Transverse PDFs at NNLO

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Outline

1. Considered problem: Resummation of $\log \frac{q_T}{M}$
2. Approach: TPDFs
3. Contribution: Calculation to NNLO in α_s

q_T spectrum, $p\bar{p} \to Z + R$

- Consider production of massive, color neutral final state at hadron collider,
- inv. mass M, transverse momentum q_T.
- E.g. $p + \bar{p} \to Z + X$.

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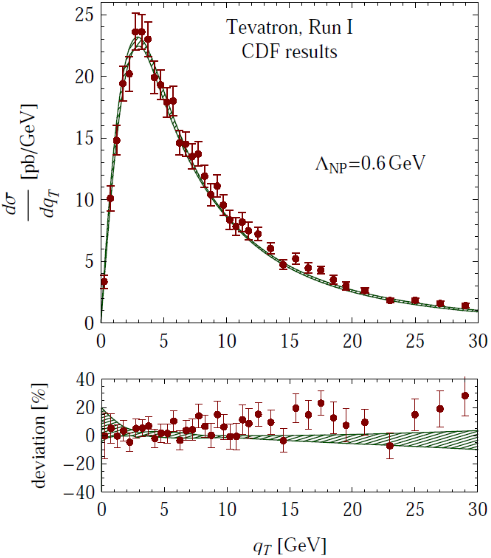
- Consider production of massive, color neutral final state at hadron collider,
- inv. mass M, transverse momentum q_T.
- E.g. $p + \bar{p} \to Z + X$.
- Focus of talk: $q_T^2 \ll M^2$.

$\Rightarrow$ Need to resum $\log \frac{q_T}{M}$

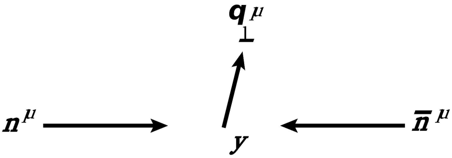
resummed result

[Becher, Neubert, Wilhelm]



Notation

Directions of incoming hadrons and outgoing vector boson:



- With $n^2, \bar{n}^2 = 0,\ n \cdot \bar{n} = 2; \qquad -q_\perp^2 = q_T^2 > 0.$
- Decompose $v^\mu = (\bar{n}v)\frac{n^\mu}{2} + (nv)\frac{\bar{n}^\mu}{2} + v_\perp^\mu\,.$

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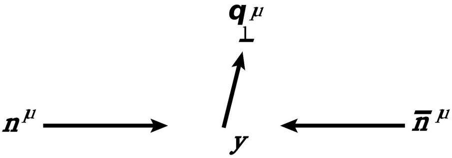

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- Decompose $v^\mu = (\bar{n}v)\frac{n^\mu}{2} + (nv)\frac{\bar{n}^\mu}{2} + v_\perp^\mu$.
- Expansion parameter $\lambda = \frac{q_T}{M}$.
- Regions:

collinear (c):	p_c	$= (\bar{n}p, np, p_\perp) \sim M(1, \lambda^2, \lambda)$
anti-collinear ($\bar{c}$):	$p_{\bar{c}}$	$\sim M(\lambda^2, 1, \lambda)$
soft (s):	p_s	$\sim M(\lambda, \lambda, \lambda)$

- SCET, for each region, fields with corresponding momentum scaling.

Factorized differential cross section

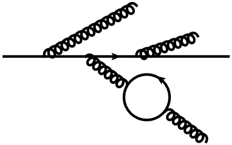
For $q_T \ll M$, $d\sigma$ can be factorized:

$$\frac{d^2\sigma_V}{dq_T^2\,dy} - \mathcal{O}\left(\frac{q_T^2}{M^2}\right) = \frac{\pi\alpha_{ew}^2}{N_c s}\left|C_V(-M^2,\mu)\right|^2 \int d^2x_\perp\, e^{-iq_\perp\cdot x_\perp} \sum_q g_q^2$$
$$\times \left[\mathcal{S}(x_\perp,\mu)\mathcal{B}_{q/N_1}(z_1, x_\perp, \mu)\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2, x_\perp, \mu) + (q \leftrightarrow \bar{q})\right]_{M^2},$$

- $C_V(-M^2,\mu)$ hard function.
- $\mathcal{S}(x_\perp,\mu)$ soft function.
- $\mathcal{B}_{q/N_1}(z_1, x_\perp, \mu)$ transverse PDF, collinear.
- $\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2, x_\perp, \mu)$ TPDF, anti-collinear.
- For last three operator definitions via SCET [Becher, Neubert].
- $z_{1,2} = \sqrt{\tau}e^{\pm y}$, $\tau = (M^2 + q_T^2)/s$.

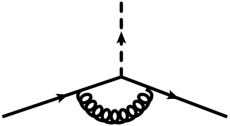
Factorized differential cross section, pictorial

$\mathcal{B}_{q/N_1}(z_1, x_\perp, \mu)$



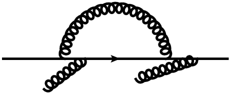
'transverse PDF'

$C_V(-M^2, \mu)$



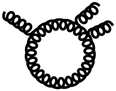
'hard function'

$\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2, x_\perp, \mu)$



'transverse PDF'

$\mathcal{S}(x_\perp, \mu)$



'soft function'

$n^\mu \longrightarrow$

c

$\longleftarrow \bar{n}^\mu$

$\bar{c}$

Transverse PDF

- Collinear (c) quark and gluon fields: $\psi_c(x)$, $A_c^\mu(x)$.
- Wilson line $W_c(x) = P\exp\left[ig_s\int_{-\infty}^0 \mathrm{d}s\, \bar{n}\cdot A_c(x+s\bar{n})\right]$
- Gauge invariant combination $\chi(x) = \left(W_c^\dagger \psi_c\right)(x)$.
- TPDF (c):

$$\mathcal{B}_{q/N}(z, x_\perp, \mu) = \frac{1}{2\pi}\int \mathrm{d}t\; e^{-izt\bar{n}\cdot p} \sum_{X_c} \frac{\bar{\not{n}}_{\alpha\beta}}{2} \langle N(p)|\, \bar{\chi}_\alpha(t\bar{n} + x_\perp)\, |X_c\rangle \langle X_c|\, \chi_\beta(0)\, |N(p)\rangle\,.$$

- In contrast to normal collinear PDF:

$$\phi_{q/N}(z, \mu) = \frac{1}{2\pi}\int \mathrm{d}t\; e^{-izt\bar{n}\cdot p} \sum_{X_c} \frac{\bar{\not{n}}_{\alpha\beta}}{2} \langle N(p)|\, \bar{\chi}_\alpha(t\bar{n})\, |X_c\rangle \langle X_c|\, \chi_\beta(0)\, |N(p)\rangle\,.$$

- Anti-collinear $\bar{\mathcal{B}}_{\bar{q}/N_2}$ correspondingly with $n \leftrightarrow \bar{n}$, $q \leftrightarrow \bar{q}$.

Gluons

For processes initiated by gluon-gluon-fusion, analogous factorization formula holds with [Becher,Neubert,Wilhelm]:

$$\mathcal{B}^{\mu\nu}_{g/N}(z, x_\perp, \mu) =$$
$$-\frac{z\bar{n}\cdot p}{2\pi}\int \mathrm{d}t\; e^{-izt\bar{n}\cdot p}\sum_{X_c} \langle N(p)|\, \mathcal{A}^{\mu,a}_{c\perp}(t\bar{n}+x_\perp)\,|X_c\rangle\,\langle X_c|\,\mathcal{A}^{\nu,a}_{c\perp}(0)\,|N(p)\rangle$$
$$= \frac{g^{\mu\nu}_\perp}{d-2}\,\mathcal{B}_{g/N}(z, x_\perp, \mu) + \left(\frac{g^{\mu\nu}_\perp}{d-2} - \frac{x^\mu_\perp x^\nu_\perp}{x^2_\perp}\right)\mathcal{B}'_{g/N}(z, x_\perp, \mu)\,,$$

gauge invariant

$$\mathcal{A}^\mu_{c\perp}(x) = \frac{1}{g_s}\left[W^\dagger_c(iD^\mu_{c\perp}W_c)\right](x)$$

constructed from c gluon fields.

Projectors to $\mathcal{B}$ and $\mathcal{B}'$ proportional to extracted tensors.

Regularization

- For $\mathcal{B}_{i/N}$ replace hadron N by parton k
  $\Rightarrow$ perturbatively calculable $\mathcal{B}_{ik}$.
- Naive definition of individual TPDFs problematic: Rapidity singularities.
  $\Rightarrow$ Need regulator in addition to dim. reg..

Regularization

- For $\mathcal{B}_{i/N}$ replace hadron N by parton k
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  $\Rightarrow$ Need regulator in addition to dim. reg..
- We chose analytic regulator α [Becher, Bell],

for each emitted parton:

$$\int d^d k\, \delta^+(k^2) \to \int d^d k\, \delta^+(k^2) \left(\frac{\nu_+}{n \cdot k}\right)^{\alpha}$$

- Mass independent.
  No new denominators.
  $\mathcal{S}(x_\perp, \mu) = 1$.
  Can expand in α (for NNLO only need up to α^0).

Collinear anomaly

- Individual $\mathcal{B}$, $\bar{\mathcal{B}}$ contain poles in α.
- Poles cancel in $\mathcal{B}\bar{\mathcal{B}}$.
- Obtain dependence on M^2.
- Due to broken rescaling invariance.
- [Becher,Neubert]: 'Collinear anomaly'.

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Refactorize $\mathcal{B}\bar{\mathcal{B}}$:

$$\left[\mathcal{B}_{i/k}(z_1, x_\perp, \mu)\bar{\mathcal{B}}_{\bar{i}/j}(z_2, x_\perp, \mu)\right]_{M^2} = \left(\frac{x_T^2 M^2}{4e^{-2\gamma_e}}\right)^{-F_{i\bar{i}}(x_\perp,\mu)} \times B_{i/k}(z_1, x_\perp, \mu) B_{\bar{i}/j}(z_2, x_\perp, \mu)\,.$$

- Defines TPDFs B, independent of hard scale M^2. F parametrizes M^2 dependence.

Matching kernel

- Perturbatively calculate $B_{i/k}$ and $\phi_{j/k}$.
- For $x_T^{-2} \gg \Lambda^2_{\rm QCD}$, have relation

Matching kernel $I_{i\leftarrow j}$

$$B_{i/k}(\rho, x_\perp, \mu) = \sum_j \int_\rho^1 \frac{dz}{z} I_{i/j}(z, x_\perp, \mu) \phi_{j/k}(\rho/z, \mu) + \mathcal{O}(\Lambda^2 x_T^2)$$

$= \sum_j \left(I_{i/j} \otimes \phi_{j/k} \right)_\rho$, in short notation.

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- Extract $I_{i/j}$ from calculation of $B_{i/k}$ and $\phi_{k/j}$.
- Renormalization: $I^{(b)}_{i/j}(z) = Z^B_i \sum_k \left(I^{(r)}_{i/k} \otimes (Z^\phi)^{-1}_{k/j}\right)_z$

$\Rightarrow$ Obtain $I^{(r)}_{i/j}$, Z^B_i and $\phi^{(r)}_{i/j} \Rightarrow P_{i/j}$.

- Last two from UV/IR poles of $I^{(b)}_{i/j}$.

Overview: Perturbative determination

- Aim: $d\sigma/dq_T$ at N^{2}LO+N^{3}LL for $q_T \ll M$.
- Calculate $\mathcal{B}_{i/j}$ and $\phi_{i/j}$ to N^{2}LO regulated by α & ϵ.
- $\mathcal{B}_{i/j}\bar{\mathcal{B}}_{\bar{i}/k} = (\ldots M^2)^{-F_{i\bar{i}}} B_{i/j} B_{\bar{i}/k}$. $\alpha \to 0$.
- Obtain $I_{i/j}$ from $B_{i/k} = \sum_j I_{i/j} \otimes \phi_{j/k}$.
- Renormalize $I_{i/j}$ and $F_{i\bar{i}}$. $\epsilon \to 0$.
- With $I_{i/j}^{(2)}$ obtained new generic ingredients for N^{3}LL resummation.

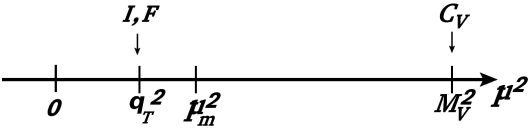
RG evolution & application

§ In factorized result each function depends only on its typical mass M_{typical} and μ.

→ Consistently determine each function in fixed order perturbation theory at $\mu \sim M_{\text{typical}}$.

→ Subsequently evolve to common scale μ_m with RG evol. eqns.

⇒ Resums logarithms.

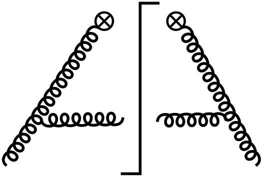


Perturbative calculation

Expansion to NNLO: $$f = \sum_{n=0,1,2} \left(\frac{\alpha_s}{4\pi}\right)^n f^{(n)} + \mathcal{O}(\alpha_s^3)$$

LO trivial: $$\mathcal{B}^{(0)}_{i/j}(z),\ \phi^{(0)}_{i/j}(z),\ I^{(0)}_{i/j}(z) = \delta_{ij}\delta(1-z)$$

NLO: example diagram



General gauge: Wilson line W
$\Rightarrow$ any number of gluons can couple to $\otimes$.
Choose: LCG $\Rightarrow$ $W = 1$.

$$\frac{\alpha_s}{4\pi}\left(\mathcal{B}^{(1)}_{i/j}, \phi^{(1)}_{i/j}\right) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}(2\pi)\delta^+(k^2)\left(\frac{\nu_+}{n\cdot k}\right)^\alpha \delta\big(\bar{n}\cdot[k-(1-z)p]\big)\left(e^{ik_T\cdot x_T}, 1\right)\mathcal{M}$$

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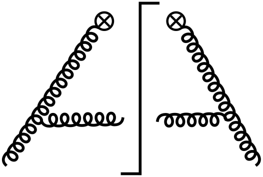

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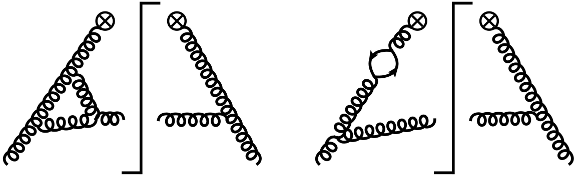
Solve in closed form. Extract I. With $L_\perp = \log\left(\frac{x_T^2\mu^2}{4e^{-2\gamma_E}}\right)$ and $P^{(0)}_{gg}(z) = \left[1/(1-z)\right]_+ + 1/z - 2 + z - z^2$ have e.g. :

$$I^{(1)}_{q/q}(z, L_\perp) = C_a\Big[-4L_\perp P^{(0)}_{gg}(z) - \delta(1-z)(\zeta_2 - L_\perp^2)\Big] .$$

Example diagrams at NNLO

Virtual-Real contribution:



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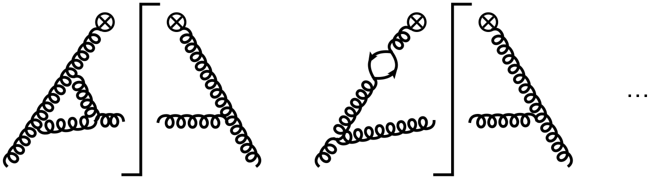
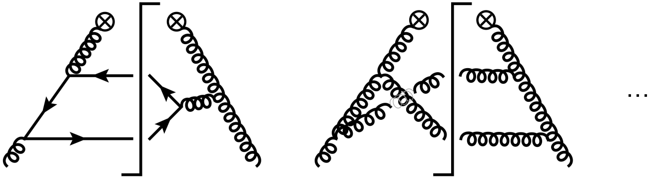


Real-Real contribution:



Hard real-real integrals as non standard:

- n & $\bar{n}$ in denominators and δ-fct.
- $\bar{n}$ through LCG or Wilson line.
- Frame dependent $e^{ik_T \cdot x_T}$.
- Two regulators.
- Non-integer denominators.

Example integral

After parametrization and some work a typical class of integrals for double real emission reads:

$$\sim \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}t \, y^{1-a_2-a_3-a_5-\epsilon}(1-y)^{1-a_1-a_2-a_6-\epsilon}\Big\{1-(1-z)(1-y)\Big\}^{-a_7}$$
$$\times (1-t)^{1-2a_2-2\epsilon}{}_2F_1(1-a_2-\epsilon, 1-a_2-2\epsilon; 1-\epsilon; t)\Big[t^{-a_3-\epsilon}\Big\{1-y(1-t)\Big\}^b$$
$$+(1-t)^{-a_1-\epsilon}\Big\{1-(1-y)(1-t)\Big\}^b\Big], \; b=-2+a_1+a_2+a_3+2\epsilon\,.$$

exponent	a_1	a_3	a_5	a_6
integer part	n_1	n_3	n_5	n_6
regulator part	α	α	$-\epsilon - m\alpha$	ϵ ,

$m = 2, 0$

NNLO results

- Determined $I_{i/j}$, $\phi_{i/j}$, $F_{i\bar{i}}$, Z_i^B up to α_s^2 for all $i,j \in \{g, q_f, q_{f'}, \bar{q}_f, \bar{q}_{f'}\}$.
- Published so far $q \leftarrow q$, [Gehrmann,TL,Yang] 1209.0682.
- $I^{(2)}_{g/g}(z, L_\perp = 0)$ on next slide.
- $L_\perp = \log\left(\frac{x_T^2 \mu^2}{4e^{-2\gamma_E}}\right)$ dependence implied by RG evol. eq.:

$$\frac{d}{d\log\mu} I_{g/g}(z, L_\perp, \alpha_s) = \left[\Gamma^A_{\text{cusp}}(\alpha_s) L_\perp - 2\gamma^g(\alpha_s)\right] I_{g/g}(z, L_\perp, \alpha_s) - \sum_k 2\left(I_{g/k} \otimes P_{kg}\right)_z .$$

Example: Scale independent part of $I^{(2)}_{g/g}$

$$
\begin{aligned}
I^{(2)}_{g/g}(z,0) = & C_a^2\Bigg[\delta(1-z)\left(\frac{1214}{81}-\frac{67}{6}\zeta_2-\frac{77}{9}\zeta_3+\frac{25}{4}\zeta_4\right) \\
& + P^{(0)}_{gg}(z)\left(8H_{0,1,0}+8H_{0,1,1}-8H_{1,0,0}+8H_{1,0,1}+8H_{1,1,0}-4H_{0,0,0}+52\zeta_3-\frac{808}{27}\right) \\
& + P^{(0)}_{gg}(-z)\Big(8H_{-1,0,0}+16H_{0,-1,0}-16H_{-1,-1,0}-4H_{0,0,0}-8H_{0,1,0}-8\zeta_2 H_{-1}+4\zeta_3\Big) \\
& - 16(1+z)H_{0,0,0}+\frac{50-22z+88z^2}{3}H_{0,0}-\frac{701+149z+536z^2}{9}H_0-\frac{2z}{3}H_1 \\
& + \left(-32[1-z]+\frac{88}{3}\left[1/z-z^2\right]\right)(H_{1,0}+\zeta_2)+\frac{232}{3}-\frac{784}{9z}-\frac{248}{3}z+\frac{844}{9}z^2\Bigg] \\
& + C_a T_f N_f\Bigg[\delta(1-z)\left(\frac{28}{9}\zeta_3+\frac{10}{3}\zeta_2-\frac{328}{81}\right)+\frac{224}{27}P^{(0)}_{gg}(z) \\
& + \left(\frac{8}{3}(1+z)H_{0,0}+\frac{4}{3}zH_1+\frac{52+40z}{9}H_0-8(1-z)+\frac{260}{27z}-\frac{332}{27}z^2\right)\Bigg] \\
& + C_f T_f N_f\left[8(1+z)H_{0,0,0}+(12+4z)H_{0,0}+24(1+z)H_0+64(1-z)-\frac{8}{3}(1/z-z^2)\right],
\end{aligned}
$$

where $P^{(0)}_{gg}(z) = \left[1/(1-z)\right]_+ + 1/z - 2 + z - z^2$,

and HPL $H_{\{m\}} \equiv H(\{m\},z)$.

Checks

- $H(\{n\}, z)$ up to weight 3.
- α^{-n} cancel in $\mathcal{B}\bar{\mathcal{B}}$.
- Consistent renormalization with same Z_i^B, $(Z_{i/j}^{\phi})^{-1}$ for all functions.
- $P_{i/j} \Leftarrow \phi_{i/j} \Leftarrow (Z_{i/j}^{\phi})^{-1}$ agree with literature.
- $I_{i/j}(z, L_\perp, \alpha_s)$ obey RG evol. eq.:

$$\frac{d}{d\log\mu} I_{i/j}(z, L_\perp, \alpha_s) = \left[\Gamma^i_{\text{cusp}}(\alpha_s) L_\perp - 2\gamma^i(\alpha_s)\right] I_{i/j}(z, L_\perp, \alpha_s) - \sum_k 2\left(I_{i/k} \otimes P_{k/j}\right)_z$$

- $F_{q\bar{q}}$, F_{gg} agree with [Becher,Neubert].
- Using our results combined with known C_V, C_t and C_H, confirmed $\mathcal{H}^{DY(2)}_{q\bar{q}\leftarrow i\bar{j}}$ and $\mathcal{H}^{H(2)}_{gg\leftarrow i\bar{j}}$ in [Catani,Cieri,de Florian,Ferrera,Grazzini].

Note on $I'_{g/j}$

- Recall: $I^{\mu\nu}_{g/j} = \frac{g^{\mu\nu}_{\perp}}{d-2}\, I_{g/j} + \left(\frac{g^{\mu\nu}_{\perp}}{d-2} - \frac{x^{\mu}_{\perp} x^{\nu}_{\perp}}{x^2_{\perp}}\right) I'_{g/j}\,.$
- Discussion above for $I_{g/j} = g_{\perp\mu\nu} I^{\mu\nu}_{g/j}$.
- Similar for 2nd function $I'_{g/j}$.
- Found: $I'^{(0)}_{g/j} = 0$, $I'^{(1)}_{g/j}$ agrees with [Becher,Bell,Neubert].
- Sufficient for $d\sigma_H/dq_T$ at N^{2}L0+N^{3}LL.
- General case, need $I'^{(2)}_{g/j}$.
- Work in progress.

$\rightarrow$ Publication in preparation.

Conclusions

- q_T resummation requires TPDFs.
- Can calculate partonic TPDFs and matching kernels perturbatively.
- Definition & calculation suffer from rapidity singularities.
- NNLO calculation using analytic regularization completed for all $I_{i/j}$.
- Generic ingredients to obtain $d\sigma/dq_T$ at small q_T to N^2LO+N^3LL precision for processes with color neutral final state.

Appendix

RG evolution equations

$$\frac{d}{d\log\mu} C_V(-M^2,\mu) = \left[\Gamma^F_{\rm cusp}(\alpha_s)\log\tfrac{-M^2}{\mu^2} + 2\gamma^q(\alpha_s)\right] C_V(-M^2,\mu)\,,$$

$$\frac{d}{d\log\mu} F_{q\bar q}(x_\perp,\mu) = 2\Gamma^F_{\rm cusp}(\alpha_{s(\mu)})\,,$$

$$\frac{d}{d\log\mu} I_{q\leftarrow q}(z,x_\perp,\mu) = \left[\Gamma^F_{\rm cusp}(\alpha_s)\log\tfrac{x_T^2\mu^2}{4e^{-2\gamma_e}} + 2\gamma^q(\alpha_s)\right] I_{q\leftarrow q}(z,x_\perp,\mu)$$
$$- \sum_k 2\int_z^1 \frac{d\xi}{\xi} I_{q\leftarrow k}(\xi,x_\perp,\mu) P_{k\leftarrow q}(z/\xi,\mu)\,,$$

where $P_{k\leftarrow q}$ are DGLAP kernels from RG evol. eq. of usual PDFs.
Taken together, differential cross section RG invariant.

Rephrased factorization formula

At $\Lambda \ll q_T \ll M$ rephrased factorization formula [Becher, Neubert]:

$$\frac{d^3\sigma}{dM^2 dq_T^2 dy} = \frac{\alpha^2}{3N_c M^2 s} \sum_{i,j=q,\bar{q},g} \sum_q e_q^2 \int_{\xi_1}^1 \frac{dz_1}{z_1} \int_{\xi_2}^1 \frac{dz_2}{z_2} \int d^2x_\perp \, e^{-iq_\perp \cdot x_\perp}$$

$$\times \left[\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_\perp,M^2,\mu)\phi_{i/N_1}(\xi_1/z_1,\mu)\phi_{j/N_2}(\xi_2/z_2,\mu) + (q\leftrightarrow\bar{q}) \right],$$

where

$$\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_\perp,M^2,\mu)$$

$$= \left|C_V(-M^2,\mu)\right|^2 \left(\frac{x_T^2 M^2}{4e^{-2\gamma_e}}\right)^{-F_{q\bar{q}}(x_\perp,\mu)} I_{q\leftarrow i}(z_1,x_\perp,\mu) I_{\bar{q}\leftarrow j}(z_2,x_\perp,\mu) \, .$$

Relation to formula by Collins, Soper, Sterman

$$\widetilde{C}_{q\bar{q}\leftarrow ij}(z_1,z_2,x_\perp,M^2,\mu)$$

$$= \left|C_V(-M^2,\mu)\right|^2 \left(\frac{x_T^2 M^2}{b_0^2}\right)^{-F_{q\bar{q}}(x_\perp,\mu)} I_{q\leftarrow i}(z_1,x_\perp,\mu) I_{\bar{q}\leftarrow j}(z_2,x_\perp,\mu)\,,$$

compare this to [Collins, Soper, Sterman, 1985]:

$$= \exp\left\{ -\int_{\mu_b}^{M^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\log \frac{M^2}{\bar{\mu}^2} A(\alpha_s(\bar{\mu})) + B(\alpha_s(\bar{\mu}))\right]\right\}$$
$$\times\ C_{qi}(z_1, \alpha_s(\mu_b)) C_{\bar{q}j}(z_2, \alpha_s(\mu_b))\,,$$

x_T dependence via $\mu_b = b_0 x_T^{-1}$ $(b_0 = 2e^{-\gamma_e})$.

Relations:

- $C_{qi}(z, \alpha_s(\mu_b)) = \left|C_V(-\mu_b^2, \mu_b)\right| I_{q/i}(z, x_\perp, \mu_b)\,,$
- A & B related to F and RG evol. eq. of F & C_V.

Dictionary CSS vs BN

Using $b_0 = 2e^{-\gamma_e}$, $\mu_b = b_0 x_T^{-1}$ and $\bar{x}_T = b_0 \bar{\mu}^{-1}$:

$$C_{qi}(z, \alpha_{s(\mu_b)}) = \left| C_V(-\mu_b^2, \mu_b) \right| I_{q/i}(z, \bar{x}_\perp, \mu_b)\,,$$

$$A(\alpha_{s(\bar{\mu})}) = \Gamma^F_{\text{cusp}}(\alpha_s) - \bar{\mu}^2 \frac{dF_{q\bar{q}}(\bar{x}_\perp, \bar{\mu})}{d\bar{\mu}^2}\,,$$

$$B(\alpha_{s(\bar{\mu})}) = 2\gamma^q(\alpha_s) + F_{q\bar{q}}(\bar{x}_\perp, \bar{\mu}) - \bar{\mu}^2 \frac{d \log |C_V(-\bar{\mu}^2, \bar{\mu})|^2}{d\bar{\mu}^2}\,,$$

where Γ^F_{cusp} and γ^q appear in RG evol. eq. of C_V, F & I.

NLO results

$$\phi^{(1)}(z,\mu) = 0 \text{ in dim. reg.} \qquad L_\perp = \log\left(\frac{x_T^2\mu^2}{4e^{-2\gamma_E}}\right)$$

$$\mathcal{B}^{(1)}_{i/j}(z,x_\perp,\mu) = e^{(\epsilon+\alpha)L_\perp-(\epsilon+2\alpha)\gamma_E}\,\frac{\Gamma(-\epsilon-\alpha)}{\Gamma(1+\alpha)}\left(\frac{\bar{n}\cdot p}{\mu}\right)^\alpha\left(\frac{\nu}{\mu}\right)^\alpha(1-z)^\alpha f_{i/j}(z,\epsilon)\,,$$

$$\bar{\mathcal{B}}^{(1)}_{\bar{i}/\bar{j}}(z,\perp,\mu) = e^{\epsilon L_\perp-\epsilon\gamma_E}\,\Gamma(-\epsilon)\left(\frac{\mu}{n\cdot\bar{p}}\right)^\alpha\left(\frac{\nu}{\mu}\right)^\alpha(1-z)^{-\alpha}\,f_{i/j}(z,\epsilon)\,,$$

$$f_{q/q}(z,\epsilon) = C_F(1-z)^{-1}\left[4z+2(1-\epsilon)(1-z)^2\right],$$

$$f_{q/g}(z,\epsilon) = 2T_F\left[1-\frac{2}{1-\epsilon}z(1-z)\right],$$

$$f_{g/q}(z,\epsilon) = C_F\left[2\frac{1+(1-z)^2}{z}-2\epsilon z\right],\quad \text{'}g_\perp\text{-part'}\,,$$

$$f_{g/g}(z,\epsilon) = C_A(1-z)^{-1}\left[4\frac{\left(1-z+z^2\right)^2}{z}\right],\quad \text{'}g_\perp\text{-part'}.$$

Another example: Scale independent part of $I^{(2)}_{q/q}$

$$
\begin{aligned}
&I^{(2)}_{q/q}(z,0) = \\
&\delta(1-z)\left[C_F^2\frac{5\zeta_4}{4} + C_F C_A\left(\frac{3032}{81} - \frac{67\zeta_2}{6} - \frac{266\zeta_3}{9} + 5\zeta_4\right) + C_F T_F n_f\left(-\frac{832}{81} + \frac{10\zeta_2}{3} + \frac{28\zeta_3}{9}\right)\right] \\
&+ P^{(0)}_{qq}(z)\left[C_F\left(12\zeta_3 + 4H_0 + \frac{3}{2}H_{0,0} + 4H_{0,1,0} + 2H_{0,1,1} - 2H_{1,0,0} + 4H_{1,0,1} + 4H_{1,1,0}\right)\right. \\
&+ C_A\left(\zeta_3 - \frac{202}{27} - \frac{38}{9}H_0 - \frac{11}{6}H_{0,0} - H_{0,0,0} - 2H_{0,1,0} - 2H_{1,0,1} - 2H_{1,1,0}\right) \\
&\left. + T_F n_f\left(\frac{56}{27} + \frac{10}{9}H_0 + \frac{2}{3}H_{0,0}\right)\right] + P^{(0)}_{qg}(z)C_F\left[-\frac{68}{27} + \frac{4\zeta_2}{3} + \frac{32}{9}H_0 - \frac{4}{3}H_{0,0} + \frac{4}{3}H_{1,0}\right] \\
&+ P^{(0)}_{gq}(z)T_F\left[\frac{86}{27} - \frac{4\zeta_2}{3} - \frac{4}{3}H_{1,0}\right] + C_F^2\left[(2-24z)H_0 + (3+7z)H_{0,0} + 2(1+z)H_{0,0,0}\right. \\
&\left. + 2zH_1 + (1-z)\left(6\zeta_2 - 22 + 4H_{0,1} + 12H_{1,0}\right)\right] + C_F C_A\left[(2+10z)H_0 - 4zH_{0,0} - 2zH_1\right. \\
&\left. + (1-z)\left(\frac{44}{3} - 6\zeta_2 - 4H_{1,0}\right)\right] + C_F T_F\left[\frac{-50+38z}{9} + \frac{20+8z}{9}H_0 + \frac{2-22z}{3}H_{0,0}\right. \\
&\left. + 4(1+z)H_{0,0,0}\right] - \frac{4}{3}C_F T_F n_f(1-z)\,,
\end{aligned}
$$

where $P^{(0)}_{qq}(z) = 2C_F\left[(1+z^2)/(1-z)\right]_+$, $P^{(0)}_{qg}(z) = 2T_F\left[z^2+(1-z)^2\right]$, $P^{(0)}_{gq}(z) = 2C_F\left[1+(1-z)^2\right]/z$,

and $H_{\{m\}} \equiv H(\{m\},z)$.