Multiloop integrals in dimensional regularization made simple

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based on

- PRL 110 (2013) [arXiv:1304.1806],
- JHEP 1307 (2013) 128 [arXiv:1306.2799] with A.V. and V.A. Smirnov
- arXiv: 1307.4083 with V.A. Smirnov

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The Abc of loop integrals

- Part 1: Method
  - iterated integrals
  - solving differential equations for loop integrals
- Part 2: Application and results
  - massless 2->2 scattering
  - Bhabha scattering
  - vector boson production

Iterated integrals

- Functions we frequently encounter in loop computations

$$\log z = \int_1^z \frac{dt}{t} \qquad \mathrm{Li}_2(z) = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2}$$

- More general integration kernels:

- harmonic polylogarithms (HPLs), 2d HPLs, ...

[Remiddi, Vermaseren, 1999]
[Gehrmann, Remiddi, 2001]

- Goncharov polylogarithms:

[Goncharov; Brown]

$$G_{a_1,\ldots a_n}(z) = \int_0^z \frac{dt}{t-a_1} G_{a_2,\ldots,a_n}(t)$$

- Chen iterated integrals

[Chen, 1977]

$$\int_C \omega_1 \omega_2 \ldots \omega_n \qquad C : [0,1] \longrightarrow M$$ (space of kinematical variables)

Alphabet: set of differential forms $\omega_i = d\log\alpha_i$

We will discuss Feynman integrals that evaluate to these functions

Using them will make the structure of the answer completely transparent

Uniform weight

- Number of iterations: weight e.g. $G_{a_1,\ldots a_n}(z)$ has weight n
- Useful concept: uniform weight

$$f_1 = \frac{1}{x}\log^2 x + \frac{1}{1+x}\mathrm{Li}_2(1-x)$$

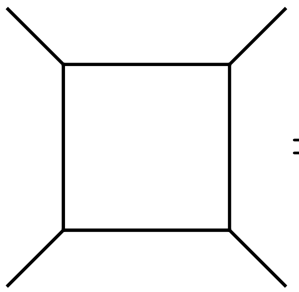
- Pure functions: $\mathbb{Q}$ -linear combinations of uniform weight functions

$$f_2(x,y) = \mathrm{Li}_4(x/y) + 3\log x\,\mathrm{Li}_3(1-y)$$

Note: derivatives of pure functions have weight reduced by one!

- Generalization: assign weight -1 to ϵ in dimensional regularization
  - discuss pure functions in this more general sense
  - extremely strong constraint!

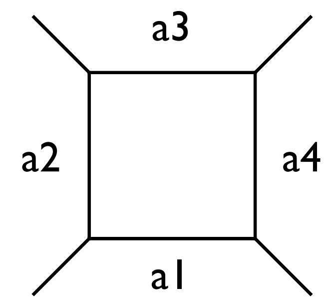
Example: one-loop box $x = t/s$



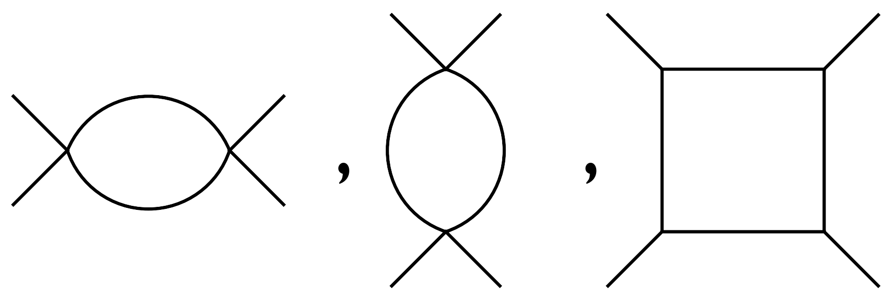
$$= -\frac{1}{(-s)^{1+\epsilon}t\epsilon^2}\left\{4 + \epsilon\left[-2\log x\right] + \epsilon^2\left[-\frac{4}{3}\pi^2\right] + \right.$$
$$\left. +\epsilon^3\left[-2H_{-1,0,0}(x) + 2H_{0,0,0}(x) - \pi^2 H_{-1}(x) + \frac{7}{6}\pi^2 H_0(x) - \frac{34}{3}\zeta_3\right]\right\} + \mathcal{O}(\epsilon^4)$$

Properties of Feynman integrals

- family of integrals: consider arbitrary integer powers of propagators



- integration-by-parts identities (IBP): linear relations [Chetyrkin, Tkachov, 1981]
- finite basis of integrals for a given family [A.V. Smirnov, Petukhov, 2010]

e.g. $f = \{$  $\}$

- integrals satisfy differential equations [Kotikov, 1991] [Gehrmann, Remiddi, 1999]

f_a basis of integrals $a = 1, \ldots N$

x_i kinematical variables, e.g. $s = (p_1 + p_2)^2\,, \quad m^2\,, \ldots$

$$\partial_i f(x_j, \epsilon) = A_i(x_j, \epsilon) f(x_j, \epsilon)$$

$(A_i)_{ab} \quad N \times N$ matrices

Differential equations

- First-order system of DE $$\boxed{\partial_i f(x_j,\epsilon) = A_i(x_j,\epsilon) f(x_j,\epsilon)}$$

- Flat connection $[\partial_i - A_i, \partial_j - A_j] = 0$

  from integrability conditions $(\partial_i \partial_j - \partial_j \partial_i) f = 0$

- Gauge transformation (change of basis)

$$f \longrightarrow B\, f$$
$$A_i \longrightarrow B^{-1} A_i B - B^{-1} (\partial_i B)$$

- Idea: choose integrals that are pure functions as basis [JMH, 2013]
  - this defines a specific gauge transformation

- Pure functions of uniform weight found systematically:
  - using (generalized) unitarity cuts
  - using explicit integral parametrizations

Solving the differential equations

- We find the simplified equations $\partial_i f(x_j, \epsilon) = \epsilon\, A_i(x_j) f(x_j, \epsilon)$
- Integrability conditions $\partial_i A_j - \partial_j A_i = 0$ $[A_i, A_j] = 0$
- pre-integration $d\, f(\epsilon, x_j) = \epsilon\, d\, \tilde{A}(x_j) f(\epsilon, x_j)$

where $\partial_i \tilde{A} = A_i$

observation: matrix $\tilde{A}$ contains only logarithms!

- Solution in terms of Chen iterated integrals

$$f = \sum_{k\geq 0} \epsilon^k f^{(k)}$$

$$f^{(0)} = \text{const} \qquad f^{(1)} = \int d\tilde{A}\, f^{(0)} \qquad f^{(2)} = \int d\tilde{A}\, f^{(1)}$$

makes weight properties manifest!

- In general: $f = P\, e^{\epsilon \int_{\mathcal{C}} d\,\tilde{A}}\, g(\epsilon)$

here $\mathcal{C}$ is a contour in the kinematical space

boundary conditions at base point from physical information

Discussion

- simplified equations

$$d\, f(\epsilon, x_j) = \epsilon\, d\, \tilde{A}(x_j) f(\epsilon, x_j)$$

[JMH, 2013]

- General solution $f = P\, e^{\epsilon \int_{\mathcal{C}} d\, \tilde{A}}\, g(\epsilon)$

$\tilde{A}$ contains (almost) all information about the solution in a compact way:

- expanding solution in ϵ is simple algebra
- contains all information about singularities, asymptotic behavior
- $\tilde{A}$ and boundary info uniquely specify solution
- $\tilde{A}$ specifies `alphabet' of differential forms,

i.e. specific class of iterated integrals;
symbol of solution follows as corollary [cf. Goncharov, Spradlin, Vergu, Volovich, 2010]

Chen iterated integrals are nice objects:

- monodromy invariance;

choice of contour related to integral identities

- underlying Hopf algebra (coproduct)

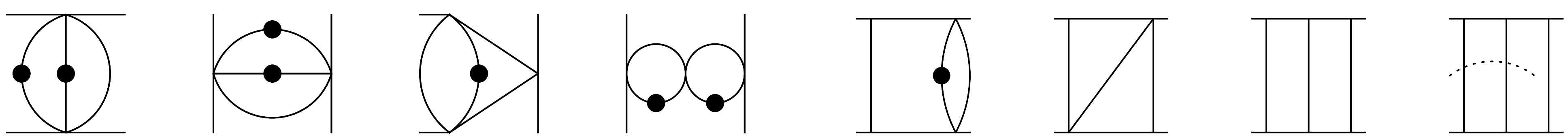
Part 2:
Applications and results

Example: massless 2 to 2 scattering

[Smirnov, 1999][Gehrmann, Remiddi, 1999]

[J.M.H., 2013]

- good choice of master integrals



- Knizhnik-Zamolodchikov equations

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f$$

$$x = t/s$$

$$a == \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ \frac{9}{2} & 3 & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix} \quad b == \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 3 & 6 & 6 & 2 & -4 & -12 & 2 & 2 \\ -\frac{9}{2} & -3 & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix}$$

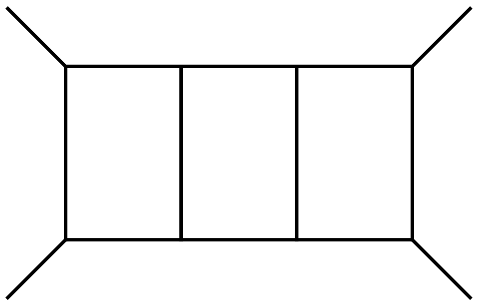
- (regular) singular points

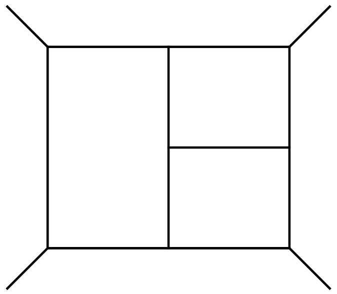
$$s = 0\,, \quad t = 0\,, \quad u = -s - t = 0$$

- boundary conditions: finiteness as $u \to 0$

Generalization to 3 loops

- all planar 2 to 2 three-loop master integrals





[J.M.H., A.V. Smirnov, V.A. Smirnov, 2013]

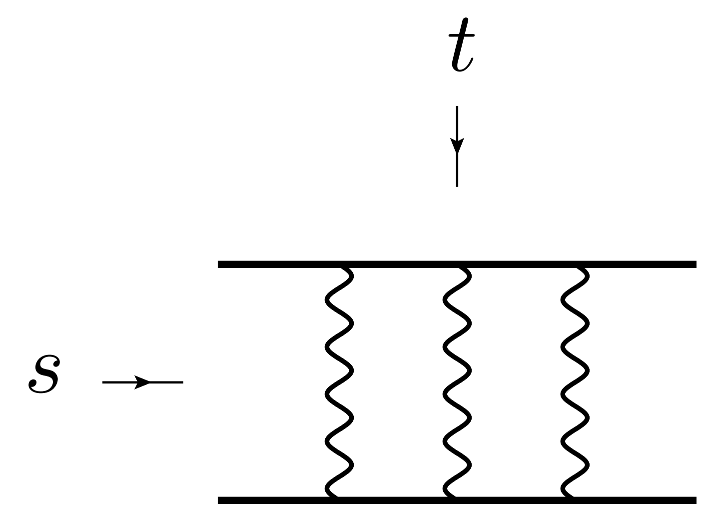
(26 master integrals) (41 master integrals)

$$\partial_x f = \epsilon \left[\frac{a}{x} + \frac{b}{1+x} \right] f \qquad x = t/s$$

- a, b constant $N \times N$ matrices, $N = 26$ or $N = 41$
- solution to arbitrary order ϵ^k in terms of HPLs of weight k by simple algebra
- boundary conditions from finiteness at $x = -1$
- single-scale form factor integrals determined algebraically!
- similar pattern for non-planar integrals [J.M.H., A.V. Smirnov, V.A. Smirnov, in progress]

Bhabha scattering

- we will discuss the integral family



$$p_i^2 = m^2$$

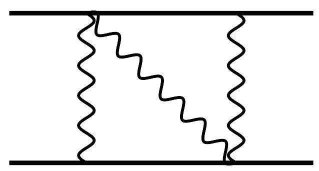
- kinematics

$$\frac{-s}{m^2} = \frac{(1-x)^2}{x} \qquad \frac{-t}{m^2} = \frac{(1-y)^2}{y}$$

- previous work (using Mellin-Barnes or DE)

[Smirnov, 2001] [Heinrich, Smirnov, 2004]
[Czakon, Gluza, Kajda, Riemann, 2004-2006]
[Bonciani, Ferroglia, Mastrolia, Remiddi, van der Bij, 2003, 2004]

many individual integrals computed, to some order in ϵ
problems with coupled differential equations



One-loop warm-up

- differential equations $d\,f = \epsilon\, d\,\tilde{A}\, f$ [J.M.H., V.A. Smirnov, 2013]

$$\tilde{A} = \Big[\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{pmatrix} \log x + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & -2 \end{pmatrix} \log(1+x) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \end{pmatrix} \log y +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \log(1+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \log(1-y) +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 \end{pmatrix} \log(x+y) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 \end{pmatrix} \log(1+xy) \Big]$$

- **alphabet:** $\{x\,, 1 \pm x\,, y\,, 1 \pm y\,, x+y\,, 1+xy\}$

(natural generalization of) two-dimensional HPLs [Gehrmann, Remiddi, 2001]

- (regular) singular points of DE have physical meaning, e.g.:

$$x = 1 \quad \leftrightarrow \quad s = 0 \qquad\qquad x = 0 \quad \leftrightarrow \quad s = \infty$$

$$x = -1 \quad \leftrightarrow \quad s = 4m^2 \qquad x = -y \quad \leftrightarrow \quad u = 0$$

Two-loop case

- differential equations $d\,f = \epsilon\, d\,\tilde{A}\,f$ [J.M.H., V.A. Smirnov, 2013]

$$
\begin{aligned}
\tilde{A} = & B_1 \log(x) + B_2 \log(1+x) + B_3 \log(1-x) + B_4 \log(y) + B_5 \log(1+y) \\
& + B_6 \log(1-y) + B_7 \log(x+y) + B_8 \log(1+xy) \\
& + B_9 \log(x+y-4xy+x^2y+xy^2) + B_{10} \log\left(\frac{1+Q}{1-Q}\right) \\
& + B_{11} \log\left(\frac{(1+x)+(1-x)Q}{(1+x)-(1-x)Q}\right) + B_{12} \log\left(\frac{(1+y)+(1-y)Q}{(1+y)-(1-y)Q}\right)
\end{aligned}
$$

Here $Q = \sqrt{\frac{(x+y)(1+xy)}{x+y-4xy+x^2y+xy^2}}$

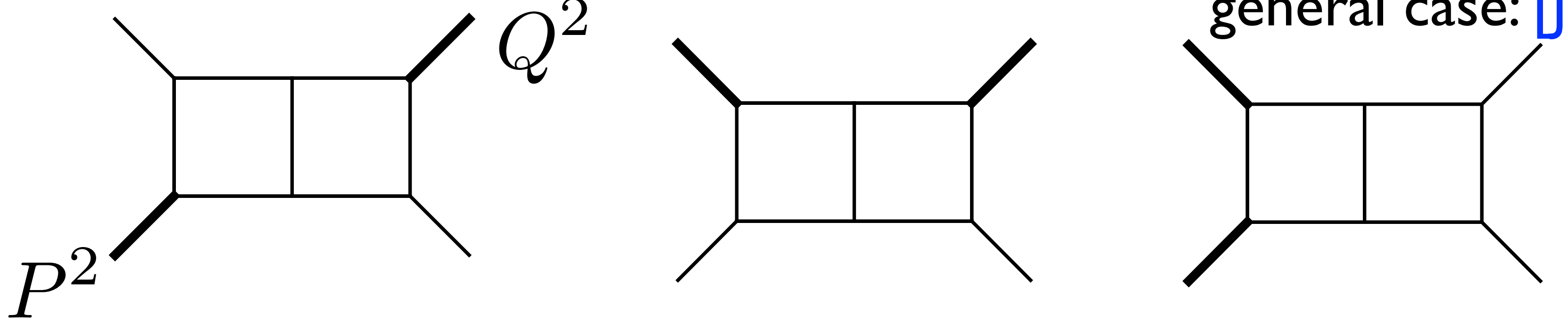
and B_i are constant 23×23 matrices

- larger, 12-letter alphabet compared to one loop
- solution to any order in terms of Chen iterated integrals
- observation: except for one integral, up to weight 4, only the one-loop alphabet is needed!

vector boson production

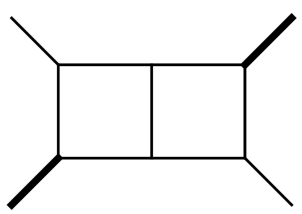
- planar integral families

equal mass case: [Gehrmann, Tancredi, Weihs, 2013]
[see L. Tancredi's talk]
general case: [J.M.H., V.A. Smirnov, to appear]



- diff. equations $\quad d\, f = \epsilon\, d\, \tilde{A}\, f$

- example:



parametrization (removes square roots):

$$\frac{t}{s} = x, \quad \frac{P^2}{s} = \frac{(x+1)z}{y+1}, \quad \frac{Q^2}{s} = \frac{(x+1)y}{(y+1)z}$$

- 14- letter alphabet: $\quad \tilde{A} = \sum_{i=1}^{14} B_i \, \log(\alpha_i)$

$$x,\ 1+x,\ y,\ 1-y,\ 1+y,\ z,\ 1-z,\ y-z,\ y-x,\ 1-xy$$
$$y-z+xy-yz,\ y+xy-xz-xyz$$
$$1+y-z-xz,\ x-z-xz+xy$$

- boundary constants from physical conditions
- other cases solved similarly

Conclusions and outlook

- criteria for finding optimal loop integral basis
  - important concept: pure functions of uniform weight
  - related questions in generalized unitarity approach
- new form of differential equations
  - make properties of functions manifest (analytic structure, discontinuities, monodromy invariance)
  - trivial to solve in terms of Chen iterated integrals
  - solution specified by constant matrices! (to all orders in ϵ)
  - monodromy invariance, coproduct structure helpful for rewriting answer (e.g. for faster numerical evaluation)
- further applications:
  - a library for NNLO integrals relevant for phenomenology
  - top quark amplitudes
  - non-planar integrals
  - phase space integrals

[von Manteuffel, Studerus 2011-2013; JMH, von Manteuffel, Smirnov, to appear]