QCD thermodynamics at 3 loops: methods and results

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recent work with Ioan Ghişoiu

and earlier work with:
F. Di Renzo, A. Hietanen, K. Kajantie, M. Laine, V. Miccio,
J. Möller, K. Rummukainen, C. Torrero, A. Vuorinen

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Motivation

Context for this talk: Thermal QCD

- study confinement and chiral symmetry breaking
- phenomenologically relevant for cosmology
- phenomenologically relevant for RHIC, LHC
- theoretical limit tractable with analytic methods
  - ▷ goal: no models - stay within QCD!
  - ▷ goal: possibility of systematic improvements
  - ▷ parameters: T, μ_q, m_q, (N_c, N_f)

Motivation

Focus on equilibrium thermodynamics of QCD

- typical questions to be addressed
  - ▷ equation of state (EoS)
  - ▷ structure of QCD phase diagram
    transition lines, order of transitions, critical points
  - ▷ medium properties: spectral functions, correlation lengths, ...

Interplay of methods

- QGP is strongly coupled system near $T_c \Rightarrow$ need e.g. LAT
- asymptotic freedom at high $T \Rightarrow$ weak-coupling approach in continuum
  - ▷ cave: strict loop expansion not well-defined
    IR divergences at higher orders [Linde 79; Gross/Pisarski/Yaffe 81]
- try to use best of both

Energy scales in hot QCD

Interactions make QCD a multi-scale system

- At asymptotically high T, $g \ll 1 \Rightarrow$ clean separation of 3 scales
- expansion parameter: $$g^2\, n_b(|k|) = \frac{g^2}{e^{|k|/T}-1} \overset{|k|\lesssim T}{\approx} \frac{g^2 T}{|k|}$$
- $|k| \sim \pi T / gT / g^2 T$
  aka hard/soft/ultrasoft scales
  are fully/barely/non- perturbative at high T
- no smaller momentum scales / larger length scales due to confinement

treatment of a multi-scale system: effective field theory !

Pressure $p(T)$ via weak-coupling expansion

- structure of pert series is non-trivial !
- $$\begin{aligned} p(T) &\equiv \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}[A^a_\mu, \psi, \bar\psi] \exp\Big(-\frac{1}{\hbar}\int_0^{\hbar/T} d\tau \int d^{3-2\epsilon}x\, \mathcal{L}^E_{\rm QCD}\Big) \\ &= c_0 + c_2 g^2 + c_3 g^3 + (c_4' \ln g + c_4) g^4 + c_5 g^5 + (c_6' \ln g + c_6) g^6 + \mathcal{O}(g^7) \end{aligned}$$

  [c_2 Shuryak 78, c_3 Kapusta 79, c_4' Toimela 83, c_4 Arnold/Zhai 94, c_5 Zhai/Kastening 95, Braaten/Nieto 96, c_6' KLRS 03]

- root cause of nonanalytic (in α_s) behavior well understood:
  above-mentioned dynamically generated scales
- clean separation best understood in effective field theory setup [here: $\mu = 0$]
  - ▷ generalizations, e.g. $\mu \neq 0$ [Vuorinen], standard model [Gynther/Vepsäläinen]
- compact (imag.) time interval ⟶ sum-integrals

  $$\sum\!\!\!\!\!\!\int_P = T \sum_{n=-\infty}^{\infty} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \;;\;\; P^2 = P_0^2 + p^2 \text{ with } P_0 = 2\pi n T \text{ (bos)}$$

  - ▷ these can be nasty objects

Effective theory prediction for $p(T)$

$$\begin{aligned}
\frac{p_{\rm QCD}(T)}{p_{\rm SB}} &= \frac{p_{\rm E}(T)}{p_{\rm SB}} + \frac{p_{\rm M}(T)}{p_{\rm SB}} + \frac{p_{\rm G}(T)}{p_{\rm SB}} \quad , \quad p_{\rm SB} = \left(16 + \frac{21}{2}N_f\right)\frac{\pi^2 T^4}{90} \\
&= 1 + g^2 \quad + g^4 \quad + g^6 \quad + \ldots \qquad \Leftarrow \text{4d QCD} \\
&\qquad + g^3 + g^4 + g^5 + g^6 + \ldots \qquad \Leftarrow \text{3d adj H} \\
&\qquad + \frac{1}{p_{\rm SB}}\frac{T}{V}\int \mathcal{D}[A_k^a]\exp\Big(-S_{\rm M}\Big) \qquad \Leftarrow \text{3d YM}
\end{aligned}$$

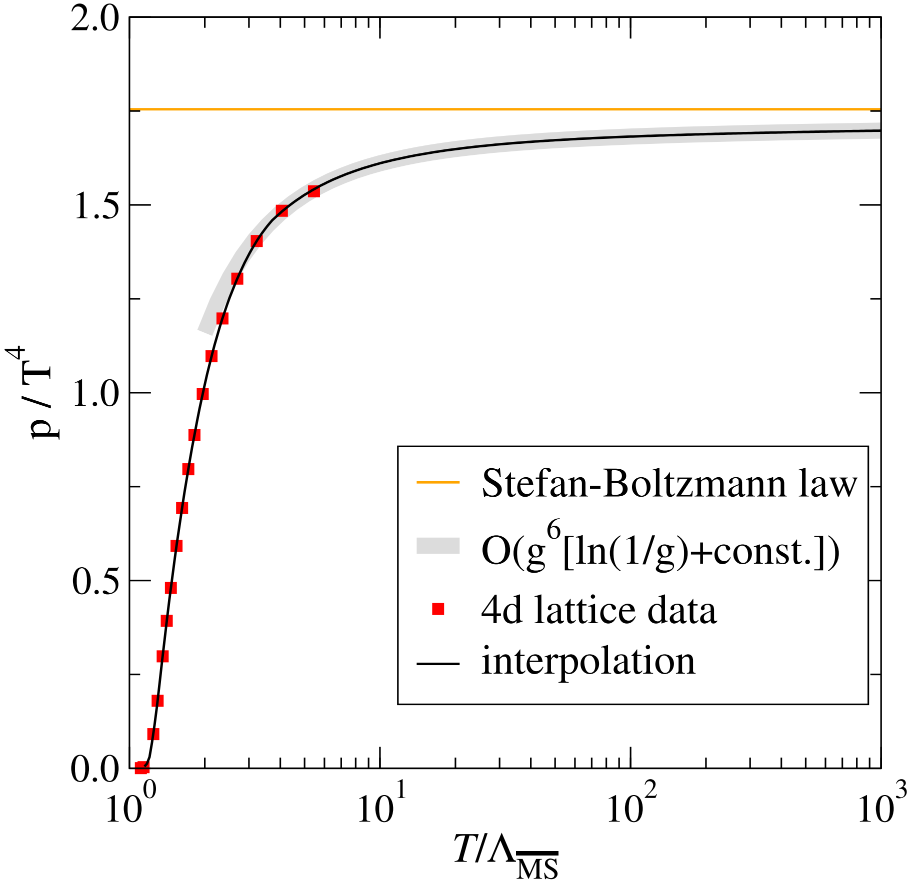
- this could be coined the physical leading-order (!) approximation
- collect contributions to $p(T)$ from all physical scales
  - ▷ weak coupling, effective field theory setup
  - ▷ faithfully adding up all Feynman diagrams
  - ▷ get long-distance input from clean lattice observable:

$$p_{\rm G}(T) \equiv \frac{T}{V}\ln\int \mathcal{D}[A_k^a]\exp\Big(-S_{\rm M}\Big) = T\#\, g_{\rm M}^6$$

only one non-perturbative (but computable!) coeff needed: 5×10^{16} flops

Estimating $p(T, N_f{=}0)$ at LO

while working on the open problems at LO ...



- fix unknown perturbative $\mathcal{O}(g^6)$ coeff
- match to lattice data [Boyd et al. 96]
  at intermediate T $\sim$ 3-5T_c
  translate via $T_c/\Lambda_{\overline{\rm MS}} \approx 1.20$
- precision on $\mathcal{O}(g^6)$ coeff?
  data to 1000T_c
  [Wuppertal group 12; LAT07; QHPD09])

$p(T)$ beyond LO: $g^6 \to g^7 \to g^8$

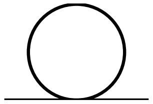
$$\begin{aligned}
\frac{p_\mathrm{E}}{p_\mathrm{SB}} &= \#_{(0)} + \#_{(2)}g^2 + \#_{(4)}g^4 + \#_{(6)}g^6 + [\text{4d 5loop 0pt}]_{(8)} + \dots_{(10)} \\
g_\mathrm{E}^2 &= T\left[g^2 + \#_{(6)}g^4 + \#_{(8)}g^6 + \#_{(10)}g^8 + \dots_{(12)}\right] \\
\lambda_\mathrm{E} &= T\left[\#_{(6)}g^4 + \#_{(8)}g^6 + \dots_{(10)}\right] \\
m_\mathrm{E}^2 &= T^2\left[\#_{(3)}g^2 + \#_{(5)}g^4 + [\text{4d 3loop 2pt}]_{(7)} + \dots_{(9)}\right] \\
\frac{p_\mathrm{M}}{p_\mathrm{SB}} &= \frac{m_\mathrm{E}^3}{T^3}\Bigg[\#_{(3)} + \frac{g_\mathrm{E}^2}{m_\mathrm{E}}\left(\#_{(4)} + \#_{(6)}\frac{\lambda_\mathrm{E}}{g_\mathrm{E}^2}\right) + \left(\frac{g_\mathrm{E}^2}{m_\mathrm{E}}\right)^2\left(\#_{(5)} + \#_{(7)}\frac{\lambda_\mathrm{E}}{g_\mathrm{E}^2} + \#_{(9)}\left(\frac{\lambda_\mathrm{E}}{g_\mathrm{E}^2}\right)^2\right) \\
&\quad + \left(\frac{g_\mathrm{E}^2}{m_\mathrm{E}}\right)^3\left(\#_{(6)} + \#_{(8)}\frac{\lambda_\mathrm{E}}{g_\mathrm{E}^2} + \#_{(10)}\left(\frac{\lambda_\mathrm{E}}{g_\mathrm{E}^2}\right)^2 + \#_{(12)}\left(\frac{\lambda_\mathrm{E}}{g_\mathrm{E}^2}\right)^3\right) \\
&\quad + [\text{3d 5loop 0pt}]_{(7)} + [\delta\mathcal{L}_\mathrm{E}]_{(7)} + [\text{3d 6loop 0pt}]_{(8)} + \dots_{(9)}\Bigg] \\
g_\mathrm{M}^2 &= g_\mathrm{E}^2\left[1 + \#_{(7)}\frac{g_\mathrm{E}^2}{m_\mathrm{E}} + \left(\frac{g_\mathrm{E}^2}{m_\mathrm{E}}\right)^2\left(\#_{(8)} + \#_{(10)}\frac{\lambda_\mathrm{E}}{g_\mathrm{E}^2}\right) + \dots_{(9)}\right] \\
\frac{p_\mathrm{G}}{p_\mathrm{SB}} &= \#_{(6)}\left(\frac{g_\mathrm{M}^2}{T}\right)^3 + [\delta\mathcal{L}_\mathrm{M}]_{(9)}
\end{aligned}$$

notation: $\#_{(n)}$ enters p_QCD at g^n

[cave: no $\frac{1}{\epsilon} + 1 + \epsilon$, no IR/UV, and no logs shown above]

Brief remarks: ultrasoft contributions

needs lattice perturbation theory



$$= \int_{-\pi}^{\pi} \frac{d^3\hat{k}}{(2\pi)^3} \frac{1}{\sum_{i=1}^{3} 4\sin^2(\hat{k}_i/2) + \hat{m}^2} = \sum_{n\geq 0} \hat{m}^{2n} \left(\{\Sigma, \xi\} + \{1\}\hat{m}\right)$$

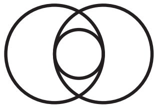
- 1loop tadpole contains elliptic integral in 3d [G.N. Watson 1939]
  - $\Sigma = 4\pi G(0) = \frac{8}{\pi}(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6})\, K^2[(2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2]$
  - later reduced to $\Sigma = \frac{\sqrt{3}-1}{48\pi^2}\,\Gamma^2(\frac{1}{24})\,\Gamma^2(\frac{11}{24})$ [Glasser, Zucker 1977; thanx to D. Broadhurst]

- open problem: classification? very little is known systematically.

- in practice: (4-loop) Numerical Stochastic Perturbation Theory [with F. Di Renzo, 04-06]
  - no diagrams! But at fixed $N_c = 3$ only (4×10^{17} flops) $\Rightarrow$ generalization?!

Brief remarks: soft contributions

for 'NLO', need

- 5-loop massive tadpoles (in 3d)
  - ▷ work in progress
  - ▷ e.g.  $/ J_1^5 = -0.518821725792769087 68 + 11.60369403761691358 9\,\epsilon + \ldots$
- higher-order operators in EFT
  - ▷ classified up to order-6 [S. Chapman, 94]

$$\frac{\delta p_{\rm QCD}(T)}{T} \sim \delta\mathcal{L}_{\rm E} \sim g^2 \frac{D_k D_l}{(2\pi T)^2}\mathcal{L}_{\rm E} \sim g^2 \frac{(gT)^2}{(2\pi T)^2}(gT)^3 \sim g^7 T^3$$

  - ▷ calculation simple: low loop orders

Debye mass: Disclaimer

- Debye mass defined as (inverse) screening length
  - ▷ via long-distance falloff of electric gluon propagator
  - ▷ Abelian plasma: screening of E; unscreened B
  - ▷ Abelian intuition fails in QCD; not a gauge-invariant concept
- gauge-invariant definition by [Arnold/Yaffe 95]
  - ▷ most easily formulated in 3d effective theory
  - ▷ classify (color-) electric/magnetic operators as odd/even under Euclidean time reflection ($A_0 \to -A_0$; CT in 4d)
- determine behavior of pairs of local gauge-invariant operators
  - ▷ can determine many different correlation lengths
  - ▷ e.g. electric operators $\mathrm{Tr}\{A_0 F_{12},\ A_0^3\}$ (4d: $\mathrm{Im\,Tr}\{P F_{12},\ P\}$)
  - ▷ e.g. magnetic operators $\mathrm{Tr}\{A_0^2\}$ (4d: $\mathrm{Re\,Tr}P$)
  - ▷ lightest electric one ≡ Debye mass, $M \approx m_\mathrm{E} + \frac{g_\mathrm{E}^2 N_c}{4\pi} \ln(C\, m_\mathrm{E}/g_\mathrm{E}^2)$
  - ▷ non-pert contributions from NLO [Rebhan 93], via 3d LAT [e.g. Laine/Philipsen 99]
- here, focus on perturbative part of Debye mass, m_E^2

Recipe to evaluate m_E^2

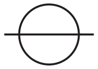
- find location of pole in static A_0 propagator
- 4d QCD: $0 = P^2 + \Pi_{00}(P)$ taken at $P_0 = 0$ and $|p| = im$
  - ▷ perturbatively, $\Pi_{00}(P) = g^2\Pi_1(P) + g^4\Pi_2(P) + \ldots$
  - ▷ so $m \sim g$ small. hence $p^2 \sim g^2$ small
  - ▷ Taylor expand! $\Pi_n(P) = \Pi_n(0) + p^2\Pi_n'(0) + \ldots$
  - ▷ iterate this double expansion

$$0 = -m^2 + g^2\Pi_1 + g^4[\Pi_2 - \Pi_1'\Pi_1] + g^6[\Pi_3 - \Pi_1'\Pi_2 - \Pi_2'\Pi_1 + \Pi_2''(\Pi_1)^2 + (\Pi_1')^2\Pi_1]$$

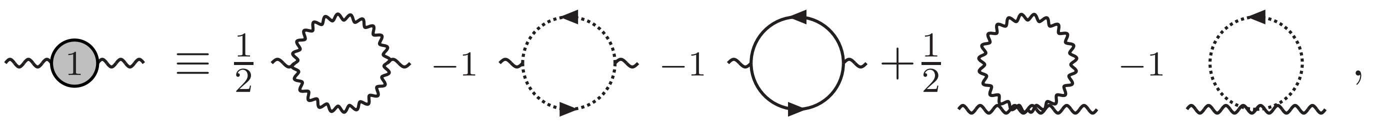
- ▷ all $\Pi = \Pi(0)$ $\Rightarrow$ need (up to) 3-loop vacuum sum-integrals

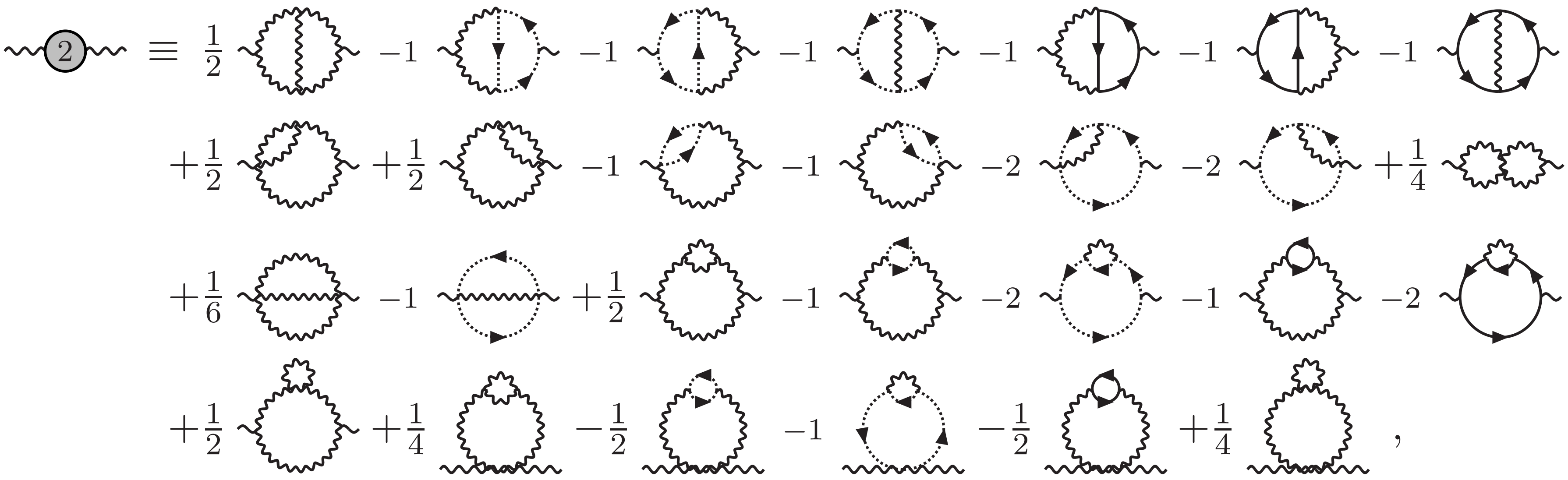
- 3d EQCD: $0 = p^2 + m_E^2 + \Pi_{EQCD}(p)$ taken at $|p| = im$
  - ▷ again double expansion ($m_E \sim m \sim g$ small)
  - ▷ but now all $\Pi_{EQCD}^{(n)}(0) = 0$ (no scale T)

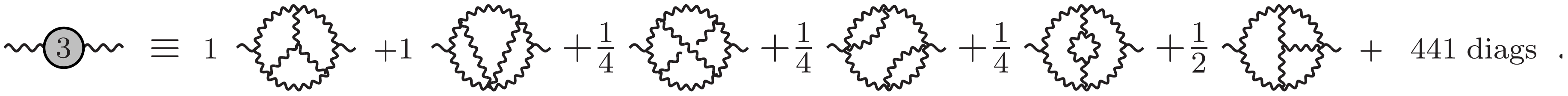
$$0 = -m^2 + m_E^2$$

- ▷ renormalization: $m_{E,R} = m_E^2 - \delta m_E^2$ (since  $\sim \frac{1}{\epsilon}$ in 3d)

Recipe to evaluate m_E^2







Details on 3-loop m_{E}^2

(a) organize the computation

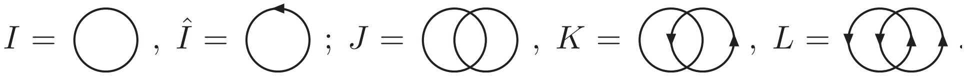
$\sim$ 450 diagrams

$\Rightarrow$ computer-algebra: diagram generation; color traces; Lorentz algebra
well-developed automatized methods; `QGRAF` `FORM`

$\sim 10^7$ sum-integrals of type

$\Rightarrow$ systematic integration by parts (IBP); Laporta algorithm

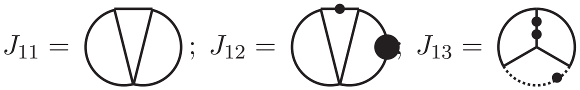
$\sim 10^2$ master sum-integrals of type



of which $\sim 10^1$ bosonic
however with divergent pre-factors

$\Rightarrow$ basis transformation via reverse IBP table lookup

$=$ 3 non-trivial bosonic master sum-integrals



(b) obtain (gauge-parameter independent!) bare result

$$
\begin{aligned}
m_{\rm E}^2 &= g^2 N_c (d-1)^2 I_1 \Big\{ 1 + g^2 N_c \frac{46 - 11d + d^2}{6} I_2 + \\
&+ g^4 N_c^2 \Big(-\frac{d-3}{4} \Big[(7d-13) J_{11}/I_1 + 32(d-4) J_{12}/I_1 + 2(d-7) J_{13}/I_1 \Big] + \\
&+ \frac{1}{6d(d-7)} \Big[\frac{p_1(d)}{5} I_3 I_1 + \frac{p_2(d)}{6(d-5)(d-2)} I_2 I_2 \Big] \Big) + \mathcal{O}(g^6) \Big\}
\end{aligned}
$$

Details on 3-loop m_E^2

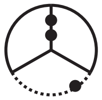
(c) expand in ϵ and renormalize

master sum-ints are complicated beasts

$\Rightarrow$ invest $\mathcal{O}(1)$ PhD year [Ioan Ghişoiu]

beautiful new methods, e.g. Tarasov at T

simple results, e.g.

$J_{13} =$  $= I_1 \frac{1}{(4\pi)^4} \left(\frac{e^\gamma}{4\pi T^2}\right)^{2\epsilon} \left(-\frac{5}{3\epsilon^2} - \frac{11}{18\epsilon} + \text{num} + \mathcal{O}(\epsilon)\right)$

renormalization is standard

$$\Rightarrow g_b^2 = \mu^{2\epsilon} g_{\rm R}^2(\bar\mu)\, Z_g \quad \text{where} \quad Z_g = 1 + \frac{g_{\rm R}^2(\bar\mu)}{(4\pi)^2}\frac{\beta_0}{2\epsilon} + \frac{g_{\rm R}^4(\bar\mu)}{(4\pi)^4}\left[\frac{\beta_1}{4\epsilon} + \frac{\beta_0^2}{4\epsilon^2}\right] + \mathcal{O}(g_{\rm R}^6)$$

$$\beta_0 = -\frac{22}{3} N_c\ , \ \beta_1 = -\frac{68}{3} N_c^2\ ; \qquad \frac{\delta m_{\rm E}^2}{(4\pi T)^2} = -\frac{10}{3\epsilon}\frac{g_{\rm R}^6 N_c^3}{(4\pi)^6}$$

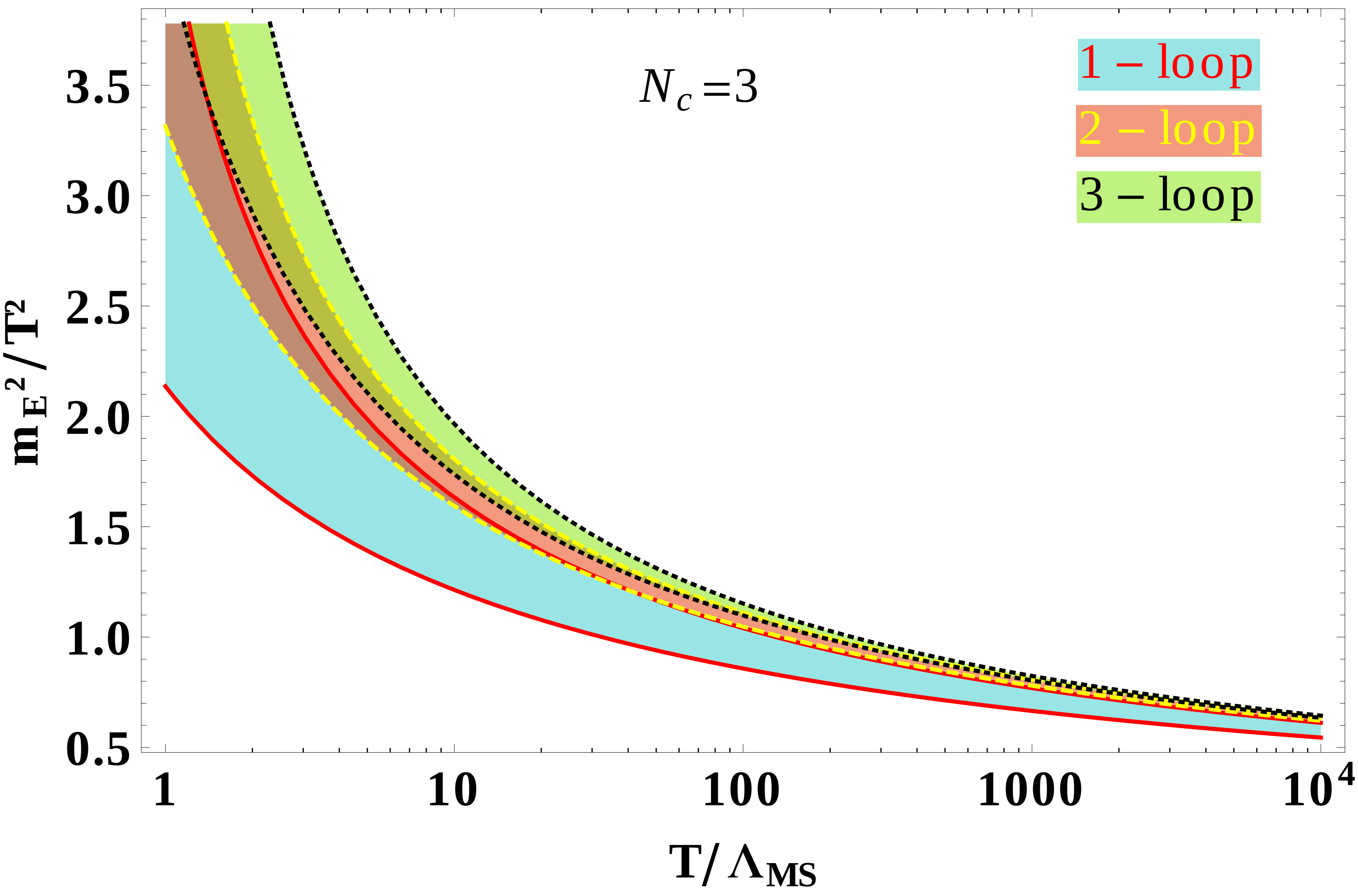
work in $\overline{\rm MS}$ scheme, use 3-loop running

(d) obtain renormalized result

$$\frac{m_{\rm E,R}^2(\bar\mu)}{(4\pi T)^2} = \frac{g_{\rm R}^2(\bar\mu)}{(4\pi)^2}\frac{N_c}{3}\left\{1 + \frac{g_{\rm R}^2(\bar\mu)}{(4\pi)^2}\frac{N_c}{3}\left(22\ln\frac{\bar\mu e^\gamma}{4\pi T} + 5\right)\right.$$

$$\left. + \left(\frac{g_{\rm R}^2(\bar\mu)}{(4\pi)^2}\frac{N_c}{3}\right)^2 \left(484\ln^2\frac{\bar\mu e^\gamma}{4\pi T} - 116\ln\frac{\bar\mu e^\gamma}{4\pi T} + \frac{1091}{2} + 180\gamma_{\rm E} - 180\frac{\zeta'(-1)}{\zeta(-1)} - \frac{56}{5}\zeta(3)\right) + \mathcal{O}(g_{\rm R}^6)\right\}$$

3-loop result for m_E^2



here, $N_c = 3$; $\Lambda_{\rm MS} \approx 200$MeV; bands from $\bar{\mu} = (0.5 \ldots 2)2\pi T$

Summary

- thermodynamic quantities of QCD are relevant for cosmology and heavy ion collisons

- these quantities can be determined
  - ▷ numerically at $T \sim 200$ MeV; analytically at $T \gg 200$ MeV
  - ▷ multi-loop sports, eff. theories convenient → systematic improvement

- 3d effective field theory opens up tremendous opportunities
  - ▷ analytic treatment of fermions (cf. LAT problems!)
  - ▷ universality, superrenormalizabilty
  - ▷ ideal playground for multi-loop methods

- much activity in determination of matching coeffs
  - ▷ $T = 0$: 4-loop lattice perturbation theory
  - ▷ $T = 0$: 5-loop massive tadpoles
  - ▷ $T \neq 0$: moments of 3-loop on-shell propagators
  - ▷ $T \neq 0$: 4-loop tadpoles