T_F^2-Contributions to 3-Loop Deep-Inelastic Wilson Coefficients and the Asymptotic Representation of Charged Current DIS

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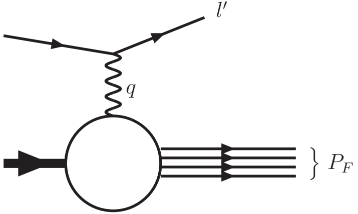
Deep-Inelastic Scattering (DIS)

DIS [inclusive, unpolarized, electromagnetic] gives a clean probe of the proton substructure.



$\rightarrow L_{\mu\nu}$ known (QED)

$\rightarrow W_{\mu\nu}$ QCD dynamics

kinematic variables: $Q^2 = -q^2$, $x = \dfrac{Q^2}{2P.q}$, $y = \dfrac{P.q}{P.l}$

parametrization of the hadronic tensor with structure functions

$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P, s\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x, Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x, Q^2)$$

$\rightarrow$ contributions of massive and massless quarks

Light cone expansion of the current commutator:

[Wilson 1969 PR, Zimmermann 1970, Brandt, Preparata 1971 NPB, Frishman 1971 AP]

$$\lim_{\xi^2 \to 0} \Big[J(\xi), J(0) \Big] \propto \sum_{i,N,\tau} \underbrace{c^N_{i,\tau}(\xi^2, \mu^2)}_{\text{Wilson coefficients}} \xi_{\mu_1} \ldots \xi_{\mu_N} \underbrace{O^{\mu_1 \ldots \mu_m}_{i,\tau}(0, \mu^2)}_{\text{local operators}} + O\left(\frac{\Lambda^2}{Q^2}\right) .$$

At leading twist τ the structure functions factorize

$$F_{(2,L)}(x, Q^2) = \sum_j \quad \mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \quad \otimes \quad f_j(x, \mu^2)$$

into perturbative Wilson coefficients and nonperturbative parton densities (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dydz \, \delta(x - yz) f(y) g(z) \ ,$$

which simplifies to a product upon Mellin transformation

$$\hat{f}(N) := \int_0^1 dx \, x^{N-1} f(x) \ .$$

$\rightarrow$ following computations in Mellin space

Divide the Wilson coefficients into massless and massive parts:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)$$

For $Q^2 \gg m^2$ ($Q^2 \gtrsim 10m^2$ für F_2) the massive Wilson coefficients factorize

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right),$$

[Buza, Matiounine, Smith, van Neerven 1996 NPB]

into massless Wilson coefficients $C_{i,(2,L)}$ and massive operator matrix elements (OMEs); OMEs are local operators O_i sandwiched between partonic states $j = q, g$

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\,.$$

The $C_{i,(2,L)}$ are known at 3 loops [Moch, Vermaseren, Vogt, 2005].

$\longrightarrow$ **compute $A_{ij}\left(\frac{m^2}{\mu^2},N\right)$ at 3 loops!**

Importance of massive quark contributions at 3 loops

Massive quark contributions:

- amount to 20–30% for small x
- contribute scaling violations which differ in shape from those of massless quarks
- are sensitive to the gluon and sea quark PDFs for small x
- allow for the determination of the strange PDF via charged current DIS
- necessary for the precision determination of α_s at 3 loops
- note: asymptotic representation holds at the 1%-level for F_2

Status of massive quark contributions

Leading Order: [Witten 1976, Babcock, Sivers 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

NLO:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

via ${}_pF_q$'s, more compact [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$-contributions (für allg. N) [Bierenbaum, Blümlein, Klein, Schneider 2008]

[Bierenbaum, Blümlein, Klein 2009]

NNLO: $Q^2 \gg m^2$

moments of F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009]

contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]

n_f-contributions to F_2 (all-N): [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011]

Known at 3 Loop:

- $A_{qq,Q}^{\mathsf{PS}}, A_{qg,Q}$: complete
- $A_{qq,Q}^{\mathsf{NS,(TR)}}, A_{gq,Q}$: complete ($\rightarrow$ A. De Freitas' talk)
- $A_{Qg}, A_{Qq}^{\mathsf{PS}}, A_{gg,Q}$: all $O(n_f T_F^2 C_{A/F})$-contributions known
- $A_{gg,Q}$: $O(T_F^2 C_{A/F})$-contributions $(m_1 = m_2) \longrightarrow$ this talk
- $A_{gg,Q}$: Moments for $(m_1 \neq m_2)$-contributions $\longrightarrow$ this talk

Computing Graphs with two massive quark lines (T_F^2-graphs, $m_1 = m_2$)

Hypergeometric series for T_F^2-graphs

Graphs with two massive lines of equal masses ($m_1^2 = m_2^2$):

- Feynman parameterization contains $(z_1 x + z_2 y(1-x))^{a+b\varepsilon}$
- Mellin-Barnes-Representation $\to B(c+N-\sigma, d+\sigma)$
- unbalanced factor $\Gamma(c+N-\sigma) \to$ sum of residues diverges
- cannot be cured by "subtracting finitely many residues"

Solution:

- observe:
$$B(c+N-\sigma, d+\sigma) = \int_0^1 dx\, x^{c+N-1}(1-x)^{d-1}\left(\frac{1-x}{x}\right)^{\sigma}$$
$\to$ direction for closing the contour depends on x
- split x-integral into two parts $x < \frac{1}{2}$ and $x \geq \frac{1}{2}$
$\to$ no beta function, integrate later
- close contour differently $\to$ convergent sum of residues

$\to$ ε-expansion and summation: `SumProduction`, `EvaluateMultiSums`, `Sigma` [C. Schneider]

Solving the last Feynman parameter integral

Idea: solve the last integral in the space of cyclotomic HPLs

The occurring cyclotomic HPLs are iterated integrals built from the alphabet:

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1-x}, \quad f_{-1}(x) = \frac{1}{1+x},$$
$$f_{(4,0)}(x) = \frac{1}{1+x^2}, \quad f_{(4,1)}(x) = \frac{x}{1+x^2},$$

e.g.: $H_{1,(4,1),0,0}(x) = \int_0^x dx_1 \; f_1(x_1) \int_0^{x_1} dx_2 \; f_{(4,1)}(x_2) \, \frac{1}{2} \ln^2(x_2).$

Furthermore, generalized letters occur: $f_{[i,\kappa]}(x) = f_i(\kappa x)$

$\rightarrow$ many properties of cycl. HPLs and cycl. S-Sums are known

[Ablinger, Blümlein, Schneider 2011 JMP]

and implemented in the package `HarmonicSums` [J. Ablinger]

Solving the last Feynman parameter integral

- summation yields cyclotomic S-Sums $\hat{=}$ cycl. HPLs

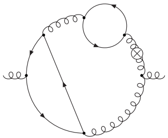
$$S_1(-x,\infty) = -H_{-1}(x)$$

$$S_{(1,0,1),(2,1,1)}(-x,1;\infty) = -\frac{4H_{(4,0)}\left(\sqrt{x}\right)}{\sqrt{x}} + \frac{H_{(4,1)}\left(\sqrt{x}\right)}{2x} - \frac{3}{2}H_{(4,1)}\left(\sqrt{x}\right) - 2H_{(4,0),(4,0)}\left(\sqrt{x}\right) + \frac{15}{4}$$

$$\ldots$$

- use generating function: $[f(x)]^N \to \frac{1}{1-\kappa f(x)}$,
- integrate
- yield representation $\sum_{\vec{a}} c_{\vec{a}} H_{\vec{a}}(\kappa)$ such that each term has a Taylor expansion for $\kappa \approx 0$
- determine N-th Taylor coefficient ($\to$ `HarmonicSums` [J. Ablinger]: Cauchy products, difference equations $\to$ `EvaluateMultiSums`, `Sigma` [C. Schneider])

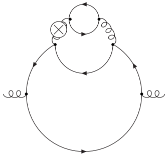
Results: A scalar graph



$$\mathrm{Res} = \frac{(-1)^N+1}{2}\Bigg\{\frac{1}{45\varepsilon^2(N+1)} - \frac{1}{\varepsilon}\left[\frac{S_1(N)}{90(N+1)} + \frac{47N^3+20N^2-67N+40}{1800(N-1)N(N+1)^2}\right]$$
$$+ \frac{105N^3-175N^2+56N+96}{13440(N+1)^2(2N-3)(2N-1)4^N}\binom{2N}{N}\left[\sum_{j=1}^{N}\frac{4^j S_1(j)}{\binom{2j}{j}j^2} - \sum_{j=1}^{N}\frac{4^j}{\binom{2j}{j}j^3} - 7\zeta_3\right]$$
$$+ \frac{\left(5264N^3-2409N^2-12770N+3528\right)S_1(N)}{100800(N+1)^2(2N-3)(2N-1)} + \frac{S_1(N)^2+S_2(N)+3\zeta_2}{360(N+1)}$$
$$+ \frac{S_3(N)-S_{2,1}(N)+7\zeta_3}{420(N+1)} + \frac{P_{13}}{2268000(N-1)^2N^2(N+1)^3(2N-3)(2N-1)}\Bigg\}$$

$\rightarrow$ new nested sums occur

Results: A QCD graph



$$
\begin{aligned}
I_{560} = {} & \frac{2P_4}{3N(N+1)^2(N+2)(2N-5)(2N-3)(2N-1)} \frac{1}{4^N}\binom{2N}{N}\left[\sum_{j=1}^{N}\frac{4^j S_1(j)}{\binom{2j}{j} j^2} - \sum_{j=1}^{N}\frac{4^j}{\binom{2j}{j} j^3} - 7\zeta_3\right] \\
& + \frac{N^2+N+2}{27(N-1)N^2(N+1)}\left[-144S_{2,1}(N) - 36\zeta_2 S_1(N) - 4S_1(N)^3 + 36S_2(N)S_1(N) + 88S_3(N) + 312\zeta_3\right] \\
& - \frac{4P_2}{6075(N-2)^2(N-1)^4N^5(N+1)^4(N+2)(2N-5)(2N-3)(2N-1)} - \frac{64\left(N^2+N+2\right)}{9\varepsilon^3(N-1)N^2(N+1)} \\
& + \frac{1}{\varepsilon^2}\left\{-\frac{32P_6}{27(N-1)^2N^3(N+1)^2(N+2)} - \frac{32\left(N^2+N+2\right)}{9(N-1)N^2(N+1)}S_1(N)\right\} \\
& + \frac{1}{\varepsilon}\left\{-\frac{8P_7}{405(N-2)(N-1)^3N^4(N+1)^3(N+2)} - \frac{16P_6}{27(N-1)^2N^3(N+1)^2(N+2)}S_1(N)\right. \\
& \left. - \frac{8\left(N^2+N+2\right)\left(S_1(N)^2 - 3S_2(N) + 3\zeta_2\right)}{9(N-1)N^2(N+1)} - \frac{8\left(55N^3+235N^2-52N+20\right)}{15(N-2)(N-1)N(N+1)^2(N+2)}\right\} \\
& - \frac{4P_5S_1(N)}{81(N-1)^3N^4(N+1)^3(N+2)(2N-5)(2N-3)(2N-1)} - \frac{4P_6\left(S_1(N)^2 - 3S_2(N) + 3\zeta_2\right)}{27(N-1)^2N^3(N+1)^2(N+2)} \\
& - \frac{4P_1}{225(N-2)^2(N-1)^2N^2(N+1)^3(N+2)}
\end{aligned}
$$

T_F^2-contributions to A_{ggQ} for $m_1 = m_2$

- inverse binomial sums occur:

$$\sum_{j=1}^{N} \frac{4^j S_1(j)}{\binom{2j}{j} j^2} - \sum_{j=1}^{N} \frac{4^j}{\binom{2j}{j} j^3} = \int_0^1 dx \, \frac{x^N - 1}{1-x} \int_x^1 dy \, \frac{1}{y\sqrt{1-y}} \times \Big[\ln(1-y) - \ln(y) + 2\ln(2) \Big]$$

- removable poles at $N = 1/2$, $N = 3/2$, $N = 5/2$
- used for the calculation of all scalar graphs consistent with $A_{gg,Q}^{(3)}$ ($\rightarrow$ proof of principle)
- applied to QCD graphs in the T_F^2-contribution to $A_{gg,Q}^{(3)}$ $\rightarrow$ [Ablinger, Blümlein, Hasselhuhn, Round, Schneider 2013]
- checks with `MATAD` [M. Steinhauser 2000]
- graphs are similar to the case $m_1 \neq m_2$ $\rightarrow$ same ideas are applicable with generalization

Moments from graphs with two massive lines of unequal mass (charm and bottom)

Moments for graphs with two massive lines ($m_1 \neq m_2$)

- There are 8 different OMES:

$$\underbrace{A_{Qg},\ A_{qq,Q}^{\mathrm{NS(TR)}},\ A_{Qq}^{\mathrm{PS}},\ A_{gg,Q},\ A_{gq,Q}}_{\text{Contain contributions with b- and c-quarks}}\ \underbrace{A_{qg,Q},\ A_{qq,Q}^{\mathrm{PS}}}_{\text{Completely known}}$$

OME	# diagrams
$A_{Qg}^{(3)}$	272
$A_{Qq}^{(3),\mathrm{PS}}$	16
$A_{qq,Q}^{(3),\mathrm{NS}}$	4
$A_{qq,Q}^{(3),\mathrm{NS,TR}}$	4
$A_{gq,Q}^{(3)}$	4
$A_{gg}^{(3)}$	76
Σ	376

- renormalization of the 2-mass case has been performed
- $m_c^2/m_b^2 \simeq 1/10$ expansion parameter
- But: Charm cannot be treated massless at the scale $\mu \simeq m_b$.

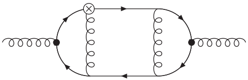
Moments for graphs with two massive lines ($m_1 \neq m_2$)

$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) = T_F^2 C_A \Bigg\{ & \frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{156992}{6615} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{68332}{6615} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{412808}{19845}\Bigg\} \\
+ T_F^2 C_F \Bigg\{ & -\frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{1980566069882672}{2467763508585375}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1230328}{138915}\Bigg\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

$\rightarrow$ `q2e/exp` [Harlander, Seidensticker, Steinhauser 1999]

Convergent massive 3-loop graphs

The α-representation and graph polynomials

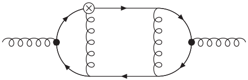


$$\hat{I}_4(N) = \int \cdots \int d\alpha_1\ d\alpha_2\ d\alpha_3\ d\alpha_4\ d\alpha_5\ d\alpha_6\ d\alpha_7\ d\alpha_8\ \frac{\sum_{j=0}^{N} T_{4\alpha}^{N-j} T_{4b}^{j}}{U^2 V^2}$$

$$\begin{aligned}
T_{4\alpha} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad + \alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad + \alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad + \alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad + \alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad + \alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
V &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
\end{aligned}$$

- The integral above is a projective integral, one α-parameter may be set 1
- use generating function ($\rightarrow$ N-th Taylor coefficient from `HarmonicSums` [J. Ablinger])
- For the above example : after applying symmetry transformations $\alpha_1 \rightarrow x_1 - \alpha_2$, $\alpha_3 \rightarrow x_2 - \alpha_4$, $\alpha_5 \rightarrow x_5 - \alpha_6$ $\alpha_2, \alpha_4, \alpha_6$ are only contained in the operator polynomials and may be integrated out at this stage.
- Generalize Brown's algorithm for the massive case and local operator insertions.

Six Massive Lines and Vertex Insertion



$$
\begin{aligned}
\hat{I}_4 \;=\;& \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 + \frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} \\
&+\frac{(-24-5N+2N^2)}{12(2+N)^2(3+N)^2}S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2S_2 \\
&+\frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} - 2\zeta_3S_{-2} - S_{-2,1}S_{-2} \\
&+\frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} + \frac{(59+42N+6N^2)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3S_1(2) \\
&-\frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3S_2 - \frac{3}{2}S_3S_2 - 2S_{2,1}S_2 + \frac{(99+225N+190N^2+65N^3+7N^4)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1} \\
&+\frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4}S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2}S_2S_1 - S_{2,3} \\
&+\frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1S_{2,1} + \frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)}S_{3,1} - 3S_{4,1} \\
&+S_{-2,1,-2} + \frac{\mathbf{2^{N+1}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right) - \frac{(-7+2N^2)}{(1+N)(2+N)(3+N)}S_{2,1,1} \\
&+5S_{2,2,1} + 6S_{3,1,1} + \frac{\mathbf{2^{N}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
&-\frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right) - \frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)
\end{aligned}
$$

Limit $N \to \infty$

The 2^N factors cancel in the large N limit (asymptotic expansion with `HarmonicSums`):

$$
\begin{aligned}
\hat{I}_4 \approx\; & \zeta_2^2\left[\frac{1115231}{20N^{10}}-\frac{74121}{4N^9}+\frac{122951}{20N^8}-\frac{40677}{20N^7}+\frac{13391}{20N^6}-\frac{873}{4N^5}+\frac{1391}{20N^4}-\frac{417}{20N^3}+\frac{101}{20\,N^2}\right]\\
&+\zeta_3\Bigg[\left(-\frac{95855}{2N^{10}}+\frac{31525}{2N^9}-\frac{10295}{2N^8}+\frac{3325}{2N^7}-\frac{1055}{2N^6}+\frac{325}{2N^5}-\frac{95}{2N^4}+\frac{25}{2N^3}-\frac{5}{2\,N^2}\right)\ln(N)\\
&-\frac{23280115}{2016N^{10}}+\frac{2093041}{1008N^9}-\frac{177251}{1008N^8}-\frac{25843}{336N^7}+\frac{2569}{48N^6}-\frac{155}{8N^5}+\frac{91}{24N^4}+\frac{2}{3N^3}-\frac{11}{12\,N^2}\Bigg]\\
&+\zeta_2\Bigg[\left(\frac{19171}{N^{10}}-\frac{6305}{N^9}+\frac{2059}{N^8}-\frac{665}{N^7}+\frac{211}{N^6}-\frac{65}{N^5}+\frac{19}{N^4}-\frac{5}{N^3}+\frac{1}{N^2}\right)\ln^2(N)\\
&+\left(\frac{103016863}{2520N^{10}}-\frac{3091261}{315N^9}+\frac{2571839}{1260N^8}-\frac{6215}{21N^7}-\frac{293}{20N^6}+\frac{2071}{60N^5}-\frac{103}{6N^4}+\frac{67}{12N^3}-\frac{1}{N^2}\right)\ln(N)\\
&+\frac{292993001621}{302400N^{10}}-\frac{4402272031}{30240N^9}+\frac{22261739}{840N^8}-\frac{78507473}{14112N^7}+\frac{180961}{144N^6}-\frac{111807}{400N^5}+\frac{629}{12N^4}-\frac{319}{72N^3}-\frac{7}{4\,N^2}\Bigg]\\
&+\left(\frac{249223}{6N^{10}}-\frac{145015}{12N^9}+\frac{10295}{3N^8}-\frac{11305}{12N^7}+\frac{1477}{6N^6}-\frac{715}{12N^5}+\frac{38}{3N^4}-\frac{25}{12N^3}+\frac{1}{6\,N^2}\right)\mathbf{\ln^3(N)}\\
&+\left(\frac{193493767}{10080N^{10}}+\frac{210658237}{10080N^9}-\frac{21541697}{2520N^8}+\frac{243269}{96N^7}-\frac{30539}{48N^6}+\frac{2123}{16N^5}-\frac{59}{3N^4}+\frac{5}{8N^3}+\frac{1}{2\,N^2}\right)\ln^2(N)\\
&+\left(-\frac{2207364771673}{4233600N^{10}}+\frac{1390655509}{352800N^9}+\frac{285594061}{22050N^8}-\frac{67234111}{14400N^7}+\frac{8617073}{7200N^6}-\frac{35209}{144N^5}+\frac{116}{3N^4}-\frac{119}{24N^3}+\frac{1}{N^2}\right)\ln(N)\\
&+\frac{1344226725047831}{889056000N^{10}}-\frac{165849841805771}{889056000N^9}+\frac{808151260279}{27783000N^8}-\frac{708430537}{120960N^7}+\frac{304474703}{216000N^6}\\
&-\frac{606811}{1728N^5}+\frac{1867}{24N^4}-\frac{1813}{144N^3}+\frac{1}{N^2}+O(N^{-11})
\end{aligned}
$$

Massive quarks in charged current DIS at 2 loops

Structure functions of charged current DIS

The cross section is parameterized introducing 3 structure functions.
For symmetry reasons consider the combinations

$$\frac{d\sigma^{\nu}}{dxdy} \pm \frac{d\sigma^{\bar{\nu}}}{dxdy} = \frac{G_F^2 s}{4\pi}\left\{(1+(1-y)^2)F_2^{W^+\pm W^-} - y^2 F_L^{W^+\pm W^-} + (1-(1-y)^2)xF_3^{W^+\pm W^-}\right\}.$$

The structure functions factorize in PDFs and Wilson coefficients

$$\begin{aligned} F_2^{W^+ + W^-} = 2\Big\{ & \left(|V_{du}|^2(d+\bar{d}) + |V_{su}|^2(s+\bar{s}) + V_u(u+\bar{u})\right)(C_{2,q}^{W^+ + W^-,\mathrm{NS}} + L_{2,q}^{W^+ + W^-,\mathrm{NS}}) \\ & + \left(|V_{dc}|^2(d+\bar{d}) + |V_{sc}|^2(s+\bar{s})\right) H_{2,q}^{W^+ + W^-,\mathrm{NS}} + 2V_c\left[H_{2,q}^{W,\mathrm{PS}}\Sigma + H_{2,g}^{W}G\right] \\ & + 2V_u\left[(C_{2,q}^{W,\mathrm{PS}} + L_{2,q}^{W,\mathrm{PS}})\Sigma + (C_{2,g}^{W} + L_{2,g}^{W})G\right]\Big\}, \\ F_3^{W^+ + W^-} = 2\Big\{ & \left(|V_{du}|^2(d+\bar{d}) + |V_{su}|^2(s+\bar{s}) - V_u(u+\bar{u})\right)(C_{3,q}^{W^+ + W^-,\mathrm{NS}} + L_{3,q}^{W^+ + W^-,\mathrm{NS}}) \\ & + \left(|V_{dc}|^2(d+\bar{d}) + |V_{sc}|^2(s+\bar{s})\right) H_{3,q}^{W^+ + W^-,\mathrm{NS}} + 2V_c\left[H_{3,q}^{W,\mathrm{PS}}\Sigma + H_{3,g}^{W}G\right]\Big\}. \end{aligned}$$

where $V_i := |V_{id}|^2 + |V_{is}|^2$, $i = u, c$, and $\Sigma = \sum_q (q + \bar{q})$.

factors (-1) due to charge antisymmetry of corresponding contributions.

2-Loop corrections in the region $Q^2 \gg m^2$

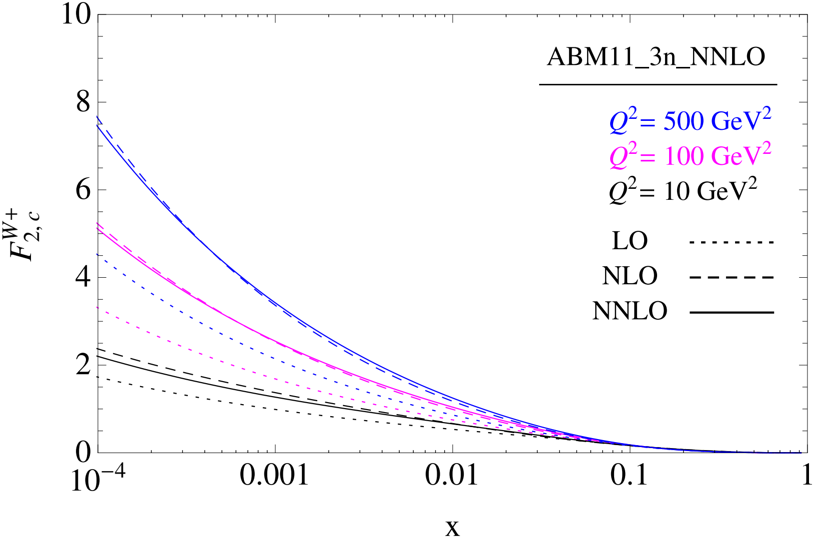
At 2-Loop order we derived the asymptotic representation using the transition to the variable flavor number scheme:

$$\begin{aligned}
L_{2,q}^{W^+\pm W^-,\mathrm{NS},(2)} &= A_{qq,Q}^{\mathrm{NS},(2)} + C_{2,q}^{W^+\pm W^-,\mathrm{NS},(2)}(n_f+1) - C_{2,q}^{W^+\pm W^-,\mathrm{NS},(2)}(n_f)\,,\\
H_{2,q}^{W^+\pm W^-,\mathrm{NS},(2)} &= A_{qq,Q}^{\mathrm{NS},(2)} + C_{2,q}^{W^+\pm W^-,\mathrm{NS},(2)}(n_f+1)\,,\\
L_{2,q}^{W,\mathrm{PS},(2)} &= C_{2,q}^{W,\mathrm{PS},(2)}(n_f+1) - C_{2,q}^{W,\mathrm{PS},(2)}(n_f) = 0\,,\\
H_{2,q}^{W,\mathrm{PS},(2)} &= \frac{1}{2}A_{Qq}^{\mathrm{PS},(2)} + C_{2,q}^{W,\mathrm{PS},(2)}(n_f+1)\,,\\
L_{2,g}^{W,(2)} &= A_{gg,Q}^{(1)}C_{2,g}^{W,(1)}(n_f+1) + C_{2,g}^{W,(2)}(n_f+1) - C_{2,g}^{W,(2)}(n_f)\,,\\
H_{2,g}^{W,(2)} &= A_{gg,Q}^{(1)}C_{2,g}^{W,(1)}(n_f+1) + C_{2,g}^{W,(2)}(n_f+1)\\
&\quad + \frac{1}{2}\left(A_{Qg}^{(2)} + A_{Qg}^{(1)}C_{2,q}^{W^+ + W^-,\mathrm{NS},(1)}(n_f+1)\right)\,,\\
L_{3,q}^{W^+\pm W^-,\mathrm{NS},(2)} &= A_{qq,Q}^{\mathrm{NS},(2)} + C_{3,q}^{W^+\pm W^-,\mathrm{NS},(2)}(n_f+1) - C_{3,q}^{W^+\pm W^-,\mathrm{NS},(2)}(n_f)\,,\\
H_{3,q}^{W^+\pm W^-,\mathrm{NS},(2)} &= A_{qq,Q}^{\mathrm{NS},(2)} + C_{3,q}^{W^+\pm W^-,\mathrm{NS},(2)}(n_f+1)\,,\\
H_{3,q}^{W,\mathrm{PS},(2)} &= -\frac{1}{2}A_{Qq}^{\mathrm{PS},(2)}\,,\\
H_{3,g}^{W,(2)} &= \frac{1}{2}\left(-A_{Qg}^{(2)} - A_{Qg}^{(1)}C_{3,q}^{W^+ + W^-,\mathrm{NS},(1)}(n_f+1)\right)\,.
\end{aligned}$$

2-Loop corrections in the region $Q^2 \gg m^2$

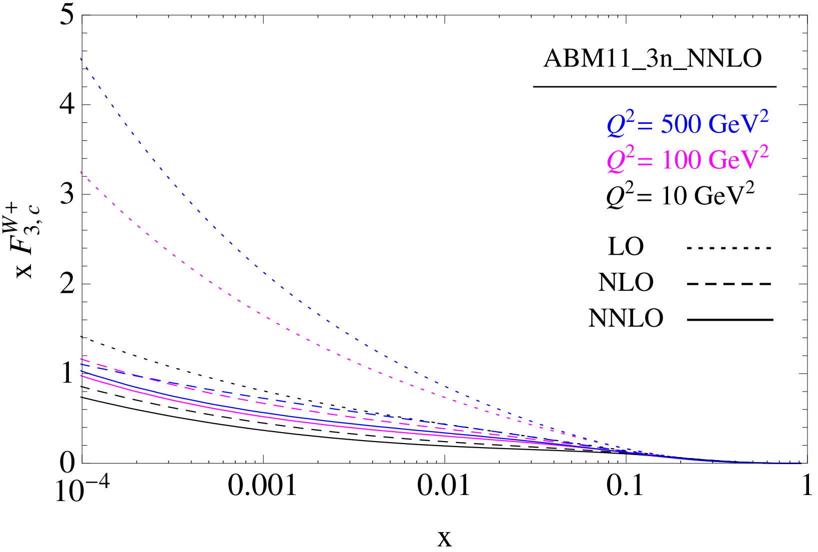
- A previous drivation [Buza, van Neerven 1997 NPB] was corrected and completed.
- Using OMEs and massless Wilson coefficients from the literature, the massive Wilson coefficients in Mellin space and x-space have been constructed.
- The Mellin transformation implemented in `HarmonicSums` [J. Ablinger] was used.
- A `FORTRAN` implementation for the application to experimental data will be published soon. [Blümlein, Hasselhuhn, Pfoh 2013]
- The asymptotic representation is accurate since experimentally $Q^2 \gtrsim 100$ GeV^2.

Results



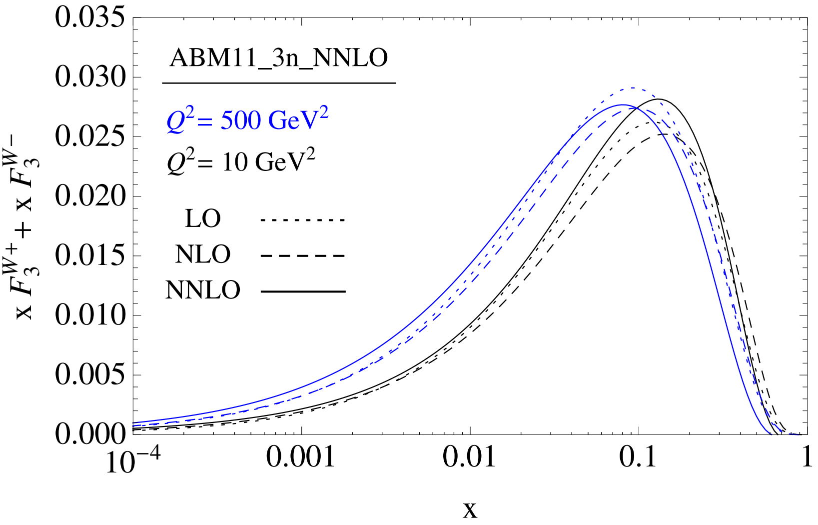
ABM11_3n_NNLO
Q^2= 500 GeV^2
Q^2= 100 GeV^2
Q^2= 10 GeV^2
LO
NLO
NNLO
$F_{2,c}^{W+}$
10
8
6
4
2
0
10^{-4}
0.001
0.01
0.1
1
x

Results



ABM11_3n_NNLO
Q^2= 500 GeV^2
Q^2= 100 GeV^2
Q^2= 10 GeV^2
LO
NLO
NNLO
x $F_{3,c}^{W+}$
x
5
4
3
2
1
0
10^{-4}
0.001
0.01
0.1
1

Results



ABM11_3n_NNLO
Q^2= 500 GeV^2
Q^2= 10 GeV^2
LO
NLO
NNLO
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
$x\,F_3^{W+} + x\,F_3^{W-}$
10^{-4}
0.001
0.01
0.1
1
x

Conclusions

- Heavy flavor contributions are needed at 3-loop order for the precise determination of α_s from DIS data.
- 5 out of 8 massive operator matrix elements have been completed recently.
- A method for the calculation of graphs with two massive lines of equal masses and operator insertions was presented, and applied to the corresponding contributions to $A_{gg,Q}^{(3)}$. The calculations will be finished soon. $\rightarrow$ [Ablinger, Blümlein, Hasselhuhn, Round 2013]
- The method can be generalized to the case of unequal masses.
- The moments for $N = 2, 4, 6$ for all graphs with two quark lines of unequal masses are now known, as well as the renormalization of these contributions.
- Convergent massive three loop graphs can be calcualted in terms of iterated integrals, allowing for a study of the functions governing the more complicated topologies like ladder graphs, benz graphs and V-graphs.

Conclusions

- On the phenomenological side, the complete 2-loop heavy flavor contributions to charged current DIS were derived from light flavor Wilson coefficients and massive OMEs. They are implemented into a Mellin space program in `FORTRAN` and will be published soon
  $\rightarrow$ [Blümlein, Hasselhuhn, Pfoh 2013].
- The calculations of new 3-loop corrections can take advantage of rigorous computer algebra packages of our friends at RISC; at the same time techniques known to physicists or newly developed on practical problems lead to improvements in general purpose programs.