Dyck Words and Multi-Quark Amplitudes

Tom Melia, CERN theory

RADCOR 2013, Durham

In this talk

Properties of tree-level QCD primitive amplitudes with many quarks.

Melia, Phys.Rev. D to appear (2013), arXiv:1304.7809

Results useful for high multiplicity & unitarity based NLO QCD calculations involving quarks.

A couple of 'new for RADCOR' results which follow from this (new tree-level colour decompositions).

Here & RADCOR 2013 Proceedings

Many jets: the LHC already sees some very high multiplicity jet events

(for quarks have to see past the gluons)

Pure jet production.

b-quark tagging.

Jet substructure to enhance light quark jets.

High pt from valence quark annihilation.

Electroweak boson(s) + jets.

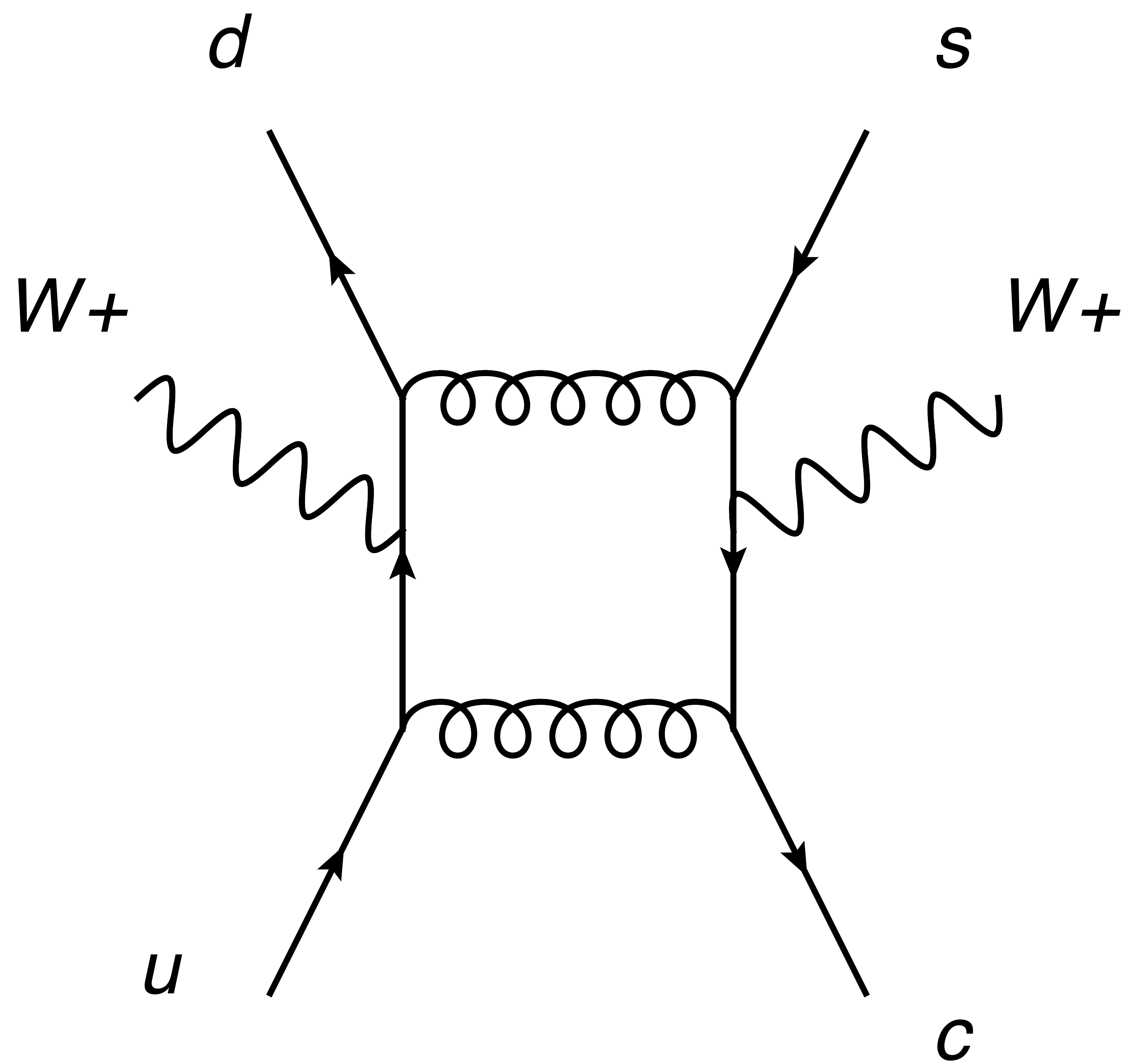
. . .

e.g. pp ⟶ W+W+/- j j

Melia, Melnikov, Rontsch, Zanderighi, 2008, 2010, +Schulze 2012

D-dimensional Generalised Unitarity

Bern, Dixon, Kosower; Cachazo, Britto, Feng; Ellis, Kunszt, Giele, Melnikov and more...



Pure kinematic

1-loop primitive

Z. Bern, L. J. Dixon, and D. A. Kosower, Nucl.Phys. B437 , 259 (1995)
V. Del Duca, L. J. Dixon, and F. Maltoni, Nucl.Phys. B571 , 51 (2000)

$$\mathcal{A}_{\text{loop}}(\bar{u}, W^+, d, \bar{s}, W^+, c)$$

e.g. pp $\longrightarrow$ W+W+/- j j

Melia, Melnikov, Rontsch, Zanderighi, 2008, 2010, +Schulze 2012

D-dimensional Generalised Unitarity

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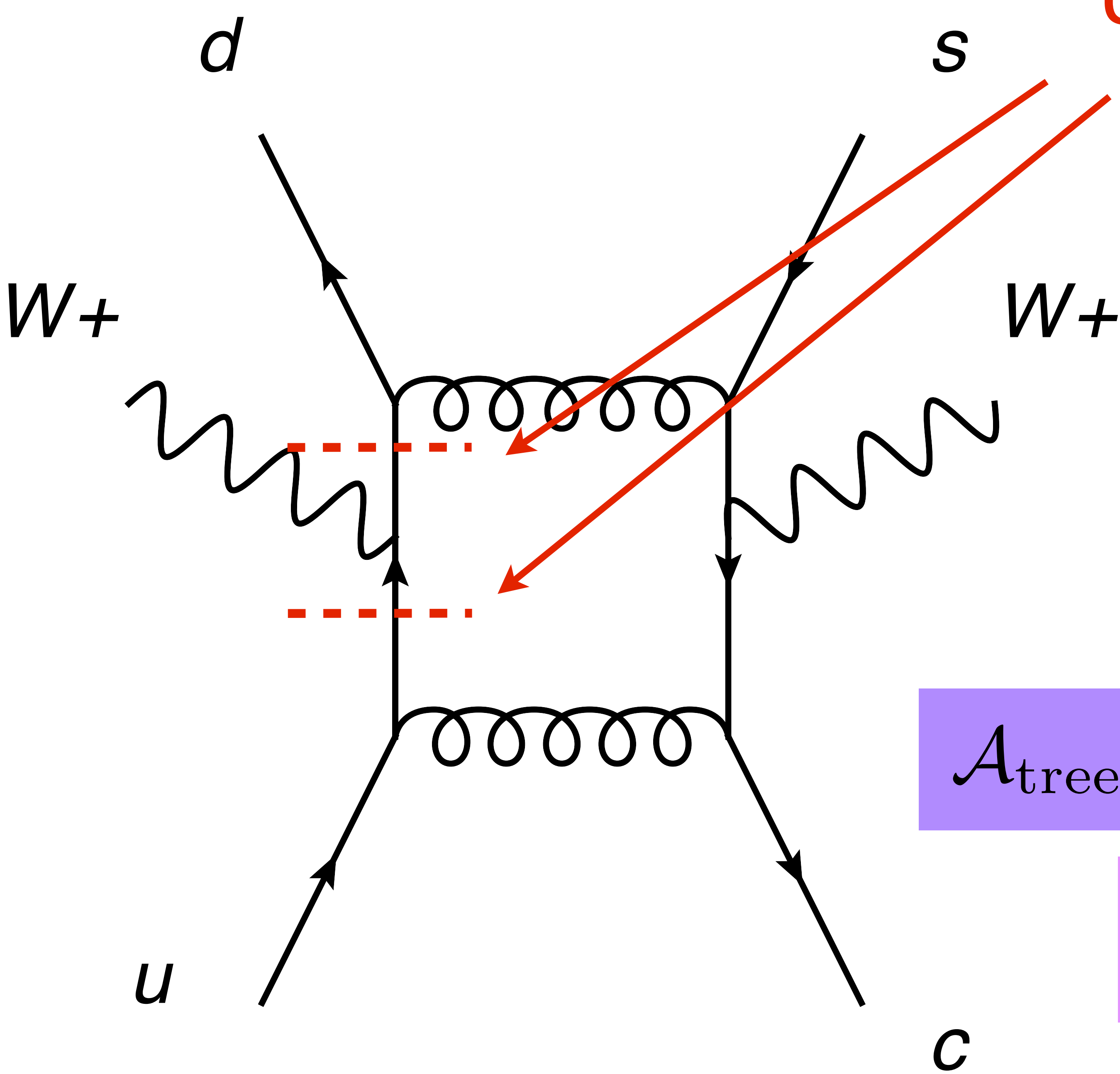
Unitarity Cut



Product of tree level primitives

$$\mathcal{A}_{\rm tree}(\bar{u}, W^+, d)$$ x

$$\mathcal{A}_{\rm tree}(\bar{d}, d, \bar{s}, W^+, c, \bar{u}, u)$$

Higher # of quarks than the 1-loop primitive

These are fundamental QCD building blocks. But their properties have not been studied in detail when there are quarks present.

For instance - how many are independent?

Pure Gluons tree-level

$$A^{\text{full}} = \sum_{\sigma \in S_{n-1}} \text{tr}(\lambda^1 \lambda^{\sigma_1} \lambda^{\sigma_2} \ldots \lambda^{\sigma_{n-1}}) \, \mathcal{A}(1, \sigma_1, \sigma_2, \ldots, \sigma_{n-1})$$

F. A. Berends and W. Giele, Nucl.Phys. B294 , 700 (1987).
M. L. Mangano, S. J. Parke, and Z. Xu, Nucl.Phys. B298 , 653 (1988).

$(n-1)!$ amplitudes

$$A^{\text{full}} = \sum_{\sigma \in S_{n-2}} [F^{\sigma_1} F^{\sigma_2} \ldots F^{\sigma_{n-2}}]^1{}_2 \, \mathcal{A}(1, 2, \sigma_1, \ldots, \sigma_{n-2}) \quad [F^a]^b{}_c = i f^{bac}$$

V. Del Duca, A. Frizzo and F. Maltoni, Nucl. Phys. B 568 (2000) 211.
V. Del Duca, L. J. Dixon, and F. Maltoni, Nucl.Phys. B571 , 51 (2000)

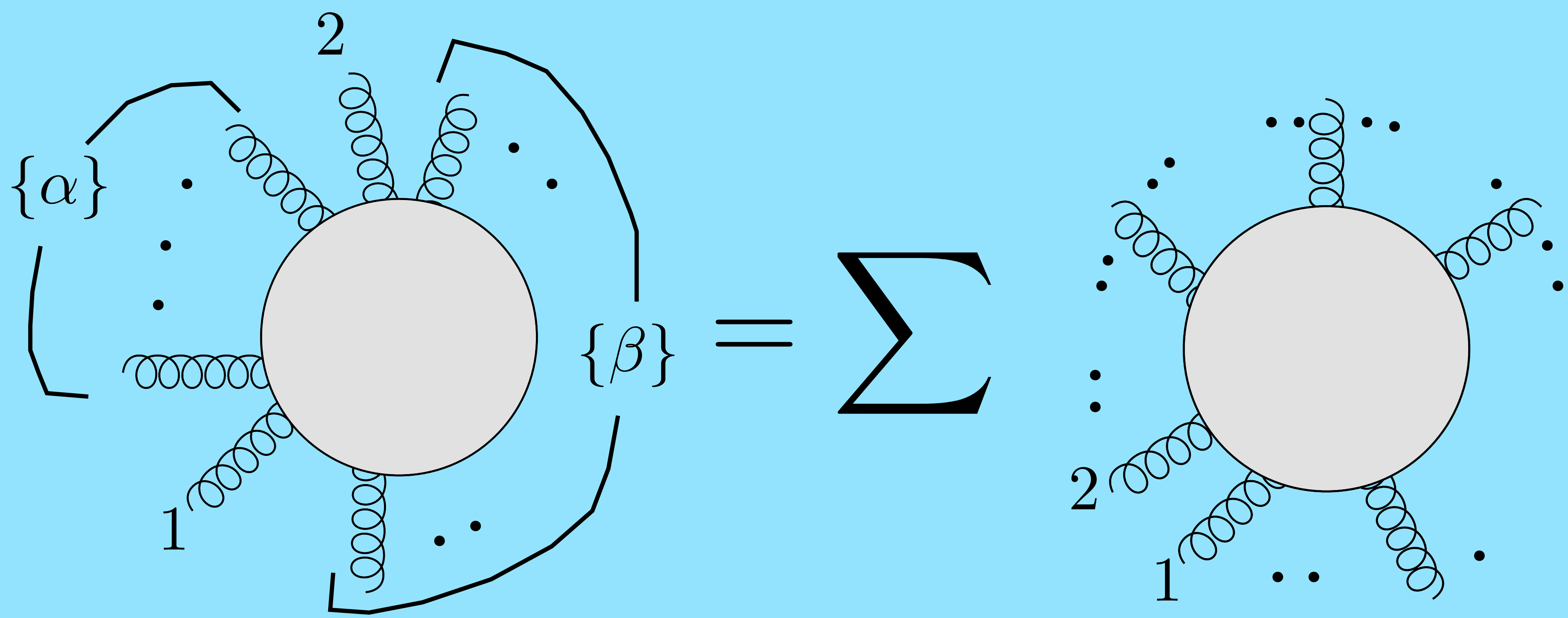
only $(n-2)!$ independent

(Not considering BCJ)

This implies the Kleiss-Kuijf relations

R. Kleiss and H. Kuijf, Nucl.Phys. B312 , 616 (1989)

$$\mathcal{A}(1,\{\alpha\},2,\{\beta\}) = \sum_{\mathrm{OP}\{A\}\{B\}} \mathcal{A}(1,2,\underbrace{\{\alpha^T\}}_{A}\underbrace{\{\beta\}}_{B})$$



$\{\alpha\} = 3\ 4\ 5$

$\{\alpha^T\} = 5\ 4\ 3$

$\{\beta\} = 6\ 7$

OP $\{\alpha^T\}\{\beta\}$

5 4 3 6 7	5 4 6 7 3	5 6 7 4 3
5 4 6 3 7	5 6 4 7 3	6 5 7 4 3
5 6 4 3 7	6 5 4 7 3	6 7 5 4 3
6 5 4 3 7		

Consider primitives with n/2 quarks + n/2 anti-quarks (put gluons back easily).

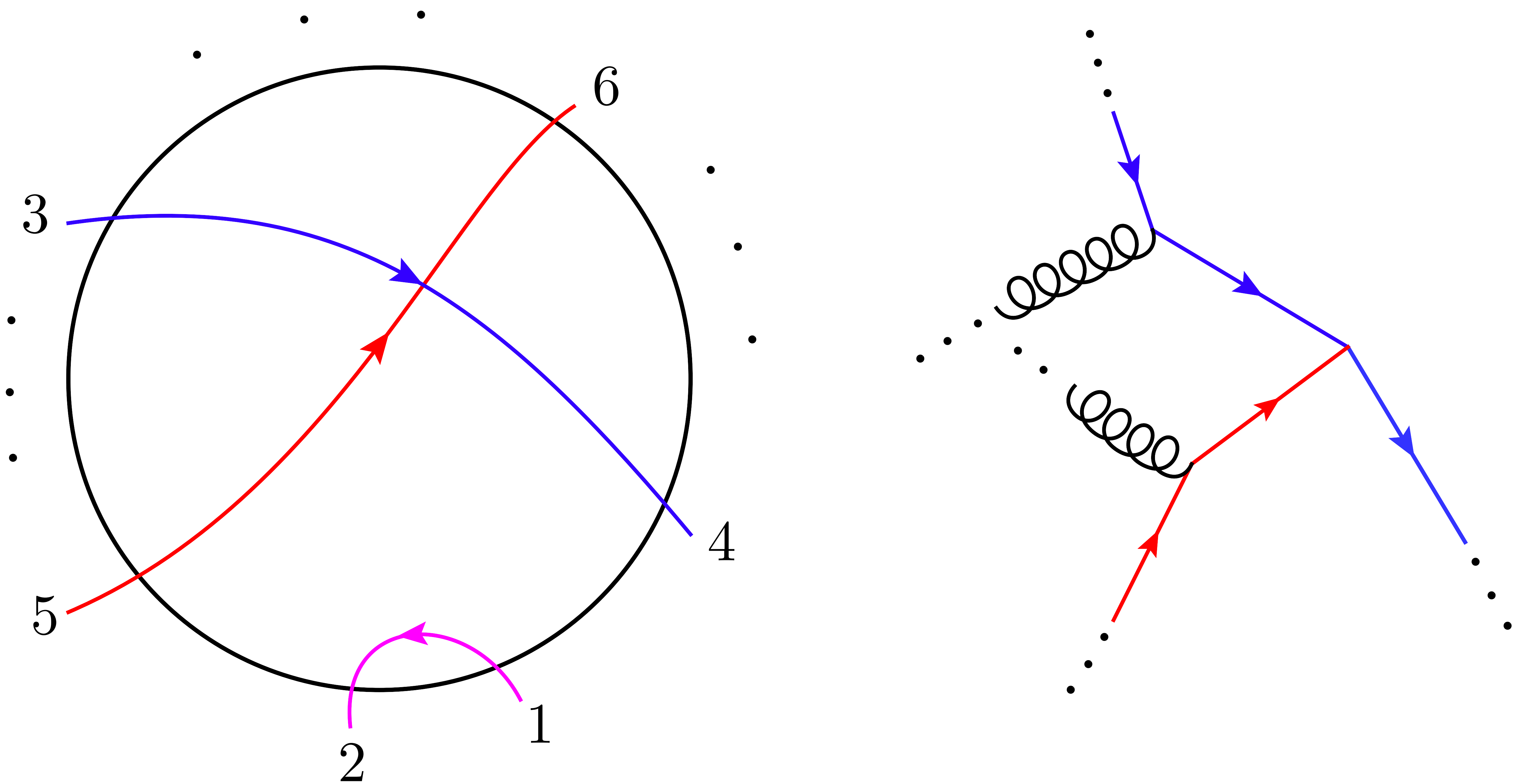
All distinct flavour. Use labeling convention:

$$(1 \rightarrow 2)\,,\ (3 \rightarrow 4)\,,\ (5 \rightarrow 6)\,,\ \ldots\,,\ (n\text{-}1 \rightarrow n)$$

Imagine they are charged under adjoint SU(3). Kinematic part unaffected.

$$\mathcal{A}(1, 2, \sigma) \quad \text{Basis?}$$

$$\mathcal{A}(1,2,\ldots,5,\ldots,3,\ldots,6,\ldots,4,\ldots)=0$$



No crossing quark lines.

Dyck Words

Walther von Dyck

German Mathematician 1856-1934

XY r=1

XXYY XYXY r=2

XXXYYY XXYXYY XXYYXY XYXXYY XYXYXY r=3

String of r Xs and r Ys such that the number of Xs is always greater than or equal to the number of Ys in any initial segment of the string.

Dyck Words

Walther von Dyck

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XY r=1

XXYY XYXY r=2

XXXYYY XXYXYY XXYYXY XYXXYY XYXYXY r=3

# of Dyck words of length 2r

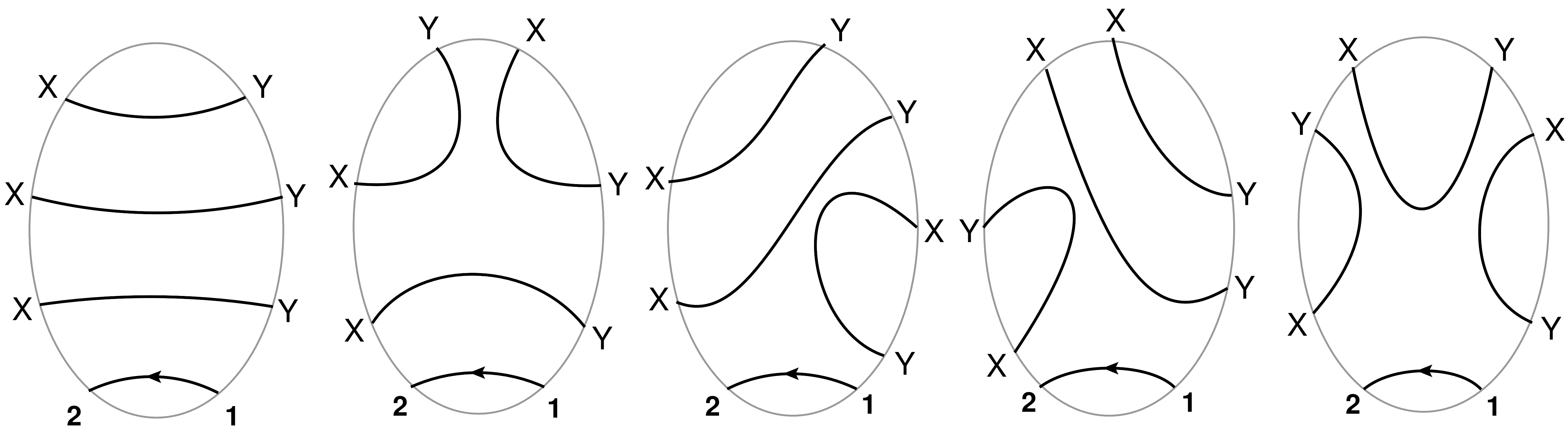
$$C_r = \frac{(2r)!}{r!(r+1)!}$$

Dyck Basis

Melia, Phys.Rev. D to appear (2013), arXiv:1304.7809

$r = n/2 - 1$

XXXYYY XXYXYY XXYYXY XYXXYY XYXYXY



No. of topologies: $\dfrac{(2r)!}{r!(r+1)!}$

Allocate flavours: $\times r!$

Draw in quark arrow: $\times 2^r$

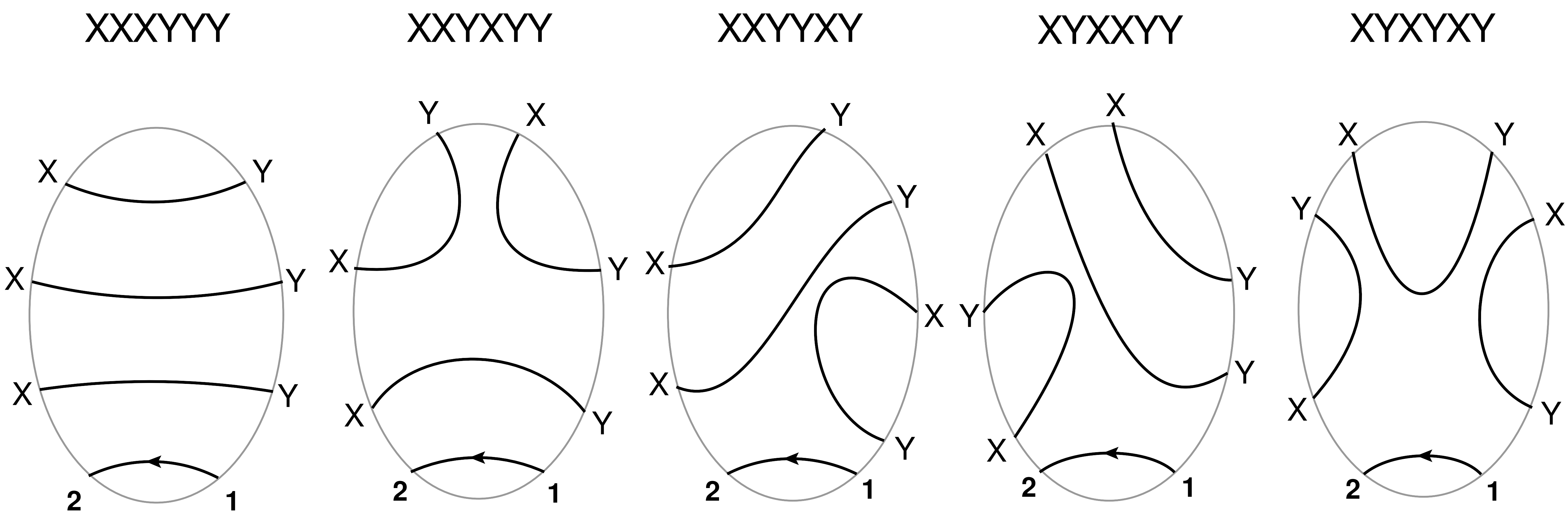
$$= 2^r \frac{(2r)!}{(r+1)!} = 2^{n/2-1} \frac{(n-2)!}{(n/2)!}$$

= #Non-zero $\mathcal{A}(1, 2, \sigma)$

Dyck Basis

Melia, Phys.Rev. D to appear (2013), arXiv:1304.7809

$r = n/2 - 1$



No. of topologies: $\dfrac{(2r)!}{r!(r+1)!}$

Allocate flavours: $\times r!$

Draw in quark arrow: $\times 2^r$

$$= 2^r \frac{(2r)!}{(r+1)!} = \cancel{2^{n/2-1}} \frac{(n-2)!}{(n/2)!}$$

One arrow configuration is independent through an SU(3) group theory relation.

Conclusion for this part (putting gluons back is simple: see proceedings)

of independent tree-level QCD primitives with k distinct quarks, k antiquarks, and n-2k gluons is $(n-2)!/k!$.

An explicit basis has a quark line structure given by

$$\mathcal{A}(1, 2, \mathrm{Dyck}_{k-1}(X_{\tau_1}, .., X_{\tau_{k-1}}, Y, .., Y))$$

for all Dyck topologies with r=k-1, and where antiquarks are assigned to the X, for all (k-1)! flavour permutations τ, and the Ys are assigned depending on the Dyck topology.

+ all gluons insertions except between 1 and 2.

Two New QCD Colour Decompositions

The general solution (all n) to relate QCD primitives to the full amplitude is known for two cases at tree-level:

n gluons

2q + n gluons

Two New QCD Colour Decompositions

The general solution (all n) to relate QCD primitives to the full amplitude is known for two cases at tree-level:

n gluons

2q + n gluons

4q + n gluons

6q + n gluons

$$\mathcal{A}_i = \sum_{j \in \text{colour ordered}} D_j$$

$$A = \sum_{i=1}^{\#\text{basis}} \mathcal{C}_i^{\text{Amp}} \mathcal{A}_i = \sum_{k=1}^{\#FD} C_k D_k$$

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Unique Feynman Diagram (UFD) Method

Use knowledge of basis:

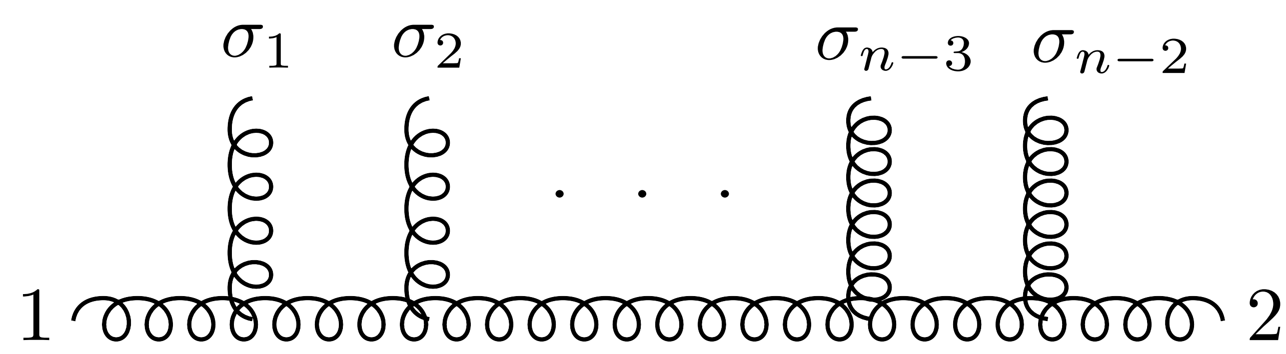
If D_k is a UFD to the primitive $\mathcal{A}_i$ then

$$\mathcal{C}_i^{\text{Amp}} = C_k$$

n gluons

Basis: $\mathcal{A}(1_g\sigma_1\sigma_2\ldots\sigma_{n-3}\sigma_{n-2}2_g) \qquad \sigma\in S_{n-2}$

UFD:



$$(F^{\sigma_1}F^{\sigma_2}\ldots F^{\sigma_{n-3}}F^{\sigma_{n-2}})^1_2 \equiv [\sigma_1\sigma_2\ldots\sigma_{n-3}\sigma_{n-2}]^1_2$$

Colour decomposition:

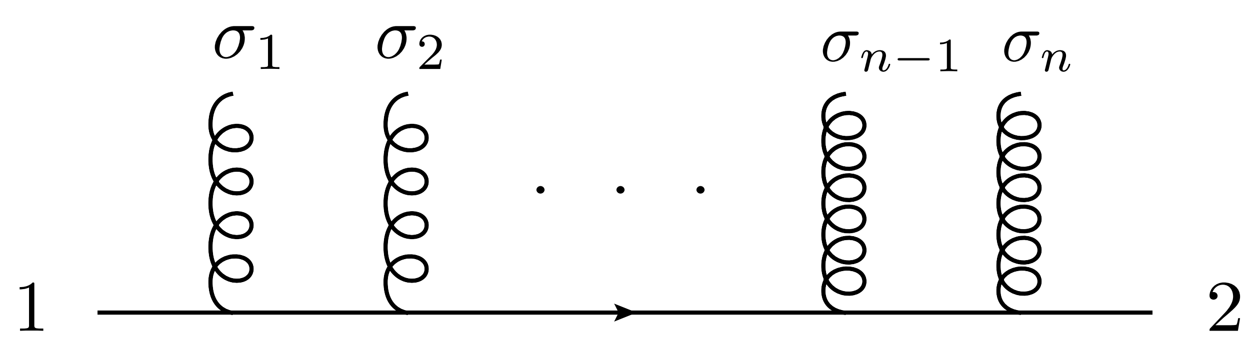
$$A^{\text{tree}} = \sum_{\sigma\in S_{n-2}} [\sigma_1\ldots\sigma_{n-2}]^1{}_2\,\mathcal{A}(1\sigma_1\ldots\sigma_{n-2}2)$$

V. Del Duca, A. Frizzo and F. Maltoni, Nucl. Phys. B 568 (2000) 211.
V. Del Duca, L. J. Dixon, and F. Maltoni, Nucl.Phys. B571 , 51 (2000)

2q+n gluons

Basis: $\mathcal{A}(1\sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_n 2) \qquad \sigma\in S_n$

UFD:



$$(\lambda^{\sigma_1}\lambda^{\sigma_2}\ldots\lambda^{\sigma_{n-1}}\lambda^{\sigma_n})^1{}_2 \equiv (\sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_n)^1{}_2$$

Colour decomposition:

$$A^{\text{tree}} = \sum_{\sigma\in S_n}(\sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_n)^1{}_2\mathcal{A}(1\sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_n)$$

D.A. Kosower, B.-H. Lee and V.P. Nair, Phys. Lett. 201B, 85 (1988);
M. Mangano, Nucl. Phys. B309, 461 (1988);

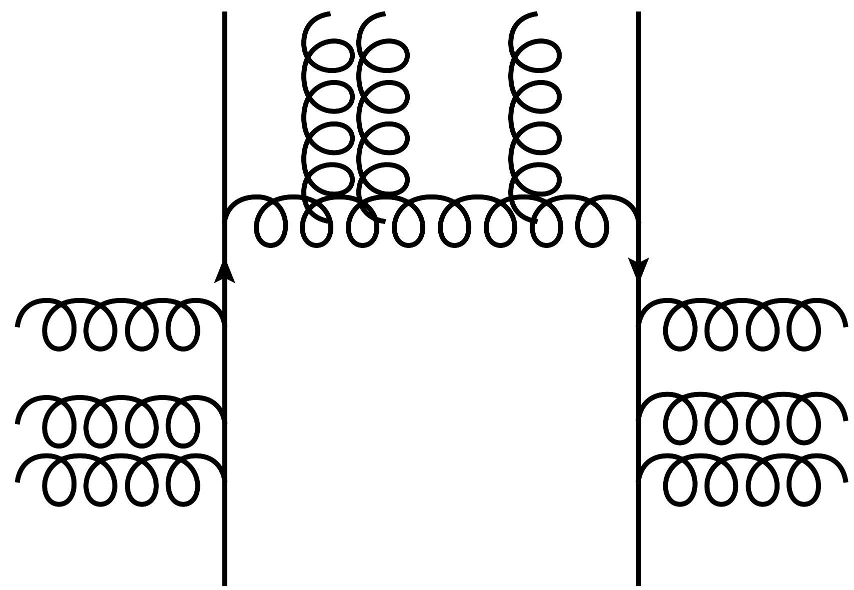
4q+n gluons

Basis: $\mathcal{A}(1, \mathbf{n_1}, 2, \mathbf{n_2}, 3, \mathbf{n_3}, 4)$

$\{\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3}\} = \mathbf{n} = \{\sigma_1\sigma_2\ldots\sigma_n\}$

$\sigma \in S_n$

UFD:



$$(\lambda^{n_1^1}\lambda^{n_1^2}\ldots\lambda^{n_1^{|n_1|}}\lambda^a)^1{}_2[F^{n_2^1}F^{n_2^2}\ldots F^{n_2^{|n_2|}}]^a{}_b(\lambda^b\lambda^{n_3^1}\lambda^{n_3^2}\ldots\lambda^{n_3^{|n_3|}})^3{}_4$$

Colour decomposition:

$$A^{\text{tree}} = \sum_{\mathbf{n}\in S_n}\ \sum_{\{\mathbf{n_1},\mathbf{n_2},\mathbf{n_3}\}=\mathbf{n}} (\mathbf{n_1}a)^1{}_2\,[\mathbf{n_2}]^a{}_b\,(b\mathbf{n_3})^3{}_4\,\mathcal{A}(1, \mathbf{n_1}, 2, \mathbf{n_2}, 3, \mathbf{n_3}, 4)$$

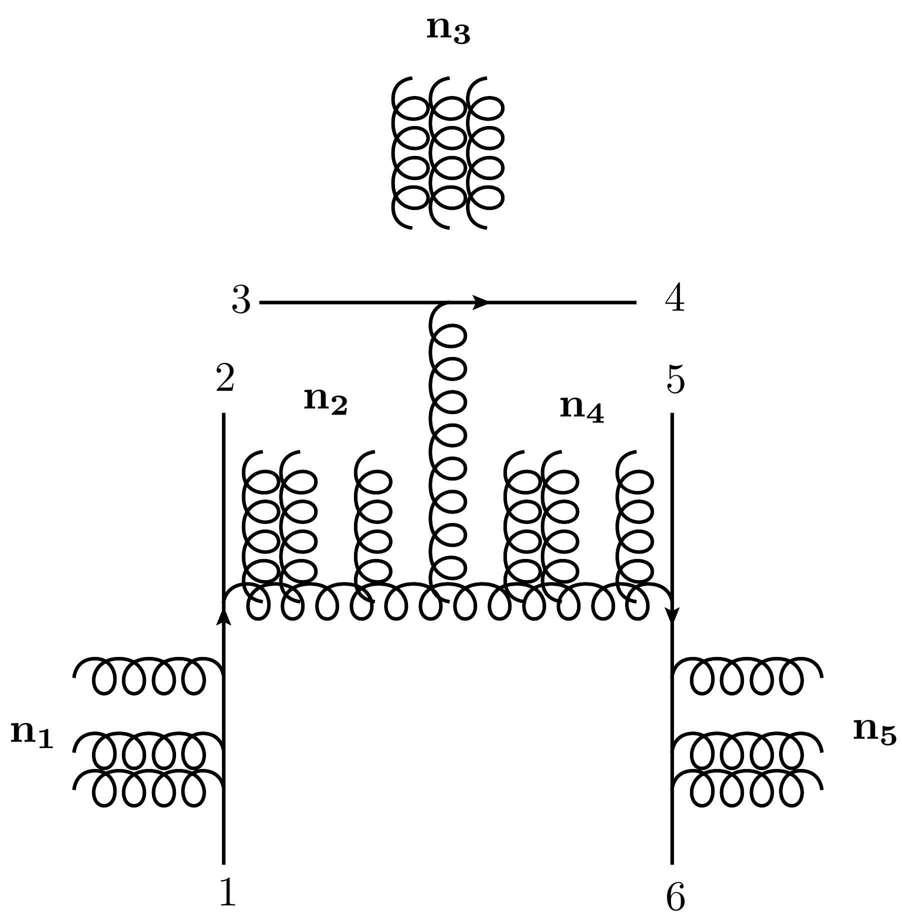
NEW!

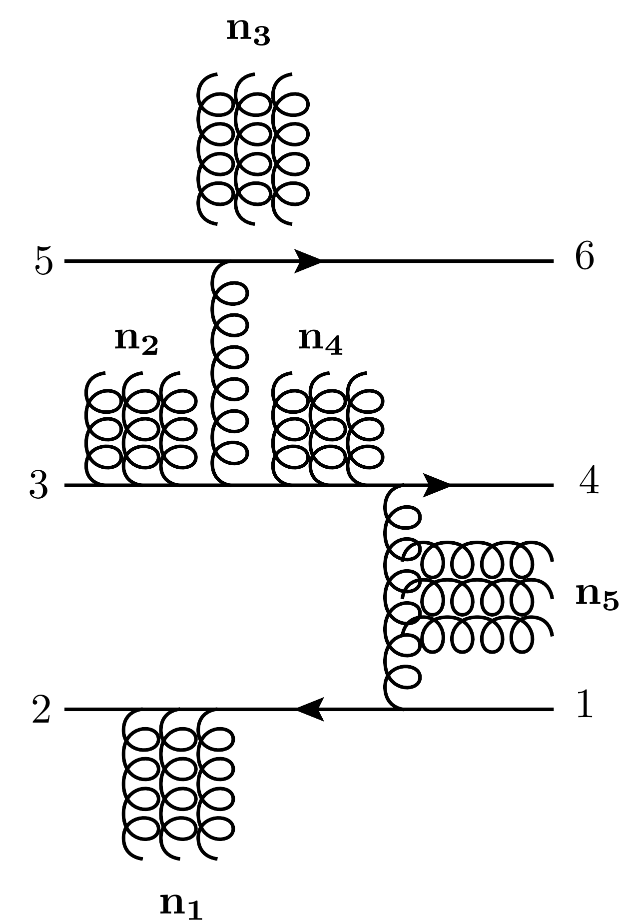
6q+n gluons

$\{\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3}, \mathbf{n_4}, \mathbf{n_5}\} = \mathbf{n} = \{\sigma_1 \sigma_2 .. \sigma_n\} \qquad \sigma \in S_n$

Basis: $\mathcal{A}(1, \mathbf{n_1}, 2, \mathbf{n_2}, 3, \mathbf{n_3}, 4, \mathbf{n_4}, 5, \mathbf{n_5}, 6)$, ($\mathcal{A}(1, \mathbf{n_1}, 2, 3, \mathbf{n_2}, 5, \mathbf{n_3}, 6, \mathbf{n_4}, 4, \mathbf{n_5})$ & cyclic (12),(34),(56))

UFDs:





Colour decomposition:

$$A^{\text{tree}} = \sum_{\mathbf{n} \in S_n} \sum_{\{\mathbf{n_1},..,\mathbf{n_5}\} = \mathbf{n}} (\mathbf{n_1} a)^1{}_2 \, [\mathbf{n_2}]^a{}_b \, \mathcal{T}_g(\mathbf{n_3}, 3, 4)^b{}_c \, [\mathbf{n_4}]^c{}_d \, (d\mathbf{n_5})^5{}_6 \, \mathcal{A}(1, \mathbf{n_1}, 2, \mathbf{n_2}, 3, \mathbf{n_3}, 4, \mathbf{n_4}, 5, \mathbf{n_5}, 6)$$

$$\Big(+ (a\mathbf{n_1})^1{}_2 \, (\mathbf{n_2})^3{}_i \, \mathcal{T}_q(\mathbf{n_3}, 5, 6)^i{}_j \, (\mathbf{n_4} b)^j{}_4 \, [\mathbf{n_5}]^b{}_a \, \mathcal{A}(1, \mathbf{n_1}, 2, 3, \mathbf{n_2}, 5, \mathbf{n_3}, 6, \mathbf{n_4}, 4, \mathbf{n_5})$$

+ cyclic (12),(34),(56))

where $\mathcal{T}_g(\mathbf{n}, \bar{q}, q)^a{}_b = \sum_{\mathbf{s_1}, \mathbf{s_2} \in \text{ord.subsets } \mathbf{n}} (a\mathbf{s_1})^q{}_{\bar{q}} [a\mathbf{s_2}]^a{}_b \qquad \mathcal{T}_q(\mathbf{n}, \bar{q}, q)^i{}_j = \sum_{\mathbf{s_1}, \mathbf{s_2} \in \text{ord.subsets } \mathbf{n}} (a\mathbf{s_1})^q{}_{\bar{q}} (a\mathbf{s_2})^i{}_j$

NEW!

New understanding of general tree-level primitives in QCD

Interesting mathematical structure.

Basis for all k and all n QCD $g^{n-2k}(q\bar{q})^k$ primitive amplitudes using Dyck word quark line structure.

Organising NLO high jet multiplicity calculations.

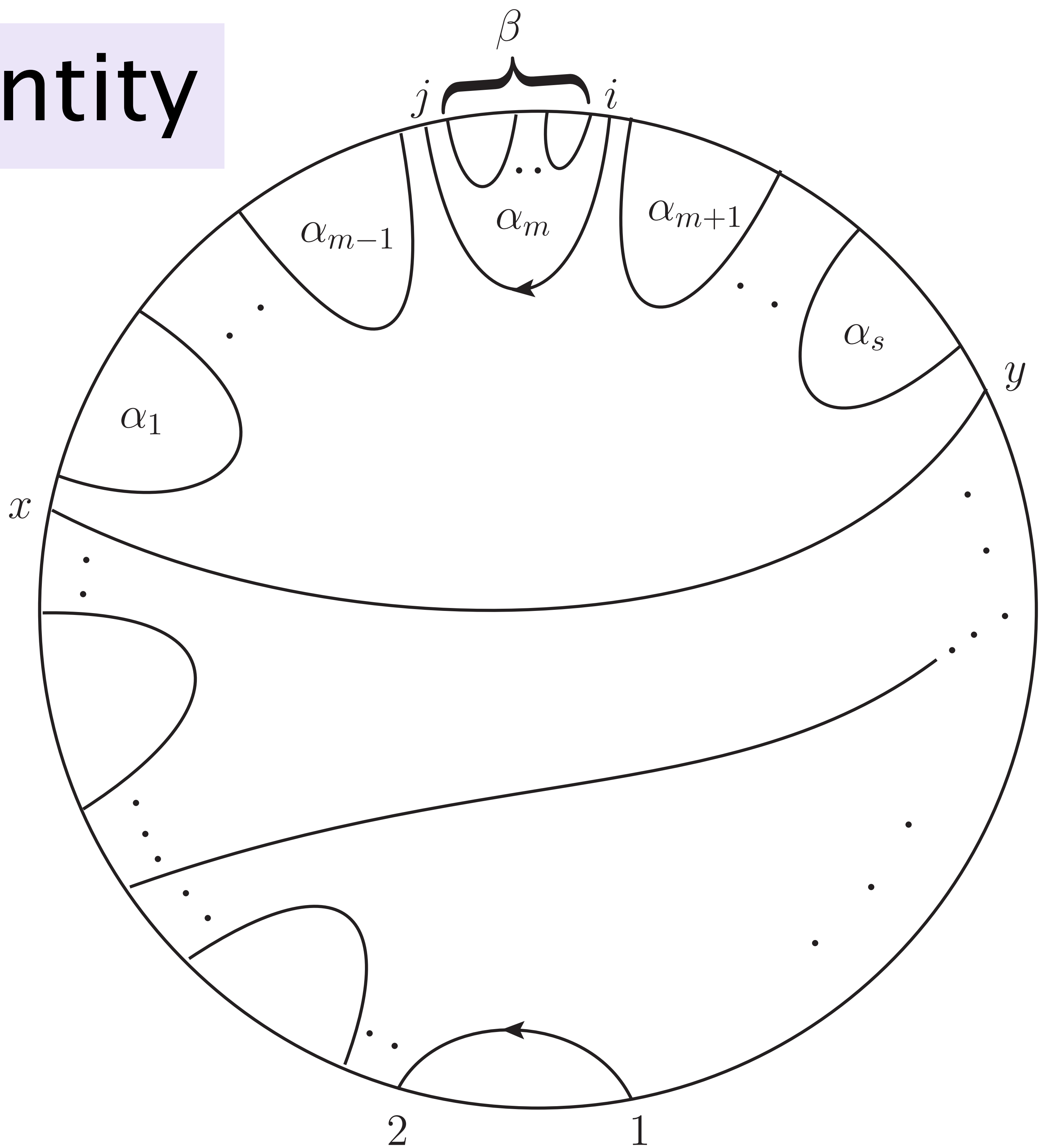
Caching procedures to take relations into account.

New (to be published RADCOR proceedings).

"All-n" 4q+ng and 6q+ng tree level colour decompositions.

Extra . . . .

SU(3) identity



$$\mathcal{A}(\ldots x\,\{\alpha_1\}..\{\alpha_{m-1}\}\,j\,\{\beta\}\,i\,\{\alpha_{m+1}\}..\{\alpha_s\}\,y\ldots) =$$

$$-\sum_{c=1}^{m}\Bigg[\sum_{\mathrm{OP}\{D_c\}\{E\}}\Bigg(\sum_{\mathrm{OP}\{A_c\}\{B\}}\mathcal{A}(\ldots x\,\{\alpha_1\}..\{\alpha_{c-1}\}\,i\,\overbrace{\underbrace{\{\alpha_c\}..\{\alpha_{m-1}\}}_{\{A_c\}}\underbrace{\{\beta^T\}}_{\{B\}}}^{\{D_c\}}\,j\,\overbrace{\{\alpha_{m+1}\}..\{\alpha_s\}}^{\{E\}}\,y\ldots)\Bigg)\Bigg]$$

Independence

$$A_{n_e} = \sum_{\mathcal{P}(q_1,q_2,\ldots,q_{n_e})} (-1)^{\mathrm{sgn}(\mathcal{P})} A_{n_e=0}$$

$$\#(\text{independent } \mathcal{A}_{n_e=0} \geq (n-2)!/k!$$

Charge parity

$$\mathcal{A}(\ldots i_{\bar{q}}^{\pm} \ldots j_{q}^{\mp} \ldots) = -\mathcal{A}(\ldots i_{q}^{\pm} \ldots j_{\bar{q}}^{\mp} \ldots)$$

A couple of KK relations