C_2(S_{m,j}) T(S_{n,j}) T(S_{m,i}) d(S_{m,k}) d(S_{n,l}) \\
- \quad & \sum_S \sum_F [\frac{25}{9} C_2(S_j) T(S_i) T(F_j) + \frac{23}{18} C_2(F_j) T(F_i) T(S_j)] d(F) d(S) \Bigg] \\
+ \quad & \ldots \Bigg\}
\end{aligned}
$$

- Yukawa and quartic scalar contributions in [Pickering, Gracey, Jones ’01]

Results for SM

$$
\begin{aligned}
\beta_1^{3l} = & \frac{\alpha_1^2}{(4\pi)^4}\Bigg\{\frac{489\alpha_1^2}{2000}+\frac{783\alpha_1\alpha_2}{200}+\frac{3401\alpha_2^2}{80}+\frac{54\alpha_1\hat{\lambda}}{25}+\frac{18\alpha_2\hat{\lambda}}{5}-\frac{36\hat{\lambda}^2}{5}-\frac{2827\alpha_1\mathrm{tr}\hat{T}}{200}\\
& -\frac{471\alpha_2\mathrm{tr}\hat{T}}{8}-\frac{116\alpha_3\mathrm{tr}\hat{T}}{5}-\frac{1267\alpha_1\mathrm{tr}\hat{B}}{200}-\frac{1311\alpha_2\mathrm{tr}\hat{B}}{40}-\frac{68\alpha_3\mathrm{tr}\hat{B}}{5}-\frac{2529\alpha_1\mathrm{tr}\hat{L}}{200}\\
& -\frac{1629\alpha_2\mathrm{tr}\hat{L}}{40}+\frac{183\mathrm{tr}\hat{B}^2}{20}+\frac{51(\mathrm{tr}\hat{B})^2}{10}+\frac{157\mathrm{tr}\hat{B}\mathrm{tr}\hat{L}}{5}+\frac{261\mathrm{tr}\hat{L}^2}{20}+\frac{99(\mathrm{tr}\hat{L})^2}{10}\\
& +\frac{3\mathrm{tr}\hat{T}\hat{B}}{2}+\frac{339\mathrm{tr}\hat{T}^2}{20}+\frac{177\mathrm{tr}\hat{T}\mathrm{tr}\hat{B}}{5}+\frac{199\mathrm{tr}\hat{T}\mathrm{tr}\hat{L}}{5}+\frac{303(\mathrm{tr}\hat{T})^2}{10}\\
& + n_G\left[-\frac{232\alpha_1^2}{75}-\frac{7\alpha_1\alpha_2}{25}+\frac{166\alpha_2^2}{15}-\frac{548\alpha_1\alpha_3}{225}-\frac{4\alpha_2\alpha_3}{5}+\frac{1100\alpha_3^2}{9}\right]\\
& + n_G^2\left[-\frac{836\alpha_1^2}{135}-\frac{44\alpha_2^2}{15}-\frac{1936\alpha_3^2}{135}\right]\Bigg\}
\end{aligned}
$$

Results for SM

$$
\begin{aligned}
\beta_1^{3l} &= \frac{\alpha_1^2}{(4\pi)^4}\left\{\frac{489\alpha_1^2}{2000}+\frac{783\alpha_1\alpha_2}{200}+\frac{3401\alpha_2^2}{80}+\frac{54\alpha_1\hat\lambda}{25}+\frac{18\alpha_2\hat\lambda}{5}-\frac{36\hat\lambda^2}{5}-\frac{2827\alpha_1\mathrm{tr}\hat T}{200}\right.\\
&- \frac{471\alpha_2\mathrm{tr}\hat T}{8}-\frac{116\alpha_3\mathrm{tr}\hat T}{5}-\frac{1267\alpha_1\mathrm{tr}\hat B}{200}-\frac{1311\alpha_2\mathrm{tr}\hat B}{40}-\frac{68\alpha_3\mathrm{tr}\hat B}{5}-\frac{2529\alpha_1\mathrm{tr}\hat L}{200}\\
&- \frac{1629\alpha_2\mathrm{tr}\hat L}{40}+\frac{183\mathrm{tr}\hat B^2}{20}+\frac{51(\mathrm{tr}\hat B)^2}{10}+\frac{157\mathrm{tr}\hat B\mathrm{tr}\hat L}{5}+\frac{261\mathrm{tr}\hat L^2}{20}+\frac{99(\mathrm{tr}\hat L)^2}{10}\\
&+ \frac{3\mathrm{tr}\hat T\hat B}{2}+\frac{339\mathrm{tr}\hat T^2}{20}+\frac{177\mathrm{tr}\hat T\mathrm{tr}\hat B}{5}+\frac{199\mathrm{tr}\hat T\mathrm{tr}\hat L}{5}+\frac{303(\mathrm{tr}\hat T)^2}{10}\\
&+ n_G\left[-\frac{232\alpha_1^2}{75}-\frac{7\alpha_1\alpha_2}{25}+\frac{166\alpha_2^2}{15}-\frac{548\alpha_1\alpha_3}{225}-\frac{4\alpha_2\alpha_3}{5}+\frac{1100\alpha_3^2}{9}\right]\\
&+ \left. n_G^2\left[-\frac{836\alpha_1^2}{135}-\frac{44\alpha_2^2}{15}-\frac{1936\alpha_3^2}{135}\right]\right\}
\end{aligned}
$$

n_G : # of generations

$$
\hat T=\frac{1}{4\pi}Y^U Y^{U\dagger}\quad,\quad \hat B=\frac{1}{4\pi}Y^D Y^{D\dagger}\quad,\quad \hat L=\frac{1}{4\pi}Y^L Y^{L\dagger}
$$

Results for SM

$$
\begin{aligned}
\beta_1^{3l} \quad = \quad & \frac{\alpha_1^2}{(4\pi)^4}\Bigg\{ \frac{489\alpha_1^2}{2000} + \frac{783\alpha_1\alpha_2}{200} + \frac{3401\alpha_2^2}{80} + \frac{54\alpha_1\hat{\lambda}}{25} + \frac{18\alpha_2\hat{\lambda}}{5} - \frac{36\hat{\lambda}^2}{5} - \frac{2827\alpha_1\mathrm{tr}\hat{T}}{200} \\
- \quad & \frac{471\alpha_2\mathrm{tr}\hat{T}}{8} - \frac{116\alpha_3\mathrm{tr}\hat{T}}{5} - \frac{1267\alpha_1\mathrm{tr}\hat{B}}{200} - \frac{1311\alpha_2\mathrm{tr}\hat{B}}{40} - \frac{68\alpha_3\mathrm{tr}\hat{B}}{5} - \frac{2529\alpha_1\mathrm{tr}\hat{L}}{200} \\
- \quad & \frac{1629\alpha_2\mathrm{tr}\hat{L}}{40} + \frac{183\mathrm{tr}\hat{B}^2}{20} + \frac{51(\mathrm{tr}\hat{B})^2}{10} + \frac{157\mathrm{tr}\hat{B}\mathrm{tr}\hat{L}}{5} + \frac{261\mathrm{tr}\hat{L}^2}{20} + \frac{99(\mathrm{tr}\hat{L})^2}{10} \\
+ \quad & \frac{3\mathrm{tr}\hat{T}\hat{B}}{2} + \frac{339\mathrm{tr}\hat{T}^2}{20} + \frac{177\mathrm{tr}\hat{T}\mathrm{tr}\hat{B}}{5} + \frac{199\mathrm{tr}\hat{T}\mathrm{tr}\hat{L}}{5} + \frac{303(\mathrm{tr}\hat{T})^2}{10} \\
+ \quad & n_G\left[-\frac{232\alpha_1^2}{75} - \frac{7\alpha_1\alpha_2}{25} + \frac{166\alpha_2^2}{15} - \frac{548\alpha_1\alpha_3}{225} - \frac{4\alpha_2\alpha_3}{5} + \frac{1100\alpha_3^2}{9} \right] \\
+ \quad & n_G^2\left[-\frac{836\alpha_1^2}{135} - \frac{44\alpha_2^2}{15} - \frac{1936\alpha_3^2}{135} \right]\Bigg\}
\end{aligned}
$$

$$
\mathrm{tr}\hat{T}\hat{B} = \mathrm{tr}\left[\begin{pmatrix}\alpha_u & 0 & 0\\ 0 & \alpha_c & 0 \\ 0 & 0 & \alpha_t\end{pmatrix} V_{\mathrm{CKM}} \begin{pmatrix}\alpha_d & 0 & 0\\ 0 & \alpha_s & 0 \\ 0 & 0 & \alpha_b\end{pmatrix} V_{\mathrm{CKM}}^\dagger\right].
$$

Numerical effects of 3-loop corrections

3 loop effects to the β functions in SM

- β_1: $\alpha_1^2\alpha_3^2$ (+90%)
- β_2: $\alpha_2^2\alpha_3^2$ (+56%)
  $\alpha_2^3\alpha_3$ (+13%)
  α_2^4 (+37%) [Pickering, Gracey, Jones '01]
- β_3: α_3^4 (+137%) [Tarasov, Vladimirov, Zharkov'80, Larin, Vermaseren'93]
  $\alpha_3^3\alpha_2$ (+45%)
  $\alpha_3^2\alpha_2^2$ (+17%)
  $\alpha_3^3\alpha_t$ (-112%) [Steinhauser'99]
  $\alpha_3^2\alpha_t^2$ (+28%)
  $\alpha_3^2\alpha_2\alpha_t$ (-16%)

Numerical analysis

SM Input parameters [PDG]:

$$
\begin{aligned}
\sin^2\Theta^{\overline{\mathrm{MS}}} &= 0.23119 \pm 0.00014\,, \\
\alpha &= 1/137.036\,, \\
\Delta\alpha^{(5)}_{\mathrm{had}} &= 0.02761 \pm 0.00015\,, \\
\alpha_s(M_Z) &= 0.1184 \pm 0.0020\,.
\end{aligned}
$$

Numerical analysis

Conversion to $\overline{\text{MS}}$ scheme & 5 active flavours

$$\Delta\alpha^{(5),\overline{\text{MS}}} - \Delta\alpha^{(5),\text{OS}} = \frac{\alpha}{\pi}\left(\frac{100}{27} - \frac{1}{6} - \frac{7}{4}\ln\frac{M_Z^2}{M_W^2}\right)$$
$$\approx 0.0072 \quad \text{[PDG]}$$

$$\alpha^{(5),\overline{\text{MS}}}(M_Z) = \frac{\alpha}{1 - \Delta\alpha^{(5)}_{\text{lep}} - \Delta\alpha^{(5)}_{\text{had}} - \Delta\alpha^{(5)}_{\text{top}} - 0.0072}$$
$$= \frac{1}{127.960 \pm 0.021}$$

[Kühn, Steinhauser '98; Steinhauser'98; Teubner et al'10] .

Numerical analysis

Top-quark threshold effects [Fanchiotti et al ’92, Chetyrkin et al ’98]:

$$\begin{aligned}
\alpha^{(6),\overline{\rm MS}}(M_Z) &= 1/(128.129 \pm 0.021)\,, \\
\sin^2\Theta^{(6),\overline{\rm MS}}(M_Z) &= 0.23138 \pm 0.00014\,, \\
\alpha_s^{(6)}(M_Z) &= 0.1173 \pm 0.0020\,.
\end{aligned}$$

$\Rightarrow$ ready for running to high energies

Numerical analysis

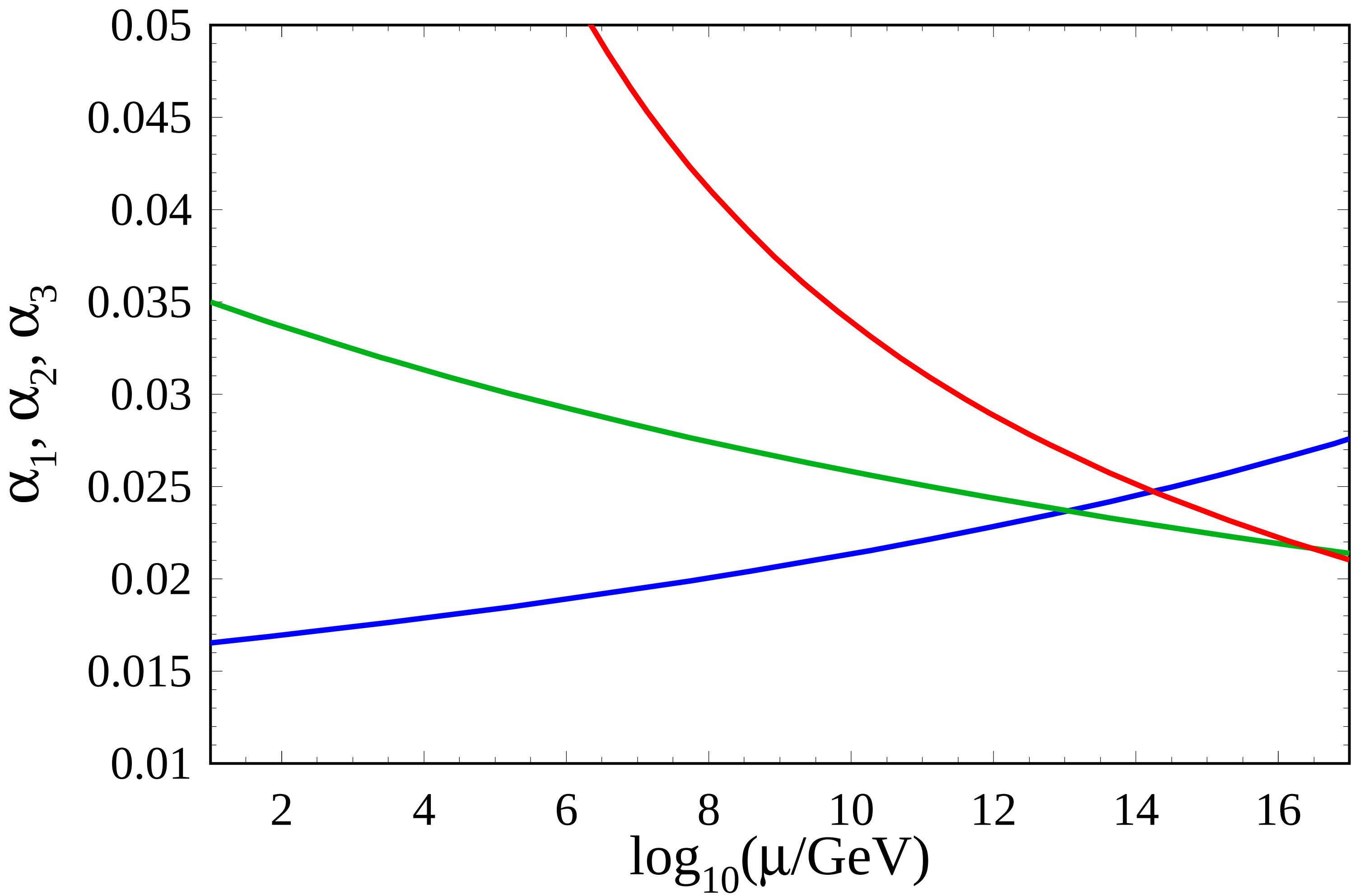
$$\begin{aligned}
\alpha_t^{\overline{\text{MS}}}(M_Z) &= 0.07514\,,\\
\alpha_b^{\overline{\text{MS}}}(M_Z) &= 0.00002064\,,\\
\alpha_\tau^{\overline{\text{MS}}}(M_Z) &= 8.077\cdot 10^{-6}\,,\\
4\pi\hat{\lambda} &= 0.13\,.
\end{aligned}$$

Numerical analysis

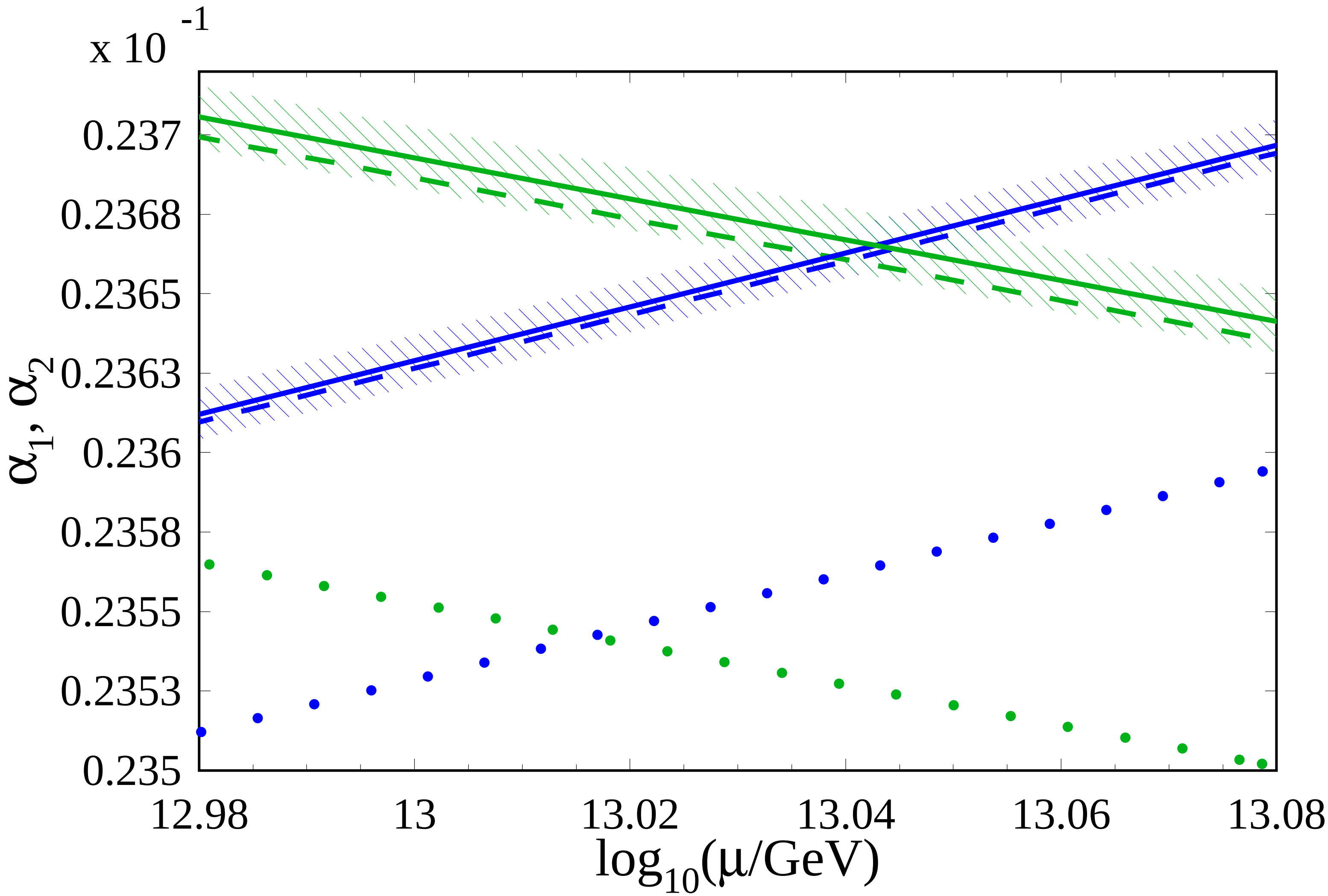
$$\alpha_t^{\overline{\text{MS}}}(M_Z) = 0.07514\,,$$
$$\alpha_b^{\overline{\text{MS}}}(M_Z) = 0.00002064\,,$$
$$\alpha_\tau^{\overline{\text{MS}}}(M_Z) = 8.077 \cdot 10^{-6}\,,$$
$$4\pi\hat{\lambda} = 0.13\,.$$

- For consistency we need:
  - 1-loop decoupling [Hempfling & Kniehl '95]
    2-loop RGEs for Yukawa couplings [see p.5]
  - no decoupling
    1-loop RGEs for Higgs quartic coupling [see p.5]

Running in SM

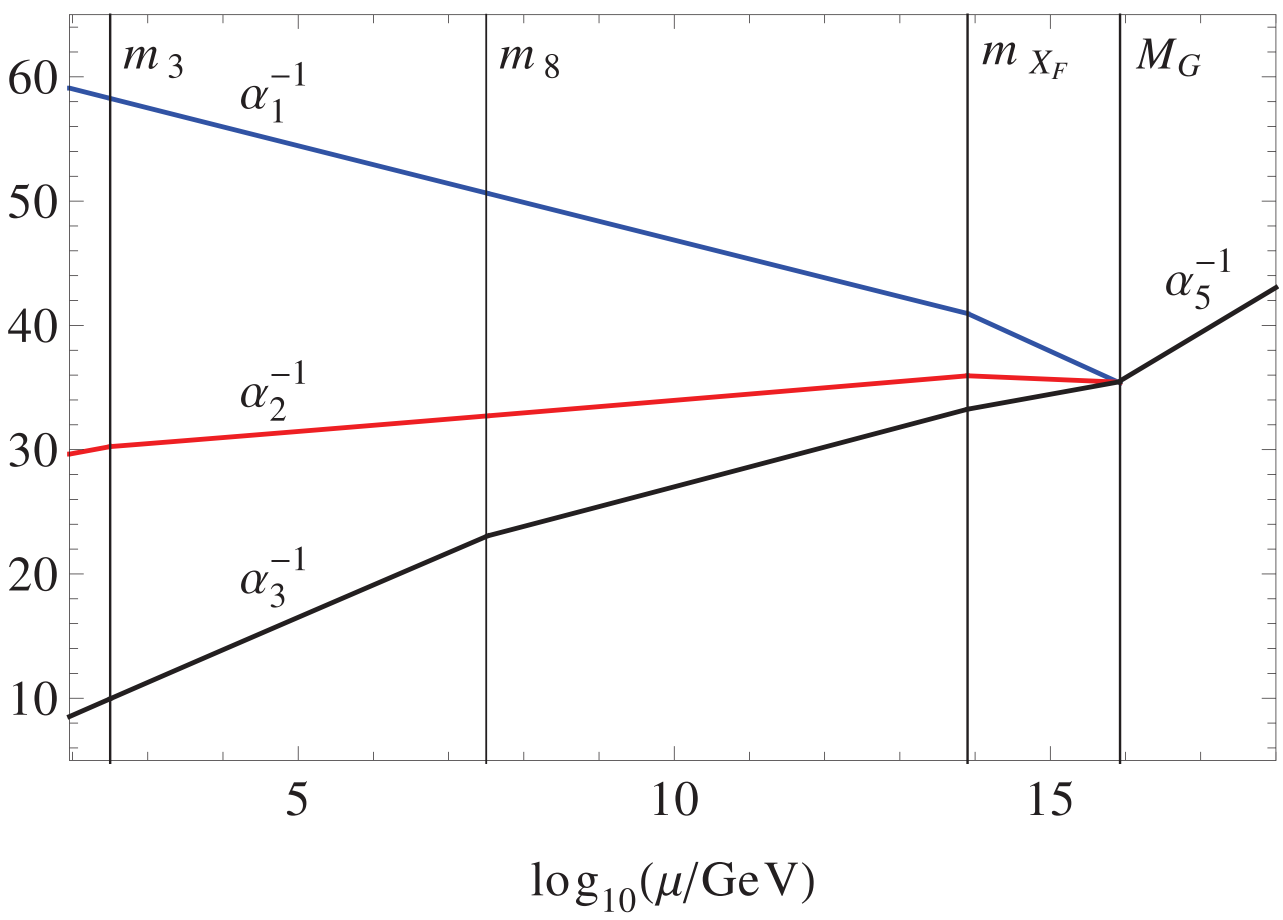


Running in SM



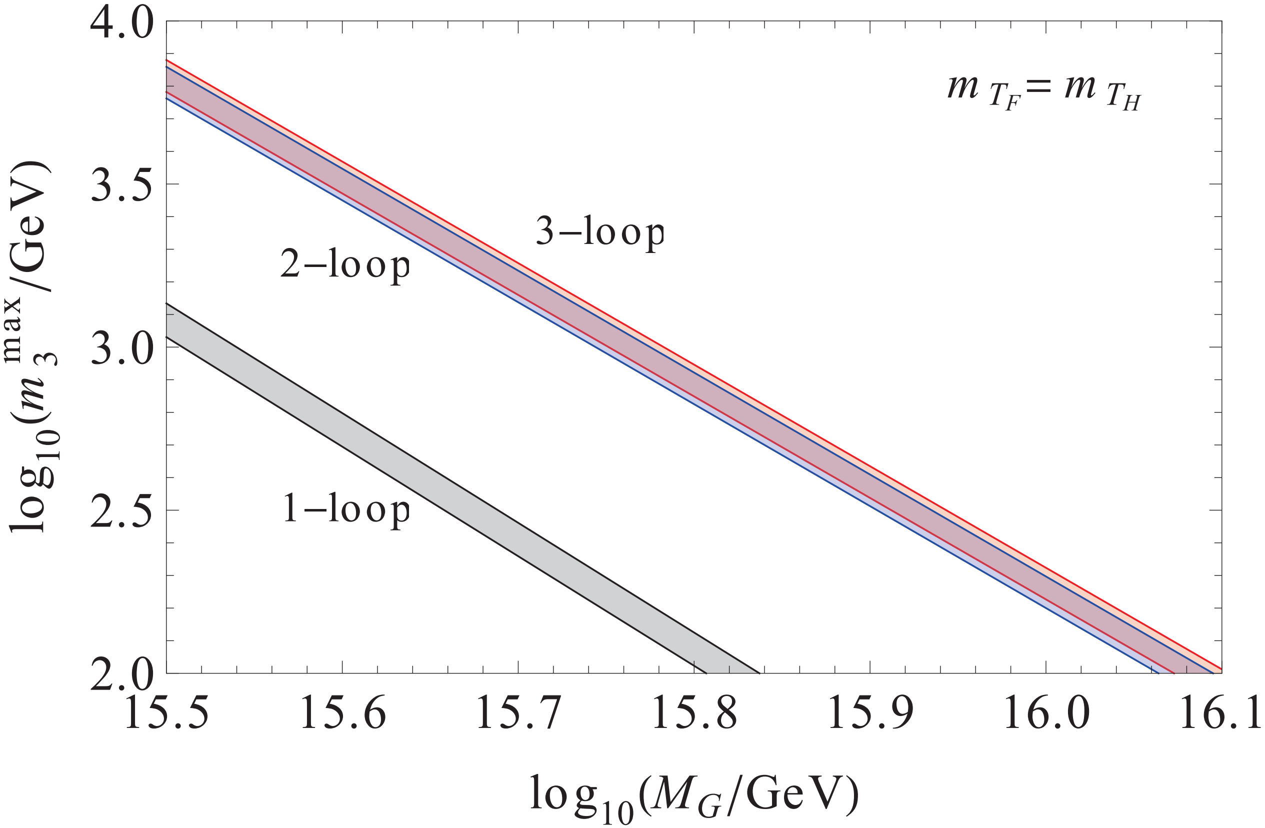
Running in SU(5)+ 24_F

SM $\rightarrow$ SM $+T_3$ $\rightarrow$ SM $+T + O_8$ $\rightarrow$ SM $+T_3 + O_8 + X_F$ $\rightarrow$ SU(5)+ 24_F



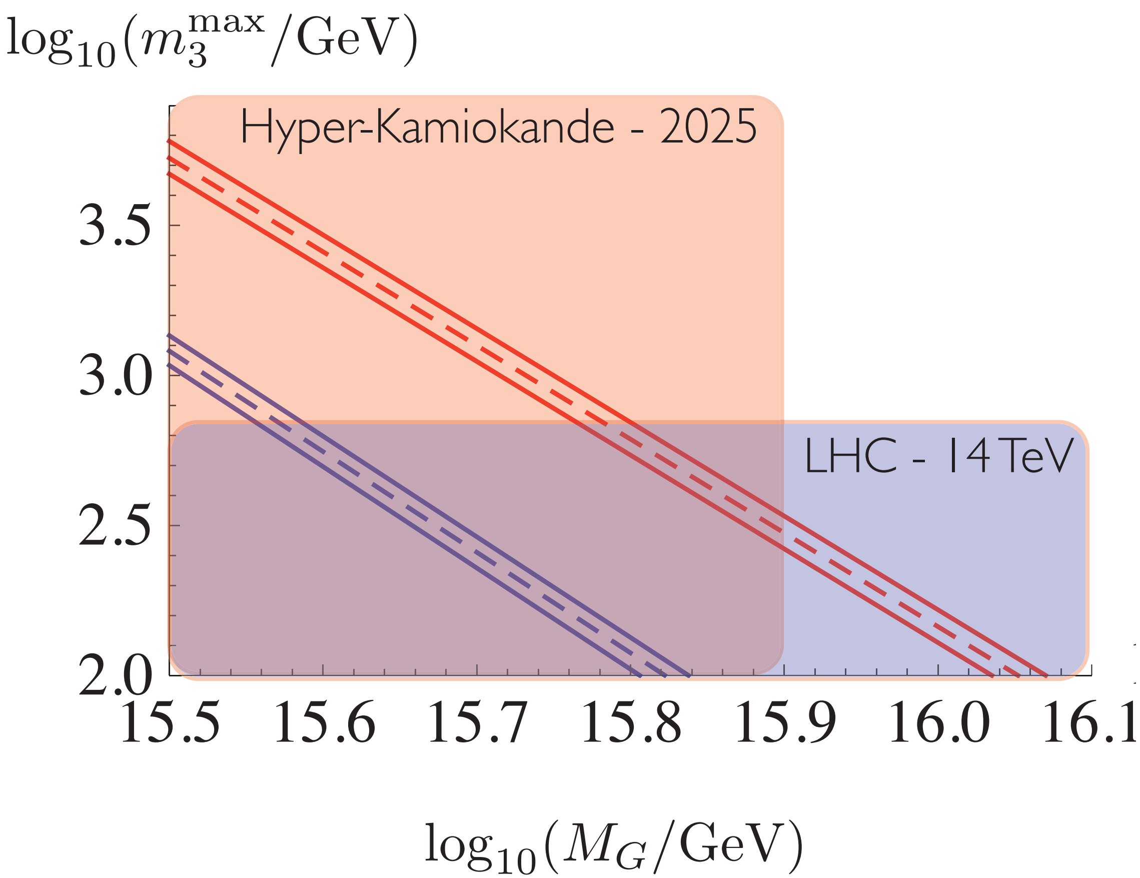
Running in SU(5)+ 24_F

SM $\rightarrow$ SM $+T_3 \rightarrow$ SM $+T + O_8 \rightarrow$ SM $+T_3 + O_8 + X_F \rightarrow$ SU(5)+ 24_F



Running in SU(5)+ 24_F

SM $\to$ SM $+T_3 \to$ SM $+T + O_8 \to$ SM $+T_3 + O_8 + X_F \to$ SU(5)+ 24_F



Conclusions

- Gauge β functions to 3 loops in terms of group invariants
- Automated setup
- Numerical effects for models based on SM:
  $\alpha_1 \, , \, \alpha_2 \;\approx\;$ present experimental uncertainty
- Essential tool for constraining BSM

Anomaly Cancellation

$$\mathcal{D}_{\alpha\beta\gamma} \equiv \mathrm{STr}(R_1^{\alpha} R_2^{\beta} R_3^{\gamma}) = 0$$

Anomaly Cancellation

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Examples

- $U(1) - U(1) - U(1)$: $\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F^3 = 0$

Anomaly Cancellation

$$\mathcal{D}_{\alpha\beta\gamma} \quad \equiv \quad \mathrm{STr}(R_1^{\alpha} R_2^{\beta} R_3^{\gamma}) = 0$$

Examples

- $U(1) - U(1) - U(1)$: $\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F^3 = 0$

$$\text{SM:} \quad 2 \times \left(\frac{1}{2}\right)^3 + (-1)^3 + 3 \times 2 \times \left(-\frac{1}{6}\right)^3 + 3 \times \left(\frac{2}{3}\right)^3 + 3 \times \left(-\frac{1}{3}\right)^3 = 0$$

Anomaly Cancellation

$$\mathcal{D}_{\alpha\beta\gamma} \equiv \mathrm{STr}(R_1^\alpha R_2^\beta R_3^\gamma) = 0$$

Examples

- $U(1) - U(1) - U(1)$: $\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F^3 = 0$
- $U(1) - G - G$:

$$\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F \mathrm{Tr}(R^\beta R^\gamma) = \sum_F Y_F C_2(R)\delta_{\beta\gamma} = 0$$

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Examples

- $U(1) - U(1) - U(1)$: $\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F^3 = 0$
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$$\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F \mathrm{Tr}(R^\beta R^\gamma) = \sum_F Y_F C_2(R)\delta_{\beta\gamma} = 0$$

SM and $\quad G = SU(3)$: $\quad C_2(R)\delta_{\beta\gamma}\left(2 \times (-\frac{1}{6}) + \frac{2}{3} - \frac{1}{3}\right) = 0$

Anomaly Cancellation

$$\mathcal{D}_{\alpha\beta\gamma} \equiv \mathrm{STr}(R_1^\alpha R_2^\beta R_3^\gamma) = 0$$

Examples

- $U(1) - U(1) - U(1)$: $\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F^3 = 0$
- $U(1) - G - G$:

$$\mathcal{D}_{\alpha\beta\gamma} = \sum_F Y_F \mathrm{Tr}(R^\beta R^\gamma) = \sum_F Y_F C_2(R) \delta_{\beta\gamma} = 0$$

- ...

Running in SU(5)+ 24_F

SM $\to$ SM $+T_3$ $\to$ SM $+T + O_8$ $\to$ SM $+T_3 + O_8 + X_F$ $\to$ SU(5)+ 24_F