RADCOR 2013

FROM WEBS TO POLYLOGS

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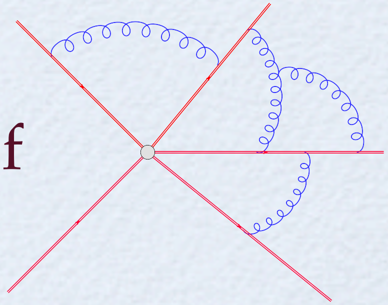
IR SINGULARITIES

Motivation

- IR singularities are key to cross section calculations (subtraction, resummation)

- Window to all-order structure of scattering amplitudes

Strategy



- Compute the renormalization Z of a product of non-lighlike Wilson lines $\langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$

- The soft anomalous dimension $$\frac{dZ}{d\ln\mu} = -Z\Gamma$$

FROM WEBS TO POLYLOGS

- Part I: The non-Abelian exponentiation theorem
- Part II: evaluating gluon-exchange webs

WEBS

DIAGRAMMATIC SOFT GLUON EXPONENTIATION

$$S = \exp w = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$$

- Abelian case (1961)

  Yennie-Frautschi-Suura

- Non-abelian, colour singlet (two-line) case (1983)

  Gatheral, Frenkel-Taylor

- Non-abelian, multi-line case (2010-2013)

  EG-Laenen-Stavenga-White & Mitov-Sterman-Sung

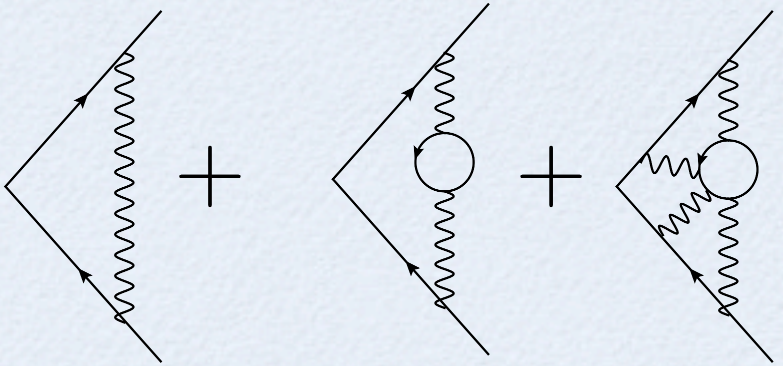
  EG-Smillie-White

EXPONENTIATION IN AN ABELIAN THEORY

- The exponent only receives contributions from *connected* diagrams:

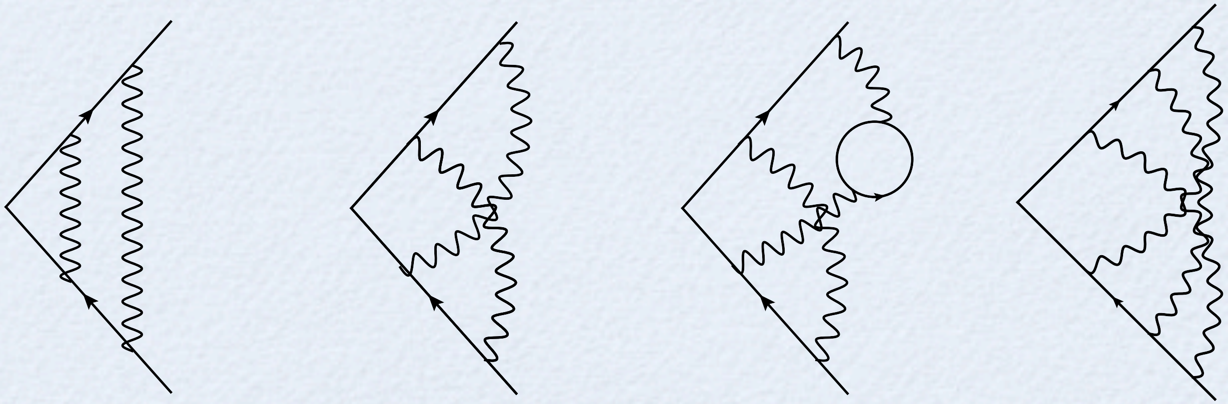
(``*connected*'' is defined after removing the Wilson lines!)

$$\mathcal{S} = \exp\Bigg\{$$



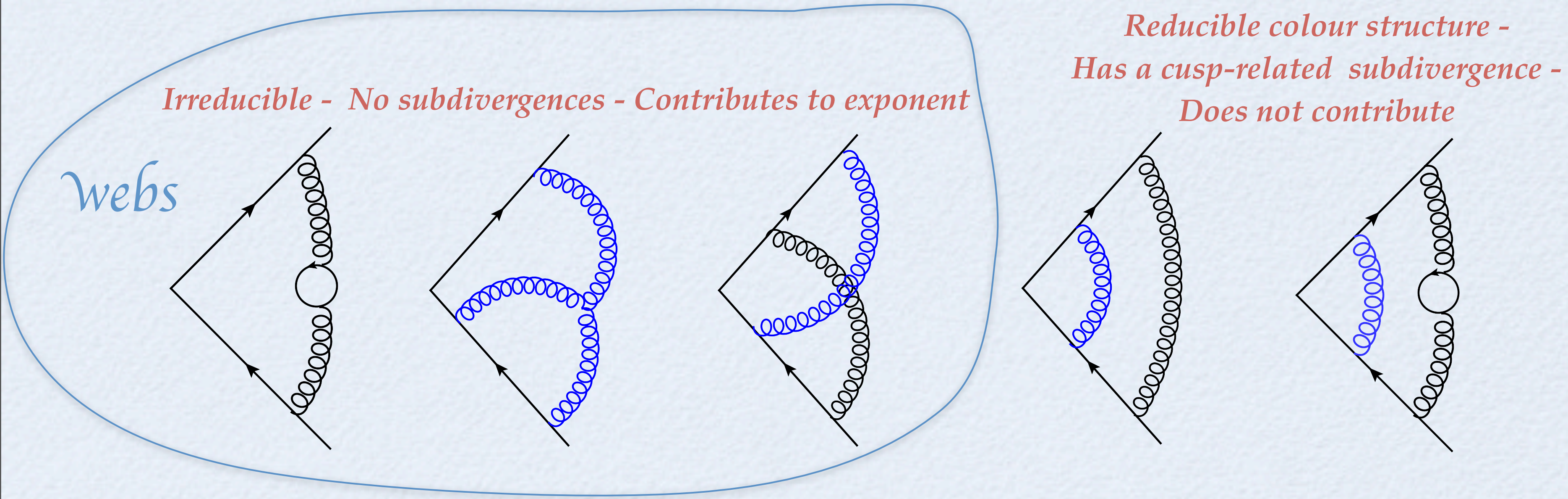
$$+ \cdots \Bigg\}$$

- Expanding the exponential exactly reproduces all disconnected diagrams:



NON-ABELIAN EXPONENTIATION IN THE TWO-LINE CASE

- Diagrams fall into two categories:



Webs

Irreducible - No subdivergences - Contributes to exponent

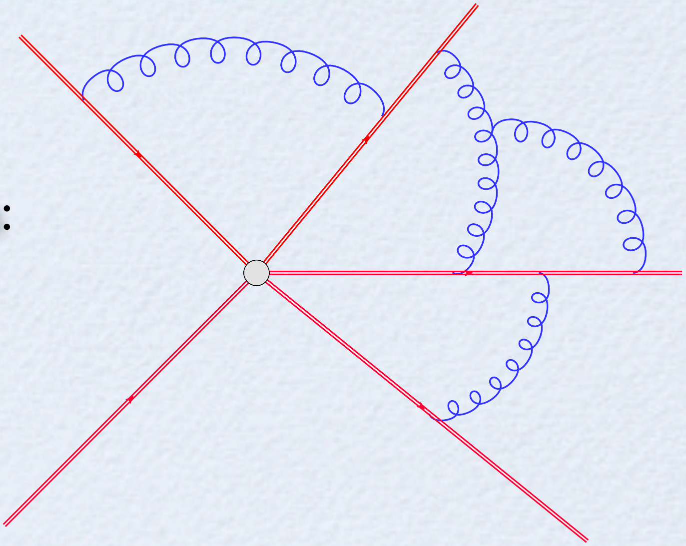
Reducible colour structure -
Has a cusp-related subdivergence -
Does not contribute

Diagrams contributing to the exponent (webs) have a non-Abelian colour factor and no subdivergences!

- This goes hand-in-hand with the renormalization of the cusp: webs have a single overall divergence

$$Z = \exp\left\{ \frac{1}{2} \int_{\mu^2}^{\infty} \frac{d\lambda^2}{\lambda^2} \, \Gamma_{\mathcal{S}}(\alpha_s(\lambda^2)) \right\} = \exp\left\{ \frac{1}{2\epsilon} \Gamma_{\mathcal{S}}^{(1)} \, \alpha_s + \frac{1}{4\epsilon} \Gamma_{\mathcal{S}}^{(2)} \, \alpha_s^2 + \frac{1}{6\epsilon} \Gamma_{\mathcal{S}}^{(3)} \, \alpha_s^3 + \mathcal{O}(\alpha_s^4) \right\}$$

NON-ABELIAN EXPONENTIATION: THE MULTI-LEG CASE



In the multi-leg case this separation breaks down:
reducible diagrams do contribute to the exponent …

- Is the colour factor still non-Abelian?
- How can we compute the exponent directly? if so what do we gain?
- Reducible diagrams have subdivergences - how is this consistent with renormalizability of the vertex?

GENERALISED NON-ABELIAN EXPONENTIATION THEOREM

$$S = \exp w = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$$

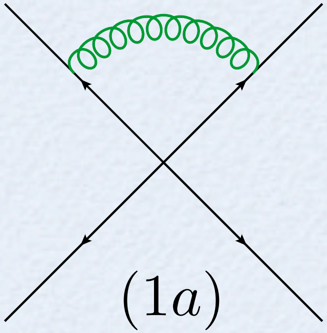
Theorem: all colour structures in the exponent w correspond to connected graphs

EG-Smillie-White (2013)

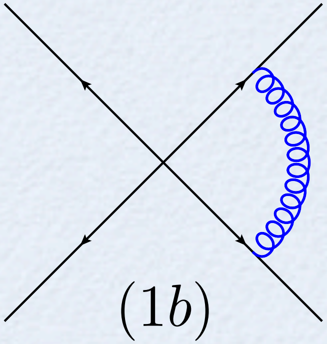
NON-ABELIAN EXPONENTIATION: MULTI-LEG CASE

Are reducible graphs reproduced by exponentiating lower order ones ?

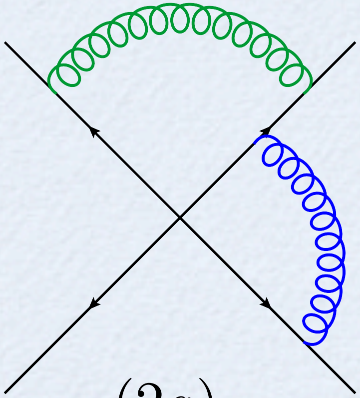
Exponentiating:


$(1a)$

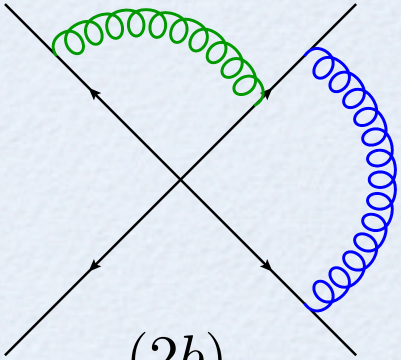
+


$(1b)$

At 2-loops, do we get


$(2a)$

+


$(2b)$

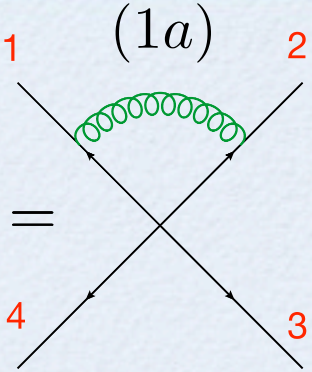
?

NON-ABELIAN EXPONENTIATION: MULTI-LEG CASE

Exponentiating:

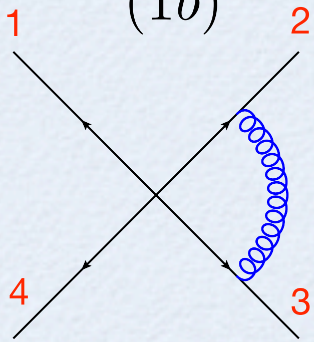
$$D_{(1a)} + D_{(1b)} =$$

$(1a)$



+

$(1b)$



$$= \mathcal{F}(1a)\mathrm{T}_1^{(a)}\mathrm{T}_2^{(a)} + \mathcal{F}(1b)\mathrm{T}_2^{(b)}\mathrm{T}_3^{(b)}$$

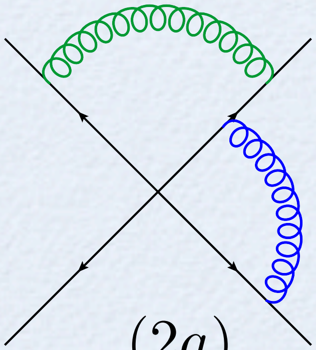
At 2-loops we get:

$$\frac{1}{2}\left[D_{(1a)} + D_{(1b)}\right]^2 = \frac{1}{2}\left[D_{(1a)}D_{(1b)} + D_{(1b)}D_{(1a)} + \cdots\right]$$

$$= \frac{1}{2}\mathcal{F}(1a)\mathcal{F}(1b)\ \mathrm{T}_1^{(a)}\left[\mathrm{T}_2^{(a)}\mathrm{T}_2^{(b)} + \mathrm{T}_2^{(b)}\mathrm{T}_2^{(a)}\right]\mathrm{T}_3^{(b)}$$

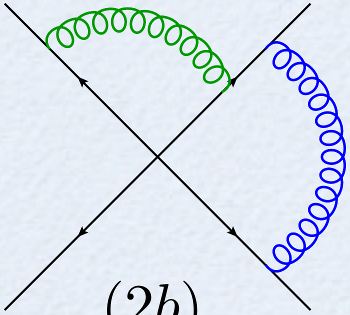
$$= \frac{1}{2}\Big[\mathcal{F}(2a) + \mathcal{F}(2b)\Big]\quad\Big[C(2a) + C(2b)\Big]$$

While



$(2a)$

+



$(2b)$

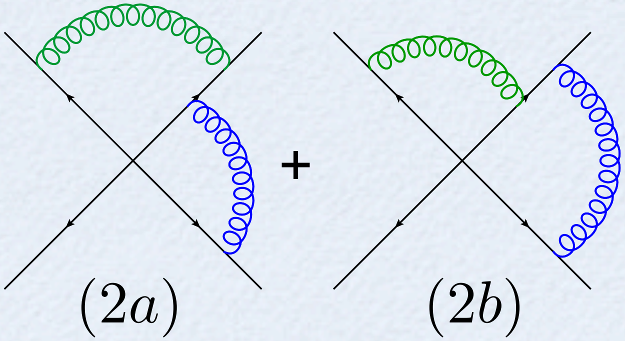
$$= \mathcal{F}(2a)\,C(2a) + \mathcal{F}(2b)\,C(2b)$$

NON-ABELIAN EXPONENTIATION: MULTI-LEG CASE

Exponentiating 1-loop diagrams yields:

$$\frac{1}{2}\left[D_{(1a)} + D_{(1b)}\right]^2 = \frac{1}{2}\Big[\mathcal{F}(2a) + \mathcal{F}(2b)\Big] \quad \Big[C(2a) + C(2b)\Big]$$

While the 2-loop amplitude is:



$$= \mathcal{F}(2a)\,C(2a) + \mathcal{F}(2b)\,C(2b)$$

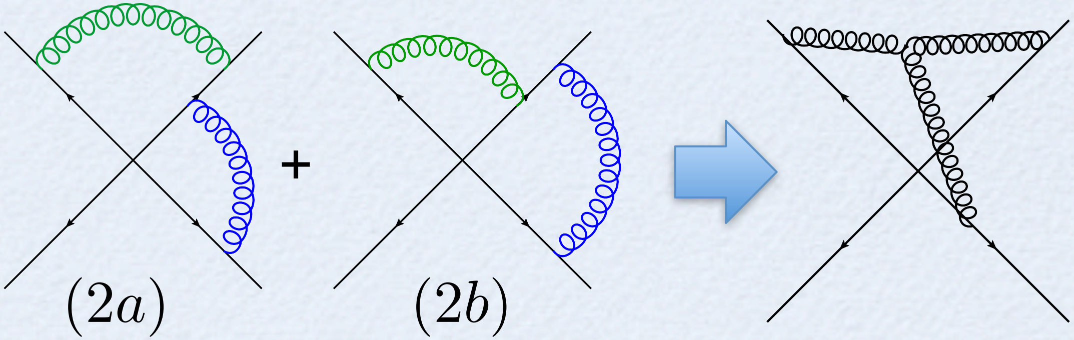
The 2-loop contribution to the exponent is therefore:

$$\frac{1}{2}\Big[\mathcal{F}(2a) - \mathcal{F}(2b)\Big] \quad \Big[C(2a) - C(2b)\Big]$$

In the multi-leg case, *reducible diagrams do contribute to the exponent*!

NON-ABELIAN EXPONENTIATION: MULTI-LEG CASE

The 2-loop contribution to the exponent



$$w^{(2)}_{121} = \frac{1}{2}\underbrace{\Big[\mathcal{F}(2a) - \mathcal{F}(2b)\Big]}_{\mathcal{O}(1/\epsilon)} \times \underbrace{\Big[C(2a) - C(2b)\Big]}_{\mathrm{i} f^{abc}\mathrm{T}_1^{(a)}\mathrm{T}_3^{(b)}\mathrm{T}_2^{(c)}}$$

These properties (single pole, maximally non-Abelian colour structure) are familiar from the two-line case.

NON-ABELIAN EXPONENTIATION: MULTI-LEG CASE

- In contrast to the 2-line case, reducible diagrams enter the exponent.

- Individual diagrams do not have ``web properties''.
  Only particular linear combinations that do - enter the exponent:

$$\mathcal{S} = \exp\left[\sum_i W_i\right] \qquad\qquad W_i = \sum_{\{D\}_i} \mathcal{F}(D)\, \widetilde{C}(D)$$

  Each diagram enters the exponent with a *modified colour factor* $\widetilde{C}(D)$

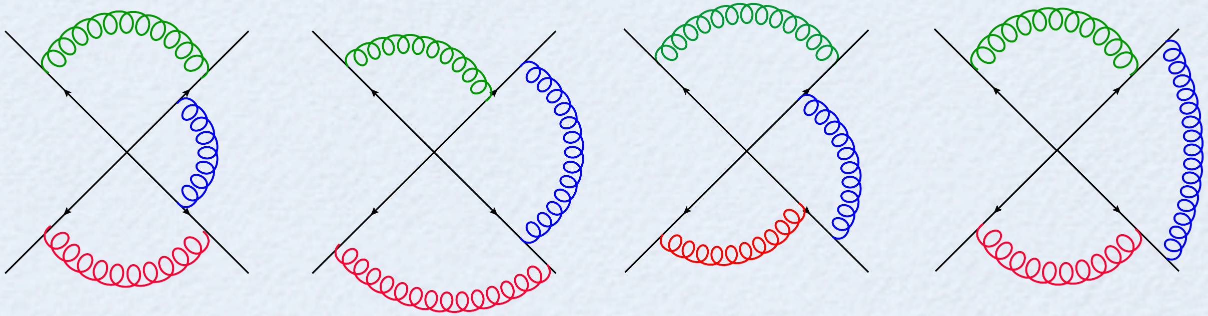
- *modified colour factors* $\widetilde{C}(D)$ are linear combination of (ordinary) colour factors of diagrams that are obtained by permuting attachments to the Wilson lines, so:

$$W_i = \sum_D \mathcal{F}(D) \underbrace{\sum_{D'} R_{DD'}\, C(D')}_{\widetilde{C}(D)} = \mathcal{F}^T \underset{\Uparrow}{R} C$$

web mixing matrix

- Using the replica trick we derived a general combinatorial formula for R

THREE-LOOP WEB: EXAMPLE



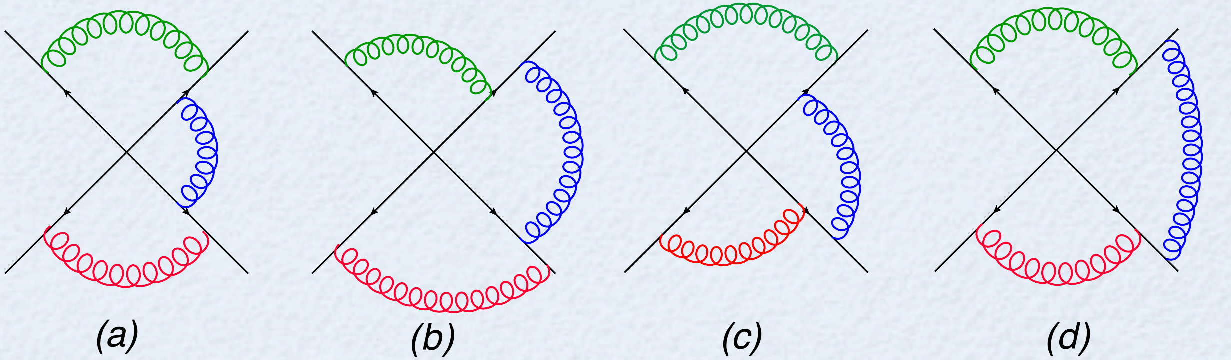
(a) (b) (c) (d)

The entire web contributes:

$$
= \begin{pmatrix} \mathcal{F}(3a) \\ \mathcal{F}(3b) \\ \mathcal{F}(3c) \\ \mathcal{F}(3d) \end{pmatrix}^T \frac{1}{6} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} C(3a) \\ C(3b) \\ C(3c) \\ C(3d) \end{pmatrix}
$$

Kinematics Web mixing matrix R Colour

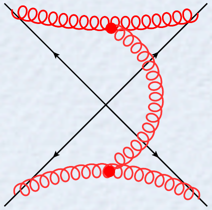
THREE-LOOP WEB: EXAMPLE



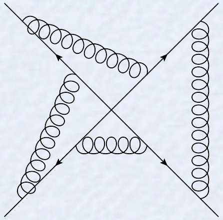
$$= \begin{pmatrix} \mathcal{F}(3a) \\ \mathcal{F}(3b) \\ \mathcal{F}(3c) \\ \mathcal{F}(3d) \end{pmatrix}^T \frac{1}{6} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -2 & 2 & 2 & -2 \\ -2 & 2 & 2 & -2 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} C(3a) \\ C(3b) \\ C(3c) \\ C(3d) \end{pmatrix}$$

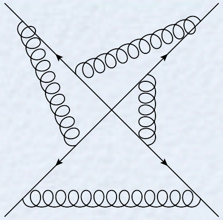
$$= \underbrace{\frac{1}{6}\Big(\mathcal{F}(3a) - 2\mathcal{F}(3b) - 2\mathcal{F}(3c) + \mathcal{F}(3d)\Big)} \times \underbrace{\Big(C(3a) - C(3b) - C(3c) + C(3d)\Big)}$$

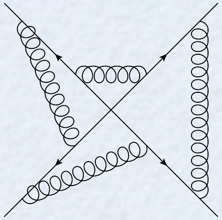
subdivergences cancel

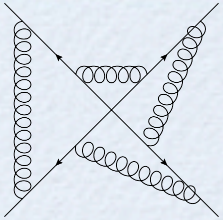


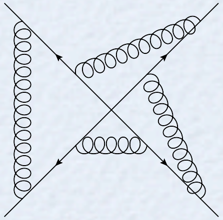
Mixing Matrices: Four-Loop Example

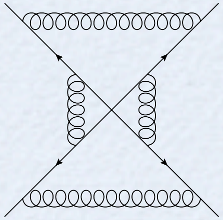

[[1,2],[3,1],[3,4],[2,4]]

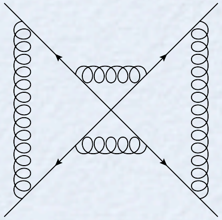

[[1,2],[2,3],[4,3],[4,1]]

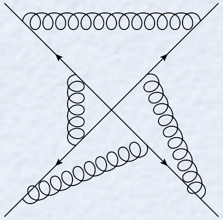

[[1,2],[3,2],[3,4],[4,1]]

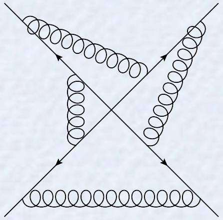

[[1,2],[2,3],[3,4],[1,4]]

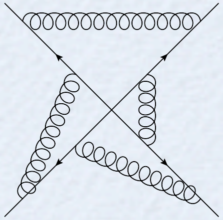

[[1,2],[3,2],[4,3],[1,4]]

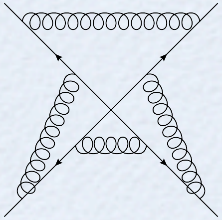

[[1,2],[1,3],[4,3],[4,2]]

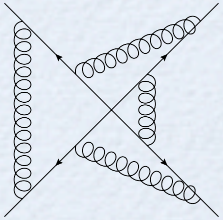

[[1,2],[3,2],[3,4],[1,4]]

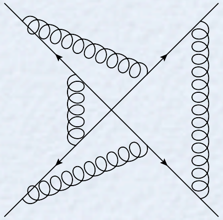

[[1,2],[1,3],[3,4],[4,2]]

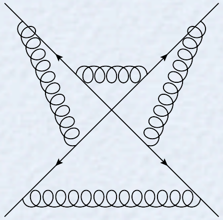

[[1,2],[3,1],[4,3],[4,2]]

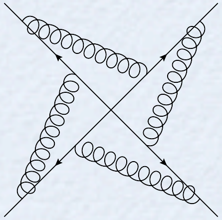

[[1,2],[1,3],[4,3],[2,4]]

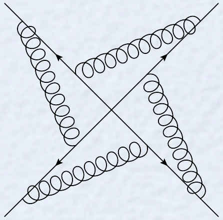

[[1,2],[1,3],[3,4],[2,4]]


[[1,2],[2,3],[4,3],[1,4]]


[[1,2],[3,1],[3,4],[4,2]]


[[1,2],[3,2],[4,3],[4,1]]


[[1,2],[3,1],[4,3],[2,4]]


[[1,2],[2,3],[3,4],[4,1]]

PROPERTIES OF WEB MIXING MATRICES

A. It is *idempotent:* $R^2 = R$

R is diagonalisable, with all its eigenvalues 0 or 1.

B. *Its rows sum to zero:*

$$\sum_{D'} R_{DD'} = 0$$

C. *Its columns, weighted by a symmetry factor* s(D), *sum to zero:*

$$\sum_{D} s(D)\, R_{DD'} = 0$$

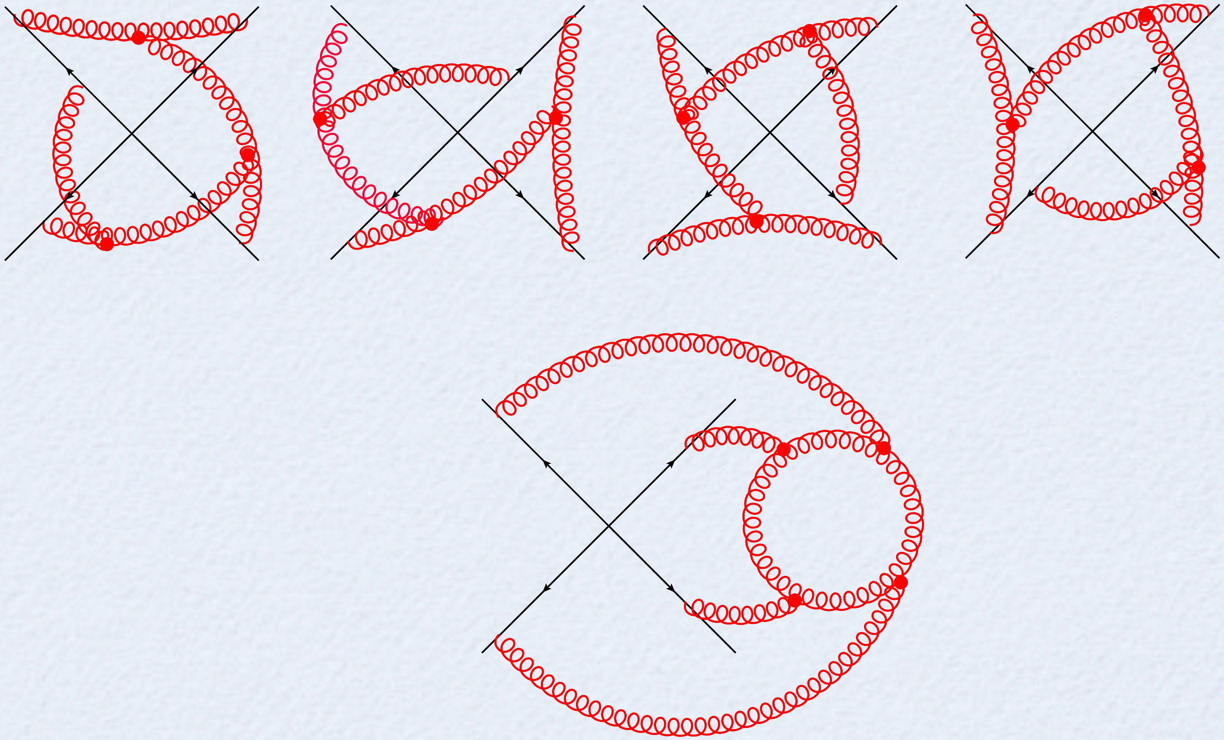
MIXING MATRICES: FOUR-LOOP EXAMPLE

The resulting mixing matrix:

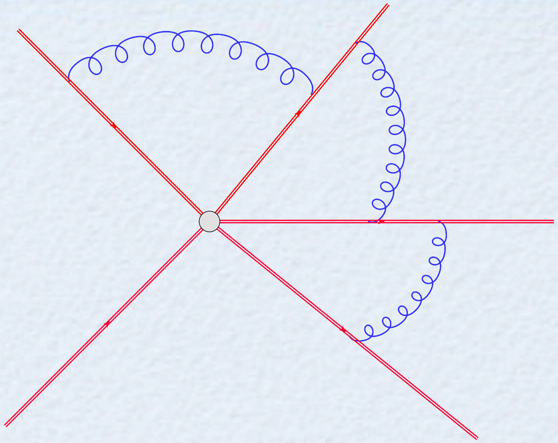
$$\widetilde{C} = RC = \frac{1}{24}\begin{pmatrix}
6 & -6 & 2 & 2 & -2 & 4 & -4 & 2 & -2 & -2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-6 & 6 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 6 & -2 & 2 & 4 & -4 & -2 & 2 & -6 & 4 & 4 & -4 & -4 & 0 & 0 \\
2 & -2 & -2 & 6 & 2 & 4 & -4 & -2 & -6 & 2 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & 2 & 6 & 4 & -4 & -6 & -2 & -2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 2 & 2 & 2 & 4 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & -2 & -2 & -4 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & -2 & -6 & -4 & 4 & 6 & 2 & 2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -6 & -2 & -4 & 4 & 2 & 6 & -2 & 4 & 4 & -4 & -4 & 0 & 0 \\
-2 & 2 & -6 & 2 & -2 & -4 & 4 & 2 & -2 & 6 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -2 & 2 & 0 & 0 & -2 & 2 & -2 & 4 & 0 & 0 & -4 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 0 & 0 & 2 & 2 & -2 & 0 & 4 & -4 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & 2 & 0 & 0 & -2 & -2 & 2 & 0 & -4 & 4 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 2 & -4 & 0 & 0 & 4 & 0 & 0 \\
-18 & -6 & -6 & -6 & -18 & 12 & 12 & -6 & -18 & -18 & 12 & 12 & 12 & 12 & 24 & 0 \\
-6 & -18 & -18 & -18 & -6 & 12 & 12 & -18 & -6 & -6 & 12 & 12 & 12 & 12 & 0 & 24
\end{pmatrix}
\begin{pmatrix}
C[[1,2],[3,1],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[4,1]] \\
C[[1,2],[3,2],[3,4],[4,1]] \\
C[[1,2],[2,3],[3,4],[1,4]] \\
C[[1,2],[3,2],[4,3],[1,4]] \\
C[[1,2],[1,3],[4,3],[4,2]] \\
C[[1,2],[3,2],[3,4],[1,4]] \\
C[[1,2],[1,3],[3,4],[4,2]] \\
C[[1,2],[3,1],[4,3],[4,2]] \\
C[[1,2],[1,3],[4,3],[2,4]] \\
C[[1,2],[1,3],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[1,4]] \\
C[[1,2],[3,1],[3,4],[4,2]] \\
C[[1,2],[3,2],[4,3],[4,1]] \\
C[[1,2],[3,1],[4,3],[2,4]] \\
C[[1,2],[2,3],[3,4],[4,1]]
\end{pmatrix}$$

Mixing Matrices: Four-Loop Example

This mixing matrix has rank 5 (5 eigenvectors with eigenvalue 1, the rest 0) corresponding to the colour factors:



RENORMALIZATION OF WILSON LINES



$$\frac{dZ}{d\ln\mu} = -Z\Gamma$$

In the multiparton case:

$$Z = \exp\left\{ \frac{1}{2\epsilon}\Gamma_{\mathcal{S}}^{(1)}\,\alpha_s + \left(\frac{1}{4\epsilon}\Gamma_{\mathcal{S}}^{(2)} - \frac{b_0}{4\epsilon^2}\Gamma_{\mathcal{S}}^{(1)}\right)\alpha_s^2 \right.$$
$$\left. + \left(\frac{1}{6\epsilon}\Gamma_{\mathcal{S}}^{(3)} + \boxed{\frac{1}{48\epsilon^2}\left[\Gamma_{\mathcal{S}}^{(1)},\Gamma_{\mathcal{S}}^{(2)}\right]} - \frac{1}{6\epsilon^2}\left(b_0\Gamma_{\mathcal{S}}^{(2)} + b_1\Gamma_{\mathcal{S}}^{(1)}\right) + \frac{b_0^2}{6\epsilon^3}\Gamma_{\mathcal{S}}^{(1)}\right)\alpha_s^3 + \mathcal{O}(\alpha_s^4) \right\}$$

multiple poles occur due to two distinct reasons:

1) running coupling
2) commutators (only in the multi-leg case, and beyond the planar limit)

Specific subdivergences of the multi-eikonal vertex survive in the exponent, BUT all multiple poles are predicted by lower orders. Only $\mathcal{O}(1/\epsilon)$ are new. In particular, there no $1/\epsilon^n$ at $\mathcal{O}(\alpha_s^n)$.

RENORMALIZATION OF WILSON LINES

To compute the anomalous dimension

$$\Gamma^{(1)} = -2w^{(1,-1)}$$

$$\Gamma^{(2)} = -4w^{(2,-1)} - 2\left[w^{(1,-1)}, w^{(1,0)}\right]$$

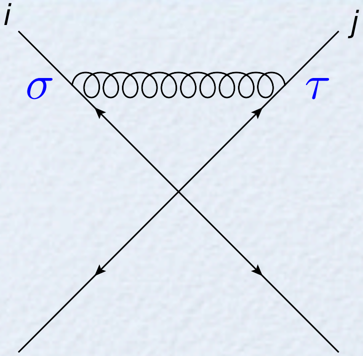
$$\Gamma^{(3)} = -6w^{(3,-1)} + \frac{3}{2}b_0\left[w^{(1,-1)}, w^{(1,1)}\right] + 3\left[w^{(1,0)}, w^{(2,-1)}\right] + 3\left[w^{(2,0)}, w^{(1,-1)}\right]$$
$$+\left[w^{(1,0)}, \left[w^{(1,-1)}, w^{(1,0)}\right]\right] - \left[w^{(1,-1)}, \left[w^{(1,-1)}, w^{(1,1)}\right]\right]$$

$w^{(n,k)}$ is the coefficient of $\alpha_s^n \epsilon^k$ in the exponent of the IR-regularised Wilson line correlator $\exp w = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$

Mitov-Sterman-Sung
EG-Smillie-White

CUSP UV SINGULARITY

IR regularized one-loop calculation



$$\mathcal{F}_{ij}^{(1)}(\gamma_{ij},\mu^2/m^2,\epsilon) = \mu^{2\epsilon} g_s^2 \mathcal{N}\, \beta_i\cdot\beta_j \int_0^{\text{“}\infty\text{”}} ds \int_0^{\text{“}\infty\text{”}} dt \Big(-(s\beta_i - t\beta_j)^2 - \mathrm{i}\varepsilon\Big)^{\epsilon-1}$$

$$\gamma_{ij} = \frac{2\beta_i\cdot\beta_j}{\sqrt{\beta_i^2\,\beta_j^2}} \longrightarrow = \kappa\,\gamma_{ij} \int_0^{\infty} d\sigma \int_0^{\infty} d\tau \Big(\sigma^2+\tau^2-\gamma_{ij}\sigma\tau - \mathrm{i}\varepsilon\Big)^{\epsilon-1} \mathrm{e}^{-\sigma-\tau}$$

$$\boxed{\begin{aligned}\lambda &= \sigma+\tau\\ x &= \frac{\sigma}{\sigma+\tau}\end{aligned}} \longrightarrow = \kappa\,\Gamma(2\epsilon)\,\gamma_{ij}\int_0^1 dx\, P(x,\gamma_{ij})$$

$$P(x,\gamma_{ij}) \equiv \Big[x^2+(1-x)^2 - x(1-x)\gamma_{ij} - \mathrm{i}\varepsilon\Big]^{\epsilon-1}$$

Korchemsky-Radyushkin (1987)

Kidonakis (2009)

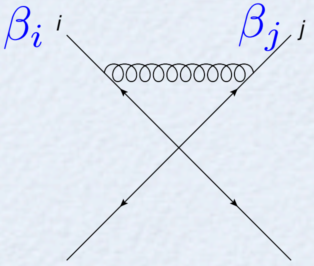
Henn-Huber (2012)

Cusp known to two loops since 1987

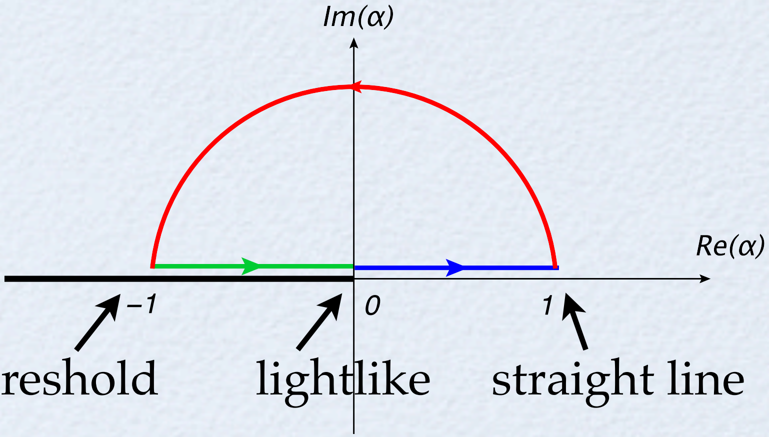
Recent formulation in terms of iterated integrals

CUSP UV SINGULARITY

Convenient kinematic variable:



$$\frac{-2\beta_i\cdot\beta_j}{\sqrt{\beta_i^2\,\beta_j^2}} = -\gamma_{ij} = \alpha_{ij} + \frac{1}{\alpha_{ij}}$$



$$\mathcal{F}^{(1,-1)}(\gamma_{ij}) = \frac{g_s^2\gamma_{ij}}{16\pi^2}\int_0^1 \mathrm{d}x\; P_0(x,\gamma_{ij})$$

$$= -\frac{g_s^2\left(\alpha_{ij}+\frac{1}{\alpha_{ij}}\right)}{16\pi^2}\int_0^1 \mathrm{d}x \frac{1}{x^2+(1-x)^2+x(1-x)(\alpha_{ij}+1/\alpha_{ij})-\mathrm{i}\varepsilon}$$

$$= \frac{g_s^2}{16\pi^2}\,\frac{1+\alpha_{ij}^2}{1-\alpha_{ij}^2}\int_0^1 \mathrm{d}x\left(\frac{1}{x-\frac{1}{1-\alpha_{ij}}-\mathrm{i}0} - \frac{1}{x+\frac{\alpha_{ij}}{1-\alpha_{ij}}+\mathrm{i}0}\right)$$

$$= \frac{g_s^2}{16\pi^2}\,2\,r(\alpha_{ij})\,\ln\left(\alpha_{ij}+\mathrm{i}0\right)$$

$$r(\alpha_{ij}) = \frac{1+\alpha_{ij}^2}{1-\alpha_{ij}^2}$$

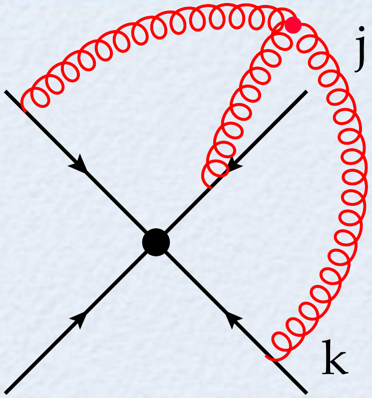
Note how the symmetry $\alpha \to 1/\alpha$ is realised.

Generalises to all multiple exchanges: each exchange yields $r(\alpha_{ij})$ times polylogs!

SOFT ANOMALOUS DIMENSION AT 2-LOOPS

Two-loop three-leg contributions:

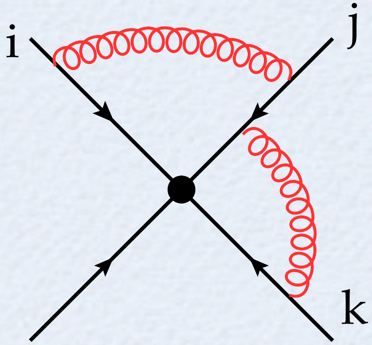
$$w_{3g}^{(2,-1)} = -\mathrm{i}f^{abc}T_i^aT_j^bT_k^c\,2\left(\frac{g_s^2}{16\pi^2}\right)^2 r(\alpha_{ij})\ln\alpha_{ij}\ln^2\alpha_{jk}\,.$$

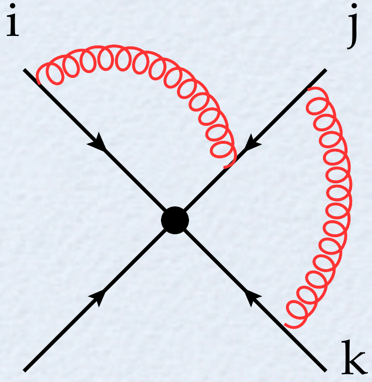


Ferroglia-Neubert-Pecjak-Yang (2009)
Mitov-Sterman-Sung (2010)

Here we are interested in double gluon-exchange diagrams:

$$w_{121}^{(2)} = \frac{1}{2}\underbrace{\Big[\mathcal{F}(2a)-\mathcal{F}(2b)\Big]}_{\mathcal{O}(1/\epsilon)}\quad\underbrace{\Big[C(2a)-C(2b)\Big]}_{\mathrm{i}f^{abc}\mathrm{T}_1^{(a)}\mathrm{T}_3^{(b)}\mathrm{T}_2^{(c)}}$$



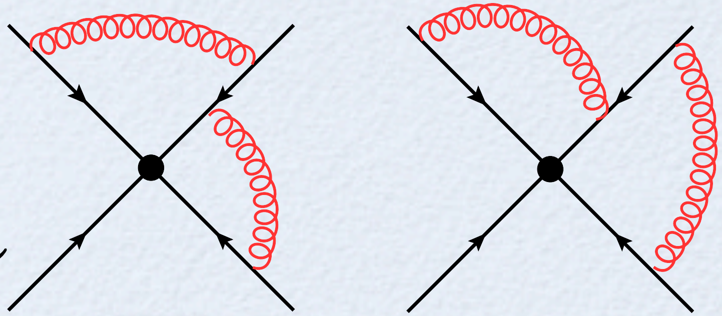


$$w_{121}^{(2,-1)}(\alpha_{ij},\alpha_{jk}) = -\,\mathrm{i}f^{abc}T_i^aT_j^bT_k^c\left(\frac{g_s^2}{16\,\pi^2}\right)^2 r(\alpha_{ij})r(\alpha_{jk})\Big(\ln(\alpha_{ij})S_1(\alpha_{jk})-\ln(\alpha_{jk})S_1(\alpha_{ij})\Big)\,,$$

$S_1(\alpha)$ is a pure function of transcendentality 2.

SOFT ANOMALOUS DIMENSION AT 2-LOOPS

Defining the subtracted web:



$$\Gamma^{(2)} = -4 w_{3g}^{(2,-1)} \underbrace{-4 w_{121}^{(2,-1)} - 2\left[w^{(1,-1)}, w^{(1,0)}\right]}_{-4\overline{w}_{121}^{(2,-1)}}$$

$$\left[w^{(1,-1)}, w^{(1,0)}\right] = -4\mathrm{i} f^{abc} T_i^a T_j^b T_k^c \left(\frac{g_s^2}{16\pi^2}\right)^2 r(\alpha_{ij})\, r(\alpha_{jk}) \Big(\ln(\alpha_{ij}) R_1(\alpha_{jk}) - \ln(\alpha_{jk}) R_1(\alpha_{ij})\Big)$$

$$\overline{w}_{121}^{(2,-1)} = -\mathrm{i} f^{abc} T_i^a T_j^b T_k^c \left(\frac{g_s^2}{16\pi^2}\right)^2 r(\alpha_{ij})\, r(\alpha_{jk}) \Big(\ln(\alpha_{ij}) U_1(\alpha_{jk}) - \ln(\alpha_{jk}) U_1(\alpha_{ij})\Big)$$

$$U_1(\alpha) = -2\mathrm{Li}_2(1-\alpha^2) - 2\ln^2(\alpha)$$

Is this a function of α^2 ?

Is there a symmetry under $\alpha \longrightarrow -\alpha$?

Recall that this would mean symmetry under crossing particles from initial to final state, or relation between timelike and spacelike kinematics

$$\frac{-2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2\, \beta_j^2}} = -\gamma_{ij} = \alpha_{ij} + \frac{1}{\alpha_{ij}}$$

SOFT ANOMALOUS DIMENSION: $\alpha \to -\alpha$ SYMMETRY

Consider the expansion near the lightlike limit:

$$\alpha_{ij} = \frac{\sqrt{1-\frac{\sqrt{m_i^2 m_j^2}}{p_i \cdot p_j}} - \sqrt{1+\frac{\sqrt{m_i^2 m_j^2}}{p_i \cdot p_j}}}{\sqrt{1-\frac{\sqrt{m_i^2 m_j^2}}{p_i \cdot p_j}} + \sqrt{1+\frac{\sqrt{m_i^2 m_j^2}}{p_i \cdot p_j}}}$$

$$\text{for } m_i^2 \to 0 \qquad \alpha_{ij} = \frac{\sqrt{m_i^2 m_j^2}}{-2p_i \cdot p_j}\left[1 + \mathcal{O}\left(\frac{m_i^2 m_j^2}{(2p_i \cdot p_j)^2}\right)\right]$$

Only the logs are sensitive to the sign, so at Symbol level the $\alpha \to -\alpha$ symmetry must be restored

$$\mathcal{S}\left[\ln(-\alpha)\right] = \otimes\alpha = \mathcal{S}\left[\ln(\alpha)\right]$$

$$\mathcal{S}\left[S_1(\alpha)\right] = 4\alpha \otimes (1-\alpha) - 2\alpha \otimes \alpha$$

$$\mathcal{S}\left[R_1(\alpha)\right] = 2\alpha \otimes (1+\alpha) - \alpha \otimes \alpha$$

$$\mathcal{S}\left[U_1(\alpha)\right] = 4\Big[\alpha \otimes (1-\alpha) + \alpha \otimes (\alpha+1) - \alpha \otimes \alpha\Big] = 2\left[\alpha^2 \otimes 1 - \alpha^2\right] - \alpha^2 \otimes \alpha^2\,,$$

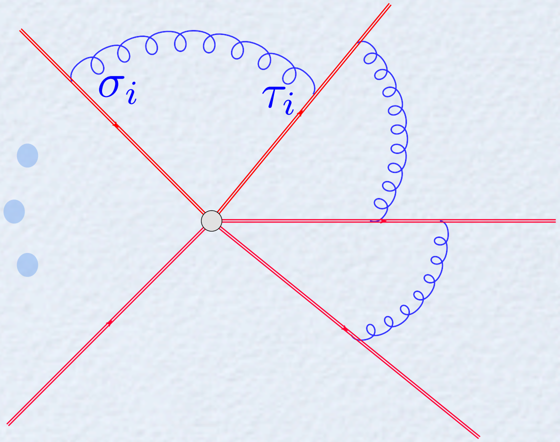
The Symbol alphabet is $\left\{\alpha^2, 1-\alpha^2\right\}$ realizing $\alpha \to -\alpha$ and $\alpha \to 1/\alpha$ symmetries.

MULTIPLE GLUON EXCHANGE GRAPHS

General structure of the integral for gluon exchange diagrams:

$$\mathcal{F}^{(n)} \sim \Gamma(2n\epsilon) \int dx_1 dx_2 \ldots dx_n \, \phi_{n-1}(x_1, x_2, \ldots, x_n; \epsilon) \prod_{k=1}^{n} p(x_k, \alpha_k)$$

$$= \Gamma(2n\epsilon) \left(\prod_{k=1}^{n} r(\alpha_k) \right) s_n(\{\alpha_k\}; \epsilon)$$



$$\lambda_i = \sigma_i + \tau_i$$

$$x_i = \frac{\sigma_i}{\sigma_i + \tau_i}$$

The kernel $\phi_{n-1}(x_1, x_2, \ldots, x_n; \epsilon)$ is obtained after $\int d\lambda_i$

We showed:

(1) the kernel is a pure function of weight $n-1$

(2) s_n is a pure function of weight $2n-1$ (Goncharov multiple-polylogs)

But there is no need to integrate individual diagrams...

MULTIPLE GLUON EXCHANGE WEBS

Subtracted webs are much simpler than individual diagrams:

$$\Gamma^{(n)} \ni \overline{w}^{(n,-1)} = \left(\frac{\alpha_s}{4\pi}\right)^n C_{i_1,i_2,\ldots i_{n+1}} \int dx_1 dx_2 \ldots dx_n \times \prod_{k=1}^{n} p_0(x_k,\alpha_k) \times$$
$$\mathcal{G}_{n-1}\Big(x_1,x_2,\ldots,x_n;q(x_1,\alpha_1),q(x_2,\alpha_2),\ldots q(x_n,\alpha_n)\Big)$$

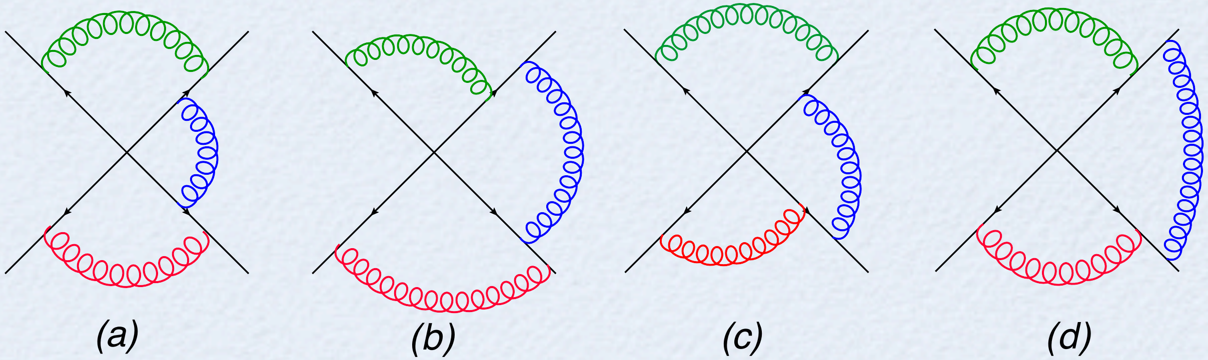
Conjecture:

(1) $\mathcal{G}_{n-1}$ is made exclusively of powers of logs

Or, equivalently,

(2) $\overline{w}^{(n,-1)}$ is a sum of products of polylogs each depending on a single α each having a Symbol with alphabet $\left\{\alpha^2, 1-\alpha^2\right\}$

THREE-LOOP WEB RESULT



$$\overline{w}^{(3)}_{1221} = - f^{abe} f^{cde} T_1^a T_2^b T_3^c T_4^d \left(\frac{g^2}{16\pi^2}\right)^3 r(\alpha_{12}) r(\alpha_{23}) r(\alpha_{34})$$

$$\Big[- 8U_2(\alpha_{12}) \ln\alpha_{23} \ln\alpha_{34} - 8U_2(\alpha_{34}) \ln\alpha_{12} \ln\alpha_{23} + 16\Big(U_2(\alpha_{23}) - 2\Sigma_2(\alpha_{23})\Big) \ln\alpha_{12} \ln\alpha_{34}$$

$$- 2\ln\alpha_{12}\, U_1(\alpha_{23})\, U_1(\alpha_{34}) - 2\ln\alpha_{34}\, U_1(\alpha_{12})\, U_1(\alpha_{23}) + 4\ln\alpha_{23}\, U_1(\alpha_{12})\, U_1(\alpha_{34})\Big]$$

EG (to appear)

CONCLUSIONS: WEBS

- The *non-Abelian exponentiation theorem* has been generalised to any number of Wilson lines in arbitrary representations
- We can directly compute the exponent in terms of webs.
- Webs include sets of non-connected diagrams related by permutations. These involve web mixing matrices with rich combinatorial complexity.
- Colour factors in the exponent - connected graphs!
- Subdivergences cancel by mixing matrices and subtraction of commutators

CONCLUSIONS: POLYLOGS

- We developed a method to compute *multiple gluon exchange webs*, and computed all three-loop 4-leg diagrams in this class. More soon.
- Contributions to the soft anomalous dimension (Subtracted webs) are simple: there is a marked advantage to computing these combinations directly!
- This class of subtracted webs can be written in terms of *polylogarithmic functions of a single variable*, each having Symbol with the alphabet: $\left\{\alpha_{ij}^2, \quad 1-\alpha_{ij}^2\right\}$ realizing the symmetries:

$$\alpha_{ij} \to 1/\alpha_{ij}$$

$$\alpha_{ij} \to -\alpha_{ij} \qquad \text{at Symbol level}$$