FROM WEBS TO POLYLOGS

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IR SINGULARITIES

Motivation

- IR singularities are key to cross section calculations
  (subtraction, resummation)

- Window to all-order structure of scattering amplitudes

Strategy

- Compute the renormalization $Z$ of a product of non-locally liklike Wilson lines $\langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$

- The soft anomalous dimension $\frac{dZ}{d \ln \mu} = -Z \Gamma$
Part I: The non-Abelian exponentiation theorem
Part II: evaluating gluon-exchange webs
 webs

Friday, 27 September 13
$$S = \exp w = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$$

- Abelian case (1961)
  Yennie-Frautschi-Suura

- Non-abelian, colour singlet (two-line) case (1983)
  Gatheral, Frenkel-Taylor

- Non-abelian, multi-line case (2010-2013)
  EG-Laenen-Stavenga-White & Mitov-Sterman-Sung
  EG-Smillie-White
The exponent only receives contributions from connected diagrams: (``connected'' is defined after removing the Wilson lines!)

\[ S = \exp \left\{ \begin{array}{c} + \\ + \\ + \cdots \end{array} \right\} \]

Expanding the exponential exactly reproduces all disconnected diagrams:
Non-Abelian exponentiation in the two-line case

- Diagrams fall into two categories:
  - Irreducible - No subdivergences - Contributes to exponent
  - Reducible colour structure - Has a cusp-related subdivergence - Does not contribute

Diagrams contributing to the exponent (webs) have a non-Abelian colour factor and no subdivergences!

- This goes hand-in-hand with the renormalization of the cusp: webs have a single overall divergence

\[
Z = \exp \left\{ \frac{1}{2} \int_{\mu^2}^{\infty} \frac{d\lambda^2}{\lambda^2} \Gamma_S(\alpha_s(\lambda^2)) \right\} = \exp \left\{ \frac{1}{2\epsilon} \Gamma_S^{(1)} \alpha_s + \frac{1}{4\epsilon} \Gamma_S^{(2)} \alpha_s^2 + \frac{1}{6\epsilon} \Gamma_S^{(3)} \alpha_s^3 + O(\alpha_s^4) \right\}
\]
In the multi-leg case **this separation breaks down**: reducible diagrams do contribute to the exponent …

- Is the colour factor still non-Abelian?
- How can we compute the exponent directly? If so, what do we gain?
- Reducible diagrams have subdivergences - how is this consistent with renormalizability of the vertex?
Theorem: all colour structures in the exponent $w$ correspond to connected graphs

EG-Smillie-White (2013)
NON-ABELIAN EXPONENTIATION: MULTI-LEG CASE

Are reducible graphs reproduced by exponentiating lower order ones?

Exponentiating:

\[(1a) + (1b)\]

\[(2a) + (2b)\]

At 2-loops, do we get + ?
Non-Abelian exponentiation: multi-leg case

Exponentiating:

\[ D_{(1a)} + D_{(1b)} = (1a) + (1b) = \mathcal{F}(1a) \mathcal{T}_{1}^{(a)} \mathcal{T}_{2}^{(a)} + \mathcal{F}(1b) \mathcal{T}_{2}^{(b)} \mathcal{T}_{3}^{(b)} \]

\[ \frac{1}{2} \left[ D_{(1a)} + D_{(1b)} \right]^2 = \frac{1}{2} \left[ D_{(1a)} D_{(1b)} + D_{(1b)} D_{(1a)} + \cdots \right] \]

\[ = \frac{1}{2} \mathcal{F}(1a) \mathcal{F}(1b) \mathcal{T}_{1}^{(a)} \left[ \mathcal{T}_{2}^{(a)} \mathcal{T}_{2}^{(b)} + \mathcal{T}_{2}^{(b)} \mathcal{T}_{2}^{(a)} \right] \mathcal{T}_{3}^{(b)} \]

\[ = \frac{1}{2} \left[ \mathcal{F}(2a) + \mathcal{F}(2b) \right] \left[ \mathcal{C}(2a) + \mathcal{C}(2b) \right] \]

At 2-loops we get:

While
Exponentiating 1-loop diagrams yields:

\[
\frac{1}{2} \left[ D_{(1a)} + D_{(1b)} \right]^2 = \frac{1}{2} \left[ \mathcal{F}(2a) + \mathcal{F}(2b) \right] \left[ C(2a) + C(2b) \right]
\]

While the 2-loop amplitude is:

\[
\frac{1}{2} \left[ \mathcal{F}(2a) - \mathcal{F}(2b) \right] \left[ C(2a) - C(2b) \right] = \mathcal{F}(2a) C(2a) + \mathcal{F}(2b) C(2b)
\]

The 2-loop contribution to the exponent is therefore:

In the multi-leg case, reducible diagrams do contribute to the exponent!
The 2-loop contribution to the exponent

\[ w_{121}^{(2)} = \frac{1}{2} \left[ \mathcal{F}(2a) - \mathcal{F}(2b) \right] \times O(1/\epsilon) \]

These properties (single pole, maximally non-Abelian colour structure) are familiar from the two-line case.
Non-Abelian exponentiation: multi-leg case

- In contrast to the 2-line case, reducible diagrams enter the exponent.

- Individual diagrams do not have "web properties". Only particular linear combinations that do - enter the exponent:

\[
S = \exp \left[ \sum_i W_i \right]
\]

\[
W_i = \sum_{\{D\}_i} \mathcal{F}(D) \tilde{C}(D)
\]

Each diagram enters the exponent with a modified colour factor \( \tilde{C}(D) \).

- modified colour factors \( \tilde{C}(D) \) are linear combination of (ordinary) colour factors of diagrams that are obtained by permuting attachments to the Wilson lines, so:

\[
W_i = \sum_D \mathcal{F}(D) \sum_{D'} R_{DD'} \ C(D') = \mathcal{F}^T \ R \ C
\]

\[
\tilde{C}(D)
\]

web mixing matrix

- Using the replica trick we derived a general combinatorial formula for \( R \).
The entire web contributes:

\[
\begin{pmatrix}
F(3a) \\
F(3b) \\
F(3c) \\
F(3d)
\end{pmatrix}^T \frac{1}{6} \begin{pmatrix}
1 & -1 & -1 & 1 \\
-2 & 2 & 2 & -2 \\
1 & -1 & -1 & 1
\end{pmatrix} \begin{pmatrix}
C(3a) \\
C(3b) \\
C(3c) \\
C(3d)
\end{pmatrix}
\]

Kinematics  Web mixing matrix  Colour
THREE-LOOP WEB: EXAMPLE

\[
\begin{array}{c}
\frac{1}{6} \begin{pmatrix}
\mathcal{F}(3a) \\
\mathcal{F}(3b) \\
\mathcal{F}(3c) \\
\mathcal{F}(3d)
\end{pmatrix}^T \\
\begin{pmatrix}
1 & -1 & -1 & 1 \\
-2 & 2 & 2 & -2 \\
1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\mathcal{C}(3a) \\
\mathcal{C}(3b) \\
\mathcal{C}(3c) \\
\mathcal{C}(3d)
\end{pmatrix}
\end{array}
\]

subdivergences cancel

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MIXING MATRICES: FOUR-LOOP EXAMPLE

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A. It is \textit{idempotent}: \[ R^2 = R \]

\( R \) is diagonalisable, with all its eigenvalues 0 or 1.

B. \textit{Its rows sum to zero:}

\[
\sum_{D'} R_{DD'} = 0
\]

C. \textit{Its columns, weighted by a symmetry factor } \( s(D) \), \textit{sum to zero:}

\[
\sum_{D} s(D) R_{DD'} = 0
\]
**MIXING MATRICES: FOUR-LOOP EXAMPLE**

The resulting mixing matrix:

$$\tilde{C} = RC = \frac{1}{24}$$

$$\begin{pmatrix}
  6 & -6 & 2 & 2 & -2 & 4 & -4 & 2 & -2 & -2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-6 & 6 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & 4 & -4 & 4 & -4 & 0 & 0 \\
 2 & -2 & 6 & 2 & 4 & -4 & -2 & 2 & -6 & 4 & 4 & -4 & 4 & 0 & 0 \\
 2 & -2 & -2 & 6 & 2 & 4 & -4 & -2 & -6 & 2 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & 2 & 6 & 4 & -4 & -6 & -2 & -2 & 4 & -4 & 4 & -4 & 0 & 0 \\
 2 & -2 & 2 & 2 & 2 & 4 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -6 & 2 & -2 & -4 & 4 & 2 & 6 & -2 & 4 & 4 & -4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -2 & -2 & -6 & -4 & 4 & 6 & 2 & 2 & -4 & 4 & 4 & 0 & 0 \\
 2 & -2 & 2 & -2 & 2 & 0 & 0 & -2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & 2 & 0 & 0 & -2 & -2 & 2 & 0 & -4 & 4 & 0 & 0 & 0 \\
 2 & -2 & -2 & 2 & -2 & 0 & 0 & 2 & 2 & -2 & 0 & 4 & -4 & 0 & 0 & 0 \\
 18 & -6 & -6 & -6 & -18 & 12 & 12 & -6 & -18 & 12 & 12 & 12 & 12 & 24 & 0 \\
-6 & -18 & -18 & -18 & -6 & 12 & 12 & -18 & -6 & 12 & 12 & 12 & 12 & 0 & 24 \\
\end{pmatrix}$$

$$C[[1, 2], [3, 1], [3, 4], [2, 4]]$$
$$C[[1, 2], [2, 3], [3, 4], [4, 1]]$$
$$C[[1, 2], [3, 2], [3, 4], [4, 1]]$$
$$C[[1, 2], [3, 2], [4, 3], [1, 4]]$$
$$C[[1, 2], [1, 3], [4, 3], [4, 2]]$$
$$C[[1, 2], [3, 2], [3, 4], [1, 4]]$$
$$C[[1, 2], [1, 3], [3, 4], [4, 2]]$$
$$C[[1, 2], [3, 1], [4, 3], [4, 2]]$$
$$C[[1, 2], [2, 3], [4, 3], [1, 4]]$$
$$C[[1, 2], [3, 1], [3, 4], [2, 4]]$$
$$C[[1, 2], [3, 2], [4, 3], [4, 1]]$$
$$C[[1, 2], [3, 1], [4, 3], [2, 4]]$$
MIXING MATRICES: FOUR-LOOP EXAMPLE

This mixing matrix has rank 5 (5 eigenvectors with eigenvalue 1, the rest 0) corresponding to the colour factors:
\[
\frac{dZ}{d \ln \mu} = -Z \Gamma
\]

In the multiparton case:

\[
Z = \exp \left\{ \frac{1}{2\epsilon} \Gamma_S^{(1)} \alpha_s + \left( \frac{1}{4\epsilon} \Gamma_S^{(2)} - \frac{b_0}{4\epsilon^2} \Gamma_S^{(1)} \right) \alpha_s^2 \right. \\
+ \left. \left( \frac{1}{6\epsilon} \Gamma_S^{(3)} + \frac{1}{48\epsilon^2} \left[ \Gamma_S^{(1)}, \Gamma_S^{(2)} \right] \right) \right. \\
- \frac{1}{6\epsilon^2} \left( b_0 \Gamma_S^{(2)} + b_1 \Gamma_S^{(1)} \right) + \frac{b_0^2}{6\epsilon^3} \Gamma_S^{(1)} \} \alpha_s^3 + \mathcal{O}(\alpha_s^4)
\]

multiple poles occur due to two distinct reasons:
1) running coupling
2) commutators (only in the multi-leg case, and beyond the planar limit)

Specific subdivergences of the multi-eikonal vertex survive in the exponent, BUT all multiple poles are predicted by lower orders. Only \(\mathcal{O}(1/\epsilon)\) are new. In particular, there no \(1/\epsilon^n\) at \(\mathcal{O}(\alpha_s^n)\).
Renormalization of Wilson Lines

To compute the anomalous dimension

\[ \Gamma^{(1)} = -2w^{(1,-1)} \]

\[ \Gamma^{(2)} = -4w^{(2,-1)} - 2 \left[ w^{(1,-1)}, w^{(1,0)} \right] \]

\[ \Gamma^{(3)} = -6w^{(3,-1)} + \frac{3}{2}b_0 \left[ w^{(1,-1)}, w^{(1,1)} \right] + 3 \left[ w^{(1,0)}, w^{(2,-1)} \right] + 3 \left[ w^{(2,0)}, w^{(1,-1)} \right] + \left[ w^{(1,0)}, \left[ w^{(1,-1)}, w^{(1,0)} \right] \right] - \left[ w^{(1,-1)}, \left[ w^{(1,-1)}, w^{(1,1)} \right] \right] \]

\( w^{(n,k)} \) is the coefficient of \( \alpha_s^n \epsilon^k \) in the exponent of the IR-regularised Wilson line correlator \( \exp w = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle \)

Mitov-Sterman-Sung
EG-Smillie-White

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IR regularized one-loop calculation

\[
\mathcal{F}_{ij}^{(1)}(\gamma_{ij}, \mu^2/m^2, \epsilon) = \mu^{2\epsilon} g_s^2 N \beta_i \cdot \beta_j \int_0^\infty ds \int_0^\infty dt \left( - (s \beta_i - t \beta_j)^2 - i \epsilon \right)^{\epsilon-1}
\]

\[
\gamma_{ij} = \frac{2 \beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}} \quad \Rightarrow \quad \lambda = \frac{\sigma + \tau}{\sigma + \tau}
\]

\[
\lambda = \sigma + \tau
\]

Cusp known to two loops since 1987
Recent formulation in terms of iterated integrals

Korchemsky-Radyushkin (1987)
Kidonakis (2009)
Henn-Huber (2012)
Convenient kinematic variable:

\[
\mathcal{F}^{(1,-1)}(\gamma_{ij}) = \frac{-2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}} = -\gamma_{ij} = \alpha_{ij} + \frac{1}{\alpha_{ij}}
\]

\[
\mathcal{F}^{(1,-1)}(\gamma_{ij}) = \frac{g_s^2 \gamma_{ij}}{16\pi^2} \int_0^1 dx \, P_0(x, \gamma_{ij}) = -\frac{g_s^2}{16\pi^2} \frac{\alpha_{ij} + \frac{1}{\alpha_{ij}}}{1 - \alpha_{ij}^2} \int_0^1 dx \left( \frac{1}{x - \frac{1}{1-\alpha_{ij}} - i0} - \frac{1}{x + \frac{1}{1-\alpha_{ij}} + i0} \right)
\]

\[
= \frac{g_s^2}{16\pi^2} 2 \, r(\alpha_{ij}) \ln (\alpha_{ij} + i0)
\]

Note how the symmetry \( \alpha \to 1/\alpha \) is realised.

Generalises to all multiple exchanges: each exchange yields \( r(\alpha_{ij}) \) times polylogs!
Soft anomalous dimension at 2-loops

Two-loop three-leg contributions:

\[ w_{3g}^{(2,-1)} = -i f_{abc} T_i^a T_j^b T_k^c 2 \left( \frac{g_s^2}{16\pi^2} \right)^2 r(\alpha_{ij}) \ln \alpha_{ij} \ln^2 \alpha_{jk}. \]

Mitov-Sterman-Sung (2010)

Here we are interested in double gluon-exchange diagrams:

\[ w_{121}^{(2)} = \frac{1}{2} \left[ F(2a) - F(2b) \right] - \left[ C(2a) - C(2b) \right] \]

\[ w_{121}^{(2,-1)} (\alpha_{ij}, \alpha_{jk}) = -i f_{abc} T_i^a T_j^b T_k^c \left( \frac{g_s^2}{16\pi^2} \right)^2 r(\alpha_{ij}) r(\alpha_{jk}) \left( \ln(\alpha_{ij}) S_1(\alpha_{jk}) - \ln(\alpha_{jk}) S_1(\alpha_{ij}) \right), \]

\[ S_1(\alpha) \] is a pure function of transcendentality 2.
Defining the subtracted web:

\[
\Gamma^{(2)} = -4w^{(2,-1)}_{3g} - 4w^{(2,-1)}_{121} - 2 \left[ w^{(1,-1)}, w^{(1,0)} \right] - 4\overline{w}^{(2,-1)}_{121}
\]

\[
\begin{align*}
[w^{(1,-1)}, w^{(1,0)}] &= -4if^{abc}T^a_i T^b_j T^c_k \left( \frac{g_s^2}{16\pi^2} \right)^2 r(\alpha_{ij})r(\alpha_{jk}) \left( \ln(\alpha_{ij})R_1(\alpha_{jk}) - \ln(\alpha_{jk})R_1(\alpha_{ij}) \right) \\
\overline{w}^{(2,-1)}_{121} &= -if^{abc}T^a_i T^b_j T^c_k \left( \frac{g_s^2}{16\pi^2} \right)^2 r(\alpha_{ij})r(\alpha_{jk}) \left( \ln(\alpha_{ij})U_1(\alpha_{jk}) - \ln(\alpha_{jk})U_1(\alpha_{ij}) \right)
\end{align*}
\]

\[
U_1(\alpha) = -2\text{Li}_2(1-\alpha^2) - 2\ln^2(\alpha)
\]

Is this a function of \(\alpha^2\)?

Recall that this would mean symmetry under crossing particles from initial to final state, or relation between timelike and spacelike kinematics:

\[
\frac{-2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}} = -\gamma_{ij} = \alpha_{ij} + \frac{1}{\alpha_{ij}}
\]
Consider the expansion near the lightlike limit:

\[
\alpha_{ij} = \frac{\sqrt{1 - \frac{m_i^2 m_j^2}{p_i \cdot p_j}} - \sqrt{1 + \frac{m_i^2 m_j^2}{p_i \cdot p_j}}}{\sqrt{1 - \frac{m_i^2 m_j^2}{p_i \cdot p_j}} + \sqrt{1 + \frac{m_i^2 m_j^2}{p_i \cdot p_j}}}
\]

for \( m_i^2 \to 0 \)

\[
\alpha_{ij} = \frac{\sqrt{m_i^2 m_j^2}}{-2 p_i \cdot p_j} \left[ 1 + \mathcal{O} \left( \frac{m_i^2 m_j^2}{(2 p_i \cdot p_j)^2} \right) \right]
\]

Only the logs are sensitive to the sign, so at Symbol level the \( \alpha \to -\alpha \) symmetry must be restored

\[
\mathcal{S} [S_1(\alpha)] = 4 \alpha \otimes (1 - \alpha) - 2 \alpha \otimes \alpha
\]

\[
\mathcal{S} [R_1(\alpha)] = 2 \alpha \otimes (1 + \alpha) - \alpha \otimes \alpha
\]

\[
\mathcal{S} [U_1(\alpha)] = 4 \left[ \alpha \otimes (1 - \alpha) + \alpha \otimes (\alpha + 1) - \alpha \otimes \alpha \right] = 2 \left[ \alpha^2 \otimes 1 - \alpha^2 \right] - \alpha^2 \otimes \alpha^2,
\]

The Symbol alphabet is \( \{\alpha^2, 1 - \alpha^2\} \) realizing \( \alpha \to -\alpha \) and \( \alpha \to 1/\alpha \) symmetries.
The general structure of the integral for gluon exchange diagrams:

\[
\mathcal{F}^{(n)} \sim \Gamma(2n\epsilon) \int dx_1 dx_2 \ldots dx_n \phi_{n-1}(x_1, x_2, \ldots, x_n; \epsilon) \prod_{k=1}^{n} p(x_k, \alpha_k)
\]

\[
= \Gamma(2n\epsilon) \left( \prod_{k=1}^{n} r(\alpha_k) \right) s_n(\{\alpha_k\}; \epsilon)
\]

\[
\lambda_i = \sigma_i + \tau_i \\
x_i = \frac{\sigma_i}{\sigma_i + \tau_i}
\]

The kernel \( \phi_{n-1}(x_1, x_2, \ldots, x_n; \epsilon) \) is obtained after \( \int d\lambda_i \)

We showed:

1. The kernel is a pure function of weight \( n - 1 \)
2. \( S_n \) is a pure function of weight \( 2n - 1 \) \hspace{1cm} (Goncharov multiple-polylogs)

But there is no need to integrate individual diagrams...

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Multiple gluon exchange webs

Subtracted webs are much simpler than individual diagrams:

\[
\Gamma^{(n)} \ni \overline{w}^{(n,-1)} = \left( \frac{\alpha_s}{4\pi} \right)^n C_{i_1,i_2,\ldots,i_{n+1}} \int dx_1 dx_2 \ldots dx_n \times \prod_{k=1}^{n} p_0(x_k, \alpha_k) \times \mathcal{G}_{n-1}(x_1, x_2, \ldots, x_n; q(x_1, \alpha_1), q(x_2, \alpha_2), \ldots q(x_n, \alpha_n))
\]

Conjecture:
(1) \( \mathcal{G}_{n-1} \) is made exclusively of powers of logs
Or, equivalently,
(2) \( \overline{w}^{(n,-1)} \) is a sum of products of polylogs each depending on a single \( \alpha \) each having a Symbol with alphabet \( \left\{ \alpha^2, 1 - \alpha^2 \right\} \)
THREE-LOOP WEB RESULT

\[
\bar{w}_{1221}^{(3)} = - f^{abe} f^{cde} T_1^a T_2^b T_3^c T_4^d \left( \frac{g^2}{16\pi^2} \right)^3 r(\alpha_{12}) r(\alpha_{23}) r(\alpha_{34}) \\
\left[ -8 U_2(\alpha_{12}) \ln \alpha_{23} \ln \alpha_{34} - 8 U_2(\alpha_{34}) \ln \alpha_{12} \ln \alpha_{23} + 16 \left( U_2(\alpha_{23}) - 2 \Sigma_2(\alpha_{23}) \right) \ln \alpha_{12} \ln \alpha_{34} \\
- 2 \ln \alpha_{12} U_1(\alpha_{23}) U_1(\alpha_{34}) - 2 \ln \alpha_{34} U_1(\alpha_{12}) U_1(\alpha_{23}) + 4 \ln \alpha_{23} U_1(\alpha_{12}) U_1(\alpha_{34}) \right]
\]

EG (to appear)
The non-Abelian exponentiation theorem has been generalised to any number of Wilson lines in arbitrary representations. We can directly compute the exponent in terms of webs. Webs include sets of non-connected diagrams related by permutations. These involve web mixing matrices with rich combinatorial complexity. Colour factors in the exponent - connected graphs! Subdivergences cancel by mixing matrices and subtraction of commutators.
CONCLUSIONS: POLYLOGS

- We developed a method to compute multiple gluon exchange webs, and computed all three-loop 4-leg diagrams in this class. More soon.

- Contributions to the soft anomalous dimension (Subtracted webs) are simple: there is a marked advantage to computing these combinations directly!

- This class of subtracted webs can be written in terms of polylogarithmic functions of a single variable, each having Symbol with the alphabet: \( \{ \alpha_{ij}^2, \ 1 - \alpha_{ij}^2 \} \)

\[
\alpha_{ij} \rightarrow 1/\alpha_{ij} \\
\alpha_{ij} \rightarrow -\alpha_{ij} \quad \text{at Symbol level}
\]