Multiloop Integrand Reduction via Multivariate Polynomial Division

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Based on:

- P. Mastrolia, E. Mirabella and **T.P.**, *Integrand reduction of one-loop scattering amplitudes through Laurent series expansion,* JHEP **1206**, 095 (2012) [arXiv:1203.0291 [hep-ph]].
- P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *Scattering Amplitudes from Multivariate Polynomial Division,* Phys. Lett. B **718**, 173 (2012) [arXiv:1205.7087 [hep-ph]].
- P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division,* Phys. Rev. D **87**, 085026 (2013) [arXiv:1209.4319 [hep-ph]].
- H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, **T.P.**, J. F. von Soden-Fraunhofen and F. Tramontano Phys. Lett. B **721**, 74 (2013) [arXiv:1301.0493 [hep-ph]].
- S. Heinemeyer,. . . ,**T.P.** *et al.* [The LHC Higgs Cross Section Working Group Collaboration], *Handbook of LHC Higgs Cross Sections: 3. Higgs Properties,* arXiv:1307.1347 [hep-ph].
- G. Cullen, H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, **T.P.** and F. Tramontano *NLO QCD corrections to Higgs boson production plus three jets in gluon fusion,* arXiv:1307.4737 [hep-ph].
- P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes,* arXiv:1307.5832 [hep-ph].
- H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola and **T.P.**, *NLO QCD corrections to Higgs boson production in association with a top quark pair and a jet,* arXiv:1307.8437 [hep-ph].

Outline

1. Introduction and motivation
2. The integrand reduction of scattering amplitudes
3. Integrand reduction via polynomial division
4. Application at one-loop
5. Higher loops
6. Conclusions

Introduction and motivation

Motivation

- Understanding the basic analytic and algebraic structure of integrands and integrals of scattering amplitudes
- Exploration of methods for obtaining theoretical predictions in perturbative Quantum Field Theory at higher orders
- Automation of the computation of loop integrals

We developed a coherent framework for the integrand decomposition of Feynman integrals

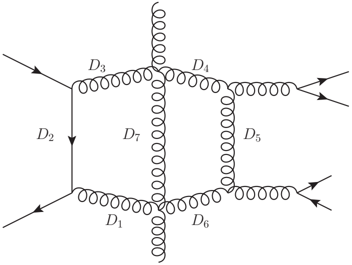
- based on simple concepts of algebraic geometry
- applicable at all loops

Integrand reduction

- Generic ℓ-loop integral:
  - is a rational function in the components of the loop momenta q_i
  - polynomial numerator $\mathcal{N}_{i_1 \ldots i_n}$

$$\mathcal{M}_n = \int d^d q_1 \ldots d^d q_\ell \;\; \mathcal{I}_{i_1 \ldots i_n}, \qquad \mathcal{I}_{i_1 \ldots i_n} \equiv \frac{\mathcal{N}_{i_1 \ldots i_n}}{D_{i_1} \ldots D_{i_n}}$$

  - quadratic polynomial denominators D_i
    - they correspond to Feynman loop propagators



$$D_i = \left(\sum_j (-)^{s_{ij}} q_j + p_i \right)^2 - m_i^2$$

Integrand reduction

The idea

Manipulate the integrand and reduce it to a linear combination of "simpler" integrands.

- The integrand-reduction algorithm leads to

$$\mathcal{I}_{i_1\cdots i_n} \equiv \frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\ldots D_{i_n}} = \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\ldots D_{i_n}} + \ldots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}} + \Delta_{\emptyset}$$

- The residues $\Delta_{i_1\ldots i_k}$ are irreducible polynomials in q_i
  - can't be written as a combination of denominators $D_{i_1}\ldots D_{i_k}$
  - universal topology-dependent parametric form
  - the coefficients of the parametrization are process-dependent

From integrands to integrals

- By integrating the integrand decomposition

$$\mathcal{M}_n = \int d^d q_1 \ldots d^d q_\ell \left(\frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\ldots D_{i_n}} + \ldots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}} + \Delta_\emptyset \right)$$

  - some terms vanish and do not contribute to the amplitude $\Rightarrow$ spurious terms
  - non-vanishing terms give Master Integrals (MIs)
- The amplitude is a linear combination of MIs
- The coefficients of this linear combination can be identified with some of the coefficients which parametrize the polynomial residues

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$\Rightarrow$ reduction to MIs $\equiv$ polynomial fit of the residues

The one-loop decomposition

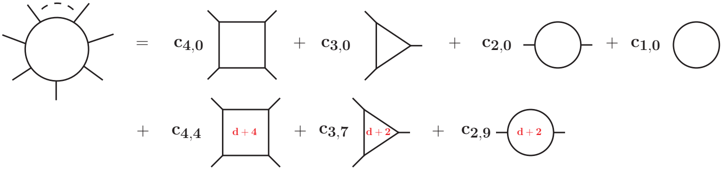
At one loop the result is well known:

- the integrand decomposition
  [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{j_1\cdots j_5} \frac{\Delta_{j_1j_2j_3j_4j_5}}{D_{j_1}D_{j_2}D_{j_3}D_{j_4}D_{j_5}} + \sum_{j_1j_2j_3j_4} \frac{\Delta_{j_1j_2j_3j_4}}{D_{j_1}D_{j_2}D_{j_3}D_{j_4}}$$

$$+ \sum_{j_1j_2j_3} \frac{\Delta_{j_1j_2j_3}}{D_{j_1}D_{j_2}D_{j_3}} + \sum_{j_1j_2} \frac{\Delta_{j_1j_2}}{D_{j_1}D_{j_2}} + \sum_{j_1} \frac{\Delta_{j_1}}{D_{j_1}}$$

- the integral decomposition



Integrand reduction and polynomials

- At ℓ-loops we want to achieve the integrand decomposition:

$$\mathcal{I}_{i_1\ldots i_n}(q_1,\ldots,q_\ell) \equiv \frac{\mathcal{N}_{i_1\ldots i_n}}{D_{i_1}\ldots D_{i_n}} = \underbrace{\frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\ldots D_{i_n}} + \ldots + \sum_{k=1}^{n} \frac{\Delta_{i_k}}{D_{i_k}}}_{\text{they must be irreducible}} + \Delta_\emptyset$$

- We trade $(q_1,\ldots,q_\ell)$ with their coordinates $\mathbf{z} \equiv (z_1,\ldots,z_m)$
  ⇒ numerator and denominators $\equiv$ polynomials in $\mathbf{z}$

$$\mathcal{I}_{i_1\ldots i_n}(\mathbf{z}) \equiv \frac{\mathcal{N}_{i_1\ldots i_n}(\mathbf{z})}{D_{i_1}(\mathbf{z})\ldots D_{i_n}(\mathbf{z})}$$

⇒ Integrand reduction $\equiv$ problem of multivariate polynomial division

The problem of the determination of the residues of a generic diagram has been solved. [Y. Zhang (2012), P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)]

Residues via polynomial division

Y. Zhang (2012), P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)

- Define the Ideal of polynomials

$$\mathcal{J}_{i_1\cdots i_n} \equiv \langle D_{i_1}, \ldots, D_{i_n} \rangle = \left\{ p(\mathbf{z}) \; : \; p(\mathbf{z}) = \sum_j h_j(\mathbf{z}) D_j(\mathbf{z}), \; h_j \in P[z] \right\}$$

- Take a Gröbner basis $G_{\mathcal{J}_{i_1\cdots i_n}}$ of $\mathcal{J}_{i_1\cdots i_n}$

$$G_{\mathcal{J}_{i_1\cdots i_n}} = \{g_1, \ldots, g_s\} \quad \text{such that} \quad \mathcal{J}_{i_1\cdots i_n} = \langle g_1, \ldots, g_s \rangle$$

- Perform the multivariate polynomial division $\mathcal{N}_{i_1\ldots i_n}/G_{\mathcal{J}_{i_1\cdots i_n}}$

$$\mathcal{N}_{i_1\cdots i_n}(z) = \underbrace{\sum_{k=1}^{n} \mathcal{N}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n}(z)\, D_{i_k}(z)}_{\text{quotient } \in \, \mathcal{J}_{i_1\cdots i_n}} + \underbrace{\Delta_{i_1\cdots i_n}(z)}_{\text{remainder}}$$

- The remainder $\Delta_{i_1\cdots i_n}$ is irreducible ⇒ can be identified with the residue

Recursive Relation for the integrand decomposition

P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)

The recursive formula

$$\mathcal{N}_{i_1\cdots i_n} = \sum_{k=1}^{n} \mathcal{N}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n}\, D_{i_k} + \Delta_{i_1\cdots i_n}$$

$$\mathcal{I}_{i_1\cdots i_n} \equiv \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}} = \sum_{k} \mathcal{I}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}}$$

- Fit-on-the-cut approach
  - from a generic $\mathcal{N}$, get the parametric form of the residues Δ
  - determine the coefficients sampling on the cuts (impose $D_i = 0$)
- Divide-and-Conquer approach
  - generate the $\mathcal{N}$ of the process
  - compute the residues by iterating the polynomial division algorithm

Fit-on-the-cut approach

[Ossola, Papadopoulos, Pittau (2007)]

The decomposition of the numerator

$$\mathcal{N}_{i_1\cdots i_n} = \Delta_{i_1\cdots i_n} + \sum_k \Delta_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n} D_{i_k} + \ldots$$

- Fit the coefficients of the residues sampling on the multiple cuts
- First step: n-ple cut
    - impose $D_{i_1} = \ldots = D_{i_n} = 0$

$$\Delta_{i_1\cdots i_n} = \mathcal{N}_{i_1\cdots i_n}$$

- Further steps: k-ple cut
    - impose $D_{i_1} = \ldots = D_{i_k} = 0$ for any subset $\{i_1 \ldots i_k\}$

$$\Delta_{i_1\cdots i_k} = \frac{\mathcal{N}_{i_1\cdots i_n} - \text{higher-point contibrutions}}{\prod_{h \neq i_1,\ldots,i_k} D_h}$$

Fit-on-the-cut approach: The reducibility criterion

What happens if a cut has no solution?

The reducibility criterion

- If a cut $D_{i_1} = \ldots = D_{i_k} = 0$ has no solutions, the associated residue vanishes. In other words, any numerator is completely reducible.
- This generally happens with overdetermined systems i.e. when the number of cut denominators is higher than the one of loop coordinates.

- When $D_{i_1} = \ldots = D_{i_k} = 0$ has no solution:

$$\Delta_{i_1\ldots i_k} = 0 \qquad \Rightarrow \text{no need to perform the fit}$$

$$\mathcal{N}_{i_1\cdots i_n} = \sum_{k=1}^{n} \mathcal{N}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n} D_{i_k}$$

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{k} \mathcal{I}_{i_1\cdots i_{k-1} i_{k+1}\cdots i_n}$$

Fit-on-the-cut approach: The maximum-cut theorem

The maximum-cut theorem

- We define maximum-cut, a cut where

$$\#(\text{cut-denominators}) \equiv \#(\text{components-of-loop-momenta})$$

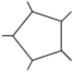
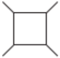
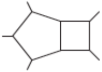
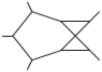
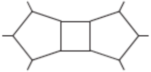
- In non-special kinematic configurations it has a finite number of solutions

$$\#(\text{coefficients-of-the-residue}) = \#(\text{solutions-of-the-cut})$$

- The fit-on-the-cut approach therefore gives a number of equations which is equal to the number of unknown coefficients.

Fit-on-the-cut approach: The maximum-cut theorem

Examples:

diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\ i}$	4
	$\sum_{i=0}^{7} c_i z^{\ i}$	8		$\sum_{i=0}^{7} c_i z^{\ i}$	8

Fit-on-the-cut approach

Pros:

- each multiple cut projects out the corresponding residue
  - ⇒ the systems of equations for the coefficients are much smaller
- can be implemented either analytically or numerically
- very successful application at one-loop

Cons:

- at higher-loops the solutions of the cuts can be difficult to find
- it cannot be applied in the presence of higher powers of denominators
  - a cut denominator might be equal to an uncut denominator

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  - it that case

$$\frac{\mathcal{N}_{i_1\cdots i_n} - \text{higher-point contibrutions}}{\prod_{h\neq i_1,\ldots,i_k} D_h} = \frac{0}{0}$$

Fit-on-the-cut approach

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  - a cut d minator
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$$\prod_{h\neq i_1,\ldots,i_k} D_h \qquad \frac{0}{0}$$

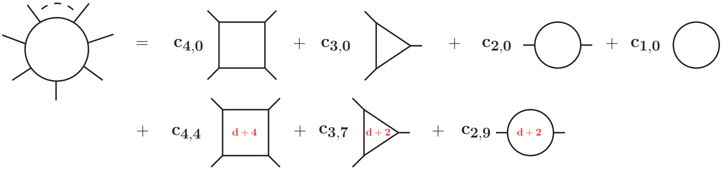
OBSERVATION:
these issues are not present
in the divide-and-conquer approach
which instead can be applied to
any integrand

One-loop decomposition from polynomial division

P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)

- Start from the most general one-loop amplitude in $d = 4 - 2\epsilon$
- Apply the recursive formula for the integrand decomposition
  - ⇒ it reproduces the OPP result [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]
- Drop the spurious terms

⇒ Get the most general integral decomposition (well knwon result)



One-loop decomposition from polynomial division

At one loop in $4-2\epsilon$ dimensions:

- 5 coordinates $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5)$
    - 4 components (z_1,z_2,z_3,z_4) of q w.r.t. a 4-dimensional basis
    - $z_5 = \mu^2$ encodes the (-2ϵ)-dependence on the loop momentum
- we start with

$$\mathcal{I}_n \equiv \mathcal{I}_{1\dots n} = \frac{\mathcal{N}_{1\dots n}(\mathbf{z})}{D_1(\mathbf{z})\dots D_n(\mathbf{z})}$$

most general 1-loop numerator

generic 1-loop denominators

- if $m>5$ any integrand $\mathcal{I}_{i_1\dots i_m}$ is reducible (reducibility criterion)

$$\mathcal{I}_{i_1\cdots i_m} = \sum_k \mathcal{I}_{i_1\dots i_{k-1} i_{k+1}\dots i_m}, \quad \Rightarrow \quad \Delta_{i_1\cdots i_m} = 0 \quad \text{for } m>5$$

- for $m \leq 5$ the polynomial-division algorithm gives the already-known parametric form of the residues $\Delta_{ijk\dots}$

- Choice of 4-dimensional basis for an m-point residue

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,, \qquad e_3^2 = e_4^2 = \delta_{m4}\,, \qquad e_3 \cdot e_4 = -(1 - \delta_{m4})$$

- Coordinates: $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$q^\mu_{\text{4-dim}} = -p^\mu_{i_1} + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu\,, \qquad q^2 = q^2_{\text{4-dim}} - \mu^2$$

- Generic numerator

$$\mathcal{N}_{i_1\cdots i_m} = \sum_{j_1,\ldots,j_5} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,, \qquad (j_1 \ldots j_5) \quad \text{such that} \quad \text{rank}(\mathcal{N}_{i_1\cdots i_m}) \leq m$$

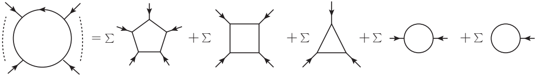
- Residues

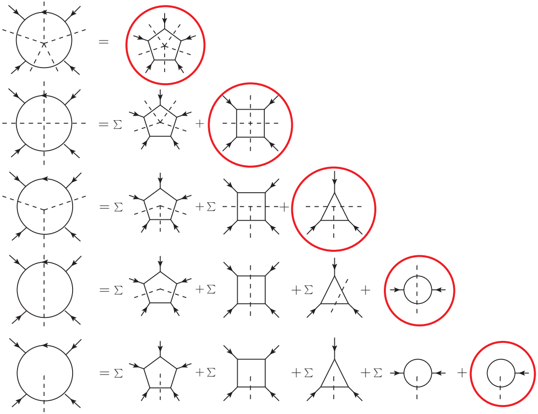
$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4) \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$

- It can be easily extended to higher-rank numerators

Fit-on-the-cut at 1-loop

Integrand decomposition:





Fit-on-the cut

fit m-point residues on m-ple cuts

Improved 1-loop Reduction with NINJA

P. Mastrolia, E. Mirabella, T.P. (2012)

The integrand reduction via Laurent expansion:

- fits residues by taking their asymptotic expansions on the cuts
- yields diagonal systems of equations for the coefficients
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  - Laurent expansions via a simplified polynomial-division algorithm
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⇒ see P. Mastrolia's talk for more details

Extension to higher loops

- The integrand-level approach to scattering amplitudes at one-loop
  - can be used to compute any amplitude in any QFT
  - has been implemented in several codes, some of which public [SAMURAI, CUTTOOLS, NGLUONS]
  - has produced (and is still producing) results for LHC [GOSAM (see P. Mastrolia's talk), FORMCALC, BLACKHAT, MADLOOP, NJETS, OPENLOOP . . .]
- At two or higher loops
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The integrand-level approach might be a tool for understanding the structure of multi-loop scattering amplitudes and a method for their evaluation.

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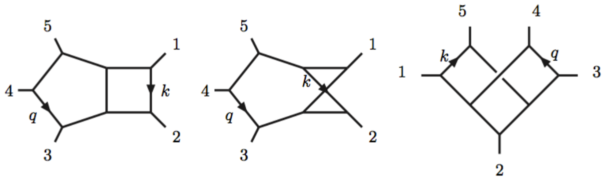
The integrand-level approach might be a tool for understanding the structure of multi-loop scattering amplitudes and a method for their evaluation.

- . . . we are moving the first steps in this direction

$\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA amplitudes

P. Mastrolia, G. Ossola (2011); P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)



- Examples in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA amplitudes ($d=4$)
  - generation of the integrand
    - graph based [Carrasco, Johansson (2011)]
    - unitarity based [U. Schubert (Diplomarbeit)]
  - fit-on-the-cut approach for the reduction
- Results:

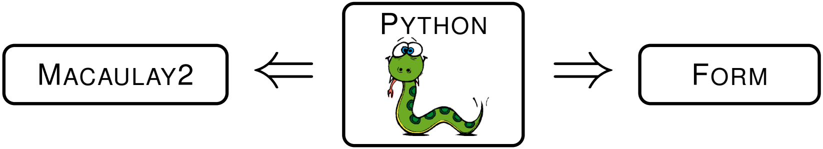
$\mathcal{N}=4$ linear combination of 8 and 7-denominators MIs
$\mathcal{N}=8$ linear combination of 8, 7 and 6-denominators MIs

Divide-and-Conquer approach

P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2013)

The divide-and-conquer approach to the integrand reduction

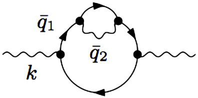
- does not require the knowledge of the solutions of the cut
- can always be used to perform the reduction in a finite number of purely algebraic operations
- has been automated in a Python package which uses Macaulay2 and Form for algebraic operations



Macaulay2 $\Leftarrow$ Python $\Rightarrow$ Form

- also works in special cases where the fit-on-the-cut approach is not applicable (e.g. in presence of double denominators)

Divide-and-Conquer approach: a simple example



$$\mathcal{I}_{11234} = \frac{\mathcal{N}_{11234}}{D_1^2 D_2 D_3 D_4}$$

$$\begin{aligned} D_1 &= \bar{q}_1^2 - m^2 \,, \\ D_2 &= (\bar{q}_1 - k)^2 - m^2 \,, \\ D_3 &= \bar{q}_2^2 \,, \\ D_4 &= (\bar{q}_1 + \bar{q}_2)^2 - m^2 \end{aligned}$$

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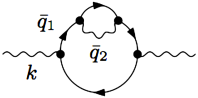

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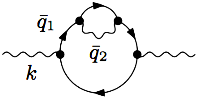
- Basis $\{e_i\} \equiv \{k, k_\perp, e_3, e_4\}$ and coordinates $\mathbf{z} = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, \mu_{11}, \mu_{12}, \mu_{22})$

$$\bar{q}_1^{(\text{4-dim})} = \sum_i x_i\, e_i, \qquad \bar{q}_2^{(\text{4-dim})} = \sum_i y_i\, e_i, \qquad (\bar{q}_i \cdot \bar{q}_j) = (\bar{q}_i^{(\text{4-dim})} \cdot \bar{q}_j^{(\text{4-dim})}) - \mu_{ij}$$

Divide-and-Conquer approach: a simple example



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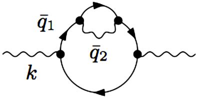
- division of $\mathcal{N}_{11234}$ modulo $\mathcal{G}_{\mathcal{J}_{11234}} (= \mathcal{G}_{\mathcal{J}_{1234}})$

$$\mathcal{N}_{11234} = \underbrace{\mathcal{N}_{1234} D_1 + \mathcal{N}_{1134} D_2 + \mathcal{N}_{1124} D_3 + \mathcal{N}_{1123} D_4}_{\text{quotients}} + \underbrace{\Delta_{11234}}_{\text{remainder}}$$

Divide-and-Conquer approach: a simple example



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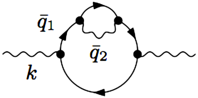
- division of $\mathcal{N}_{i_1 i_2 i_3 i_4}$ modulo $\mathcal{G}_{\mathcal{J}_{i_1 i_2 i_3 i_4}}$, e.g.

$$\mathcal{N}_{1234}/\mathcal{G}_{\mathcal{J}_{1234}} \quad \Rightarrow \quad \mathcal{N}_{1234} = \underbrace{\mathcal{Q}_{234}^{(1234)} D_1 + \mathcal{Q}_{134}^{(1234)} D_2 + \mathcal{Q}_{124}^{(1234)} D_3 + \mathcal{Q}_{123}^{(1234)} D_4}_{\text{quotients}} + \underbrace{\Delta_{1234}}_{\text{remainder}}$$

$$\mathcal{N}_{1134}/\mathcal{G}_{\mathcal{J}_{1134}} \quad \Rightarrow \quad \mathcal{N}_{1134} = \underbrace{\mathcal{Q}_{134}^{(1134)} D_1 + \mathcal{Q}_{114}^{(1134)} D_3 + \mathcal{Q}_{113}^{(1134)} D_4}_{\text{quotients}} + \underbrace{\Delta_{1134}}_{\text{remainder}}$$

Divide-and-Conquer approach: a simple example



$$\mathcal{I}_{11234} = \frac{\mathcal{N}_{11234}}{D_1^2 D_2 D_3 D_4}$$

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- Basis $\{e_i\} \equiv \{k, k_\perp, e_3, e_4\}$ and coordinates $\mathbf{z} = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, \mu_{11}, \mu_{12}, \mu_{22})$

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$$\mathcal{N}_{11234} = \underbrace{\mathcal{N}_{1234} D_1 + \mathcal{N}_{1134} D_2 + \mathcal{N}_{1124} D_3 + \mathcal{N}_{1123} D_4}_{\text{quotients}} + \underbrace{\Delta_{11234}}_{\text{remainder}}$$

- division of $\mathcal{N}_{i_1 i_2 i_3 i_4}$ modulo $\mathcal{G}_{\mathcal{J}_{i_1 i_2 i_3 i_4}}$

$$\mathcal{N}_{11234} = \underbrace{\mathcal{N}_{234} D_1^2 + \mathcal{N}_{134} D_1 D_2 + \mathcal{N}_{124} D_1 D_3 + \mathcal{N}_{123} D_1 D_4 + \mathcal{N}_{114} D_2 D_3 + \mathcal{N}_{113} D_2 D_4}_{\text{(sums of) quotients}}$$

$$+ \underbrace{\Delta_{1234} D_1 + \Delta_{1134} D_2 + \Delta_{1124} D_3 + \Delta_{1123} D_4}_{\text{remainders}} + \Delta_{11234}$$

Divide-and-Conquer approach: a simple example

- after a further step (division $\mathcal{N}_{i_1 i_2 i_3}/\mathcal{G}_{\mathcal{J}_{i_1 i_2 i_3}}$) no quotient remains

$$\mathcal{N}_{11234} = \Delta_{11234} + \Delta_{1234} D_1 + \Delta_{1134} D_2 + \Delta_{1124} D_3 + \Delta_{1123} D_4 + \Delta_{234} D_1^2 + \Delta_{114} D_2 D_3 + \Delta_{113} D_2 D_4$$

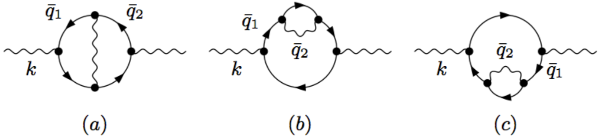
- the integrand decomposition becomes

$$\begin{aligned}
\mathcal{I}_{11234} = \frac{\mathcal{N}_{11234}}{D_1^2 D_2 D_3 D_4} = {} & \frac{\Delta_{11234}}{D_1^2 D_2 D_3 D_4} + \frac{\Delta_{1234}}{D_1 D_2 D_3 D_4} + \frac{\Delta_{1134}}{D_1^2 D_3 D_4} + \frac{\Delta_{1124}}{D_1^2 D_2 D_4} \\
& + \frac{\Delta_{1123}}{D_1^2 D_2 D_3} + \frac{\Delta_{234}}{D_2 D_3 D_4} + \frac{\Delta_{114}}{D_1^2 D_4} + \frac{\Delta_{113}}{D_1^2 D_3}
\end{aligned}$$

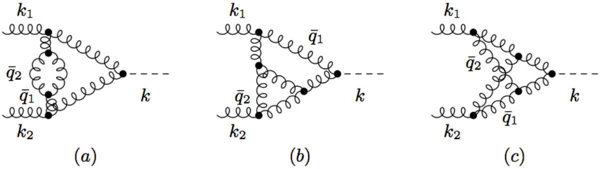
$$\begin{aligned}
\Delta_{11234} &= 16 m^2 \left(k^2 + 2\, m^2 - k^2 \epsilon \right) , \\
\Delta_{1234} &= 16 \left[(q_2 \cdot k)(1-\epsilon)^2 + m^2 \right] , \\
\Delta_{1124} &= -\Delta_{1123} = 8\,(1-\epsilon) \left[k^2 (1-\epsilon) + 2\, m^2 \right] , \\
\Delta_{1134} &= -16 m^2 \, (1-\epsilon) , \\
\Delta_{113} &= -\Delta_{114} = \Delta_{234} = 8 \, (1-\epsilon)^2 .
\end{aligned}$$

Examples of divide-and-conquer approach

- Photon self-energy in massive QED, $(4-2\epsilon)$-dimensions



- Diagrams entering $gg \to H$, in $(4-2\epsilon)$-dimensions



Conclusions and Outlook

- Conclusions
  - We developed a general framework for the reduction at the integrand level
    - can be applied to any amplitude in any QFT, at every loop order
  - At one loop
    - naturally reproduces known results (OPP)
    - allows to express any amplitude in terms of known MIs
    - can be improved with the Laurent-expansion approach (NINJA)
  - At higher loops
    - it gives a recursive formula for the integrand decomposition
    - generates the form of the residue for every cut
    - can decompose any integrand with purely algebraic operations (divide-and-conquer approach)

- Outlook
  - application to a full 2-loop QED/QCD process
  - combine integrand reduction with other techniques (e.g. IBP)

THANK YOU
FOR YOUR ATTENTION