

Four-Loop On-shell Integrals: $\overline{\text{MS}}$ -onshell relation and $g - 2$

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DESY
in collaboration with

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T. Liu, A. Kurz and M. Steinhauser (in preparation)



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Outline

- 1 Introduction
- 2 Technicalities
- 3 \overline{MS} – on-shell relation for quark masses in QCD
- 4 muon $g - 2$ – electron contributions
- 5 muon $g - 2$ – tau contributions
- 6 Conclusions and Outlook

Application I: $\overline{\text{MS}}$ – on-shell relation

- Fundamental relation between different renormalization schemes.
- Last missing renormalization constant at four loops in QCD.
- Improved precision needed e.g. for the measurement of the top quark mass (the PS mass) at a linear collider
 - Aim: top-mass measurement with $\Delta M_t \approx 100 \text{ MeV}$
 - three-loop correction in the $\overline{\text{MS}}$ –on-shell relation $\approx 300 \text{ MeV}$

Application II: lepton anomalous magnetic moment

- Best experimentally measured and theoretically predicted quantity

$$a_e|_{\text{exp}} = 0.00115965218073(28)$$

$$a_e|_{\text{theo}} = 0.00115965218178(6)(4)(3)(77)$$

Application II: lepton anomalous magnetic moment

- Best experimentally measured and theoretically predicted quantity

$$a_e|_{\text{exp}} = 0.00115965218073(28)$$

$$a_e|_{\text{theo}} = 0.00115965218178(6)(4)(3)(77)$$

$$a_\mu|_{\text{exp}} = 1.16592080(54)(33)[63] \cdot 10^{-3}$$

$$a_\mu|_{\text{theo}} = 1.16591790(65) \cdot 10^{-3} \quad 3.2\sigma \text{ diff.}$$

- QED contributions known numerically up to 5 loops but starting from four loops not checked by an independent calculation
- difference between experiment and theory prediction of the order of the four-loop QED contribution

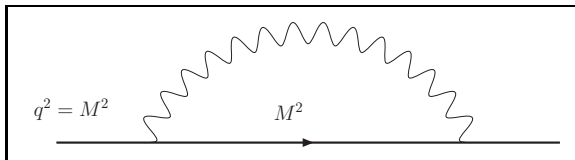
Technicalities

Why discussing $\overline{\text{MS}}$ – on-shell relation and $g - 2$ together?

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Why discussing $\overline{\text{MS}}$ – on-shell relation and $g - 2$ together?

Both lead to the same type of topologies / integrals : on-shell integrals!

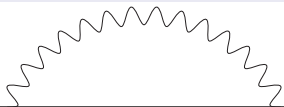


Integral classes

- At four-loop level two classes of integrals have been studied extensively: **massive tadpoles** and **massless propagators**.
- Both classes have many phenomenological applications
- We are now ready for the treatment of a new class of integrals: **on-shell integrals!**
- This new class of integrals has not been studied in any detail.

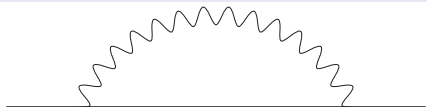
Order of Complexity

$$Z_m : \Sigma(q^2, M^2)|_{q^2=M^2}$$

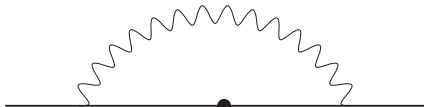


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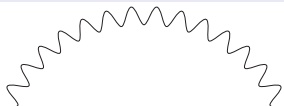


$$Z_2 : \frac{d}{dq^2} \Sigma(q^2, M^2)|_{q^2=M^2}$$

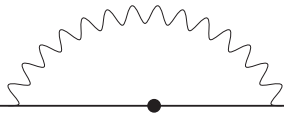


Order of Complexity

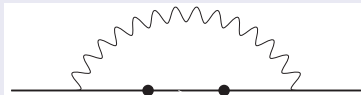
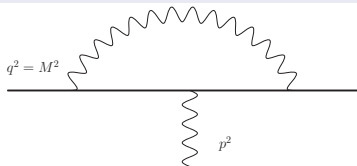
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$$Z_2 : \frac{d}{dq^2} \Sigma(q^2, M^2)|_{q^2=M^2}$$



$$g - 2 : \frac{d}{dp^2} \Gamma(p^2, q^2 = M^2, M^2)|_{p^2=0}$$



Setup

- common setup for both calculations
- tools used include
 - `qgraf`
Generation of Feynman diagrams
 - `q2e, exp`
Expansion / Mapping to topologies
 - FORM
Algebra
 - CRUSHER, FIRE
Reduction to master integrals
 - FIESTA
Calculation of master integrals
- two independent calculations

[Nogueira]

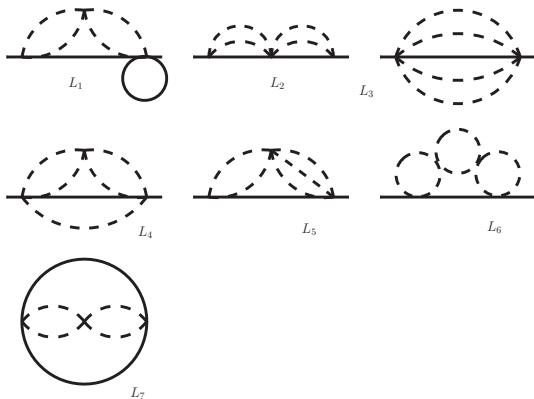
[Harlander, Seidensticker, Steinhauser]

[Vermaseren]

[Seidel, PM / Smirnov, Smirnov]

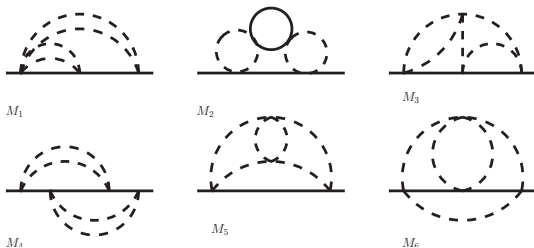
[Smirnov, Smirnov]

Master Integrals: simple



- Expressible through Gamma functions for arbitrary dimension D !

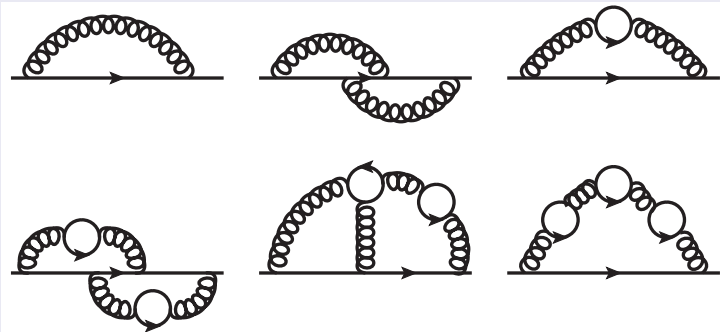
Master Integrals: difficult



- Calculated analytically in expansion in $\epsilon = (4 - D)/2$ using the DRA (dimensional recurrence and analyticity) method and checked using FIESTA!
- Calculated up to $\mathcal{O}(\epsilon^3)$

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$\overline{\text{MS}}$ – on-shell relation n_l^3 and n_l^2 part

Previous works

The $\overline{\text{MS}}$ – on-shell relation has been studied extensively

- two loop [Broadhurst, Grafe, Gray, Schilcher 1990]
- three loop
 - numerical [Chetyrkin, Steinhauser 1999]
 - analytical [Melnikov, van Rittbergen 2000; PM, Mihaila, Piclum, Steinhauser]
- large β_0 approximation [Beneke, Braun]

Results: analytical

$$z_m^{\text{OS}} = \frac{\bar{m}(\mu)}{M} = 1 + \dots + \left(\frac{\alpha_s(\mu)}{\pi} \right)^4 \delta z_m^{(4)} + \mathcal{O}(\alpha_s^5)$$

$$\delta z_m^{(4)} = \delta z_m^{(40)} + n_l \delta z_m^{(41)} + n_l^2 \delta z_m^{(42)} + n_l^3 \delta z_m^{(43)}$$

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$$\begin{aligned} \delta z_m^{(43)} = & C_F T^3 \left(\frac{\ell_M^4}{144} + \frac{13\ell_M^3}{216} + \left(\frac{89}{432} + \frac{\pi^2}{36} \right) \ell_M^2 \right. \\ & + \ell_M \left(\frac{\zeta_3}{3} + \frac{1301}{3888} + \frac{13\pi^2}{108} \right) \\ & \left. + \frac{317\zeta_3}{432} + \frac{71\pi^4}{4320} + \frac{89\pi^2}{648} + \frac{42979}{186624} \right), \ell_M = \log \frac{\mu^2}{M^2} \end{aligned}$$

$$\delta z_m^{(42)} = \dots$$

Results: Numerics

$$\begin{aligned}
 z_m^{\text{OS}} &= 1 - A_s 1.333 + A_s^2 (-14.229 - 0.104 n_h + 1.041 n_l) \\
 &+ A_s^3 (-197.816 - 0.827 n_h - 0.064 n_h^2 \\
 &\quad + 26.946 n_l - 0.022 n_h n_l - 0.653 n_l^2) \\
 &+ A_s^4 (-43.465 n_l^2 - 0.017 n_h n_l^2 + 0.678 n_l^3 + \dots) + \mathcal{O}(A_s^5),
 \end{aligned}$$

with $n_l = 5$, $n_h = 1$:

$$\begin{aligned}
 z_m^{\text{OS}} &= 1 - A_s 1.333 + A_s^2 (-14.332 + 5.207 n_l) \\
 &+ A_s^3 (-198.707 + 134.619 n_l - 16.317 n_l^2) \\
 &+ A_s^4 (-1087.060 n_l^2 + 84.768 n_l^3 + \dots) + \mathcal{O}(A_s^5).
 \end{aligned}$$

large β_0 approximation

$$\left. \frac{M_q}{\bar{m}_q(\bar{m}_q)} \right|_{\text{large-}\beta_0} = 1 + a_s 1.333 + a_s^2 (17.186 - 1.041 n_f) \\ + a_s^3 (177.695 - 21.539 n_f + 0.653 n_f^2) \\ + a_s^4 (3046.294 - 553.872 n_f + 33.568 n_f^2 - 0.678 n_f^3)$$

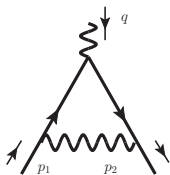
[Beneke, Braun 1995]

$$\frac{M_q}{\bar{m}_q(\bar{m}_q)} = 1 + a_s 1.333 + a_s^2 (13.443 - 1.041 n_f) \\ + a_s^3 (190.595 - 26.655 n_f + 0.653 n_f^2) \\ + a_s^4 (c_0 + c_1 n_f + 43.396 n_f^2 - 0.678 n_f^3)$$

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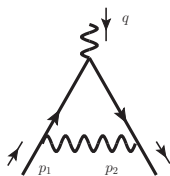
Definition: anomalous magnetic moment a_μ



$$= (-ie)\bar{u}(p_2) \left\{ \gamma^\mu F_E(q^2) + i\frac{\sigma^{\mu\nu} q^\nu}{2m} F_M(q^2) \right\} u(p_1)$$

$$a_\mu = F_M(0)$$

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$$a_\mu = F_M(0)$$

Important!

- The calculation is performed with a **massive** muon and a **massless** electron
- Therefore only correct up to power correction
- Logarithmic contributions can be obtained through renormalization

Previous works

- analytical results

- one loop: $a_{\mu}^{(1)} = \frac{1}{2}$

[Schwinger 1948]

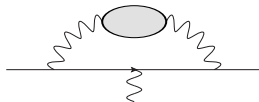
- two loop

[Petermann; Sommerfeld 1957]

- three loop

[Laporta, Remiddi 1996]

- four loops: only partial results, mainly contributions due to corrections to the vacuum polarization function of the photon



- numerical results

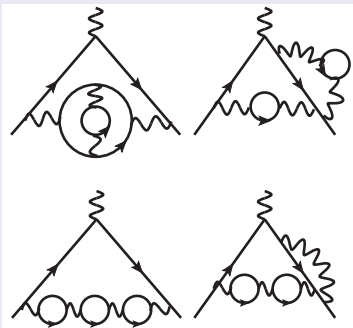
- four loop

[Kinoshita et al]

- five loop

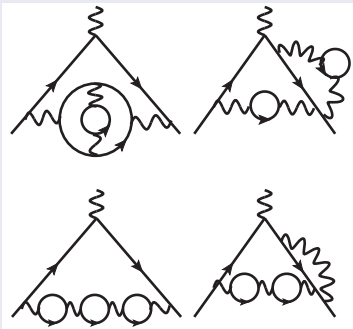
[Aoyama, Hayakawa, Kinoshita, Nio 2012]

Diagrams

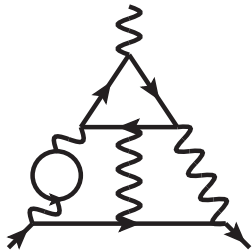
 n_f^3 and n_f^2 part

Diagrams

n_1^3 and n_1^2 part



light-by-light



- calculated for massless electrons!
- light-by-light contribution not finite in this approximation!

Results

$$\begin{aligned} a_{\mu}^{(43)} &= \frac{1}{54} L_{\mu e}^3 - \frac{25}{108} L_{\mu e}^2 + \left(\frac{317}{324} + \frac{\pi^2}{27} \right) L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \\ &\approx 7.19666, \end{aligned}$$

[Laporta; Aguilar, Greynat, De Rafael]

Results

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$$\approx 7.19666,$$

[Laporta; Aguilar, Greynat, De Rafael]

$$a_{\mu}^{(42)} = a_{\mu}^{(42)a} + n_h a_{\mu}^{(42)b}$$

$$a_{\mu}^{(42)a} = L_{\mu e}^2 \left[\pi^2 \left(\frac{5}{36} - \frac{\log 2}{6} \right) + \frac{\zeta_3}{4} - \frac{13}{24} \right] + \dots \approx -3.62427,$$

$$a_{\mu}^{(42)a} \Big|_{\text{num}} = -3.64204(112),$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]

Results

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$$a_{\mu}^{(42)a} \Big|_{\text{num}} = -3.64204(112),$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]

$$a_{\mu}^{(42)b} = \left(\frac{119}{108} - \frac{\pi^2}{9} \right) L_{\mu e}^2 + \left(\frac{\pi^2}{27} - \frac{61}{162} \right) L_{\mu e} - \frac{4\pi^4}{45} + \frac{13\pi^2}{27} + \frac{7627}{1944}$$

$$\approx 0.49405$$

[Laporta; Aguilar, Greynat, De Rafael]

Results: n_l part

- decompose $a_\mu^{(41)}$ further

$$a_\mu^{(41)} = a_\mu^{(41)a} + n_h a_\mu^{(41)b} + n_h^2 a_\mu^{(41)c}$$

- Preliminary** result for $a_\mu^{(41)b}$ and $a_\mu^{(41)c}$

$$a_\mu^{(41)b} = -1.06(5)$$

$$a_\mu^{(41)c} = 0.0280$$

compare with [\[Aoyama, Hayakawa, Kinoshita, Nio 2012\]](#)

$$a_\mu^{(41)b} = -1.046$$

$$a_\mu^{(41)c} = 0.0280$$

- $n_l n_h^0$ part calculated but further cross checks necessary !

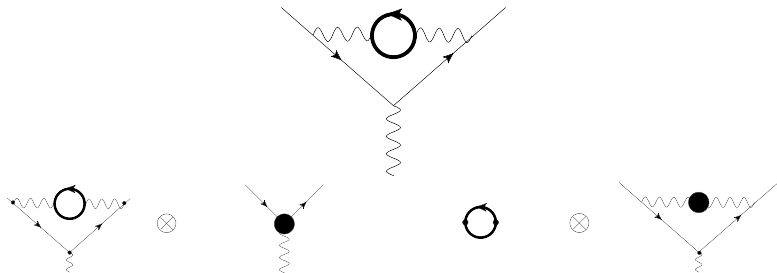
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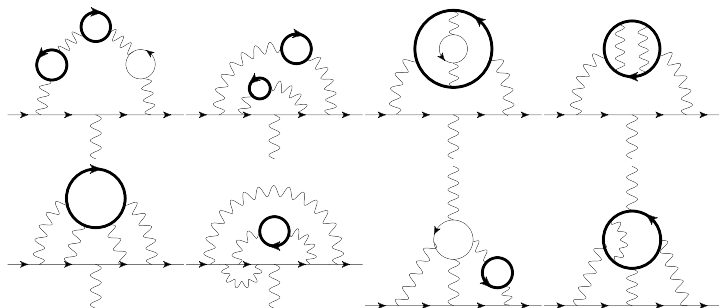
Method

- contributions due to heavy leptons can be calculated in an asymptotic expansion in M_μ/M_τ
- asymptotic expansion leads to at most vacuum diagrams at four loops
- expansion up to including terms of order $(M_\mu/M_\tau)^7$
- all required master integrals are known
- same tool set used as in case of electron contributions

Asymptotic expansion: Example



Preliminary result



Preliminary! [Liu,Kurz,PM,Steinhauser]

$$a_{2,\mu}^{(4)}(M_\mu/M_\tau) = 0.0421670 + 0.0003257 + 0.0000015 = 0.0424941$$

$$\text{cmp. with } a_{2,\mu}^{(4)}(M_\mu/M_\tau) = 0.04234(12)$$

[Aoyama et al 2012]

Conclusions and Outlook

- Progress in the calculation of $g - 2$ and the $\overline{\text{MS}}$ -on-shell relation at four loops.
- Setup for the full calculation established
- Main obstacle: Calculation of $\mathcal{O}(600)$ master integrals
 - Numerical solution for needed master integrals
 - Analytical solution of needed master integrals still missing
- Contributions from τ loops to $g - 2$ calculated with improved precision.