Towards QCD running in 5 loops: quark mass anomalous dimension

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in collaboration to

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RADCOR 2013

RENORMALIZATION-GROUP (all started in 1953)

Stückelberg and Petermann; Gell-Mann and Low; Bogoliubov and Shirkov

and after only 50 years

$\Downarrow$

Nobel Prise in Physics in 2004!

$$\beta_0 = \frac{33 - 2N_F}{12}$$

Multiloop RG: Current Status

Any 5-loop RG functions (that is β-functions and anomalous dimensions) are ***analytically*** **computable in any model (in the minimal subtraction scheme)**

Recent advances at 5-loop level:

- QED-beta function (including corrections due to the quark-gluon interaction) / Baikov, K. Ch. , P, J. Kühn, J. Rittinger, 2008-2012/
- ghost and quark field and quark mass anomalous dimensions as well as (a significant piece of) ghost-ghost-gluon vertex anomalous dimension are ready /this talk/
- but the full QCD β-function is not (yet!) available (gluon field renormalization $\rightarrow$ main technical challenge, due to # of diagrams and over-complicated IR structure)

Motivations:

$\beta(\alpha_s)$ and $\gamma_m(\alpha_s)$ at 5 loops will be useful for

- the analysis of the τ-decay rate within so-called CIPT (a host of **new** terms will be added to the current theoretical prediction)

- various QCD "optimization" schemes like PMS and PMC (the Principles of Maximal Sensitivity P. Stevenson, 1981) and of Maximal Conformality (S. Brodsky, X. G. Wu,L. Di Giustino,M. Mojaza, 2012) ... will benefit from the knowledge of β-function at 5 loops

- construction of a self-consistent prediction for $H \to \bar{b}b/\bar{c}c$ at $\mathcal{O}(\alpha_s^4)$ from the corresponding result for the scalar correlator /P. Baikov, K.Ch. and J. Kühn, (2006)/ **and** the quark mass anom. dim. $\gamma_m(\alpha_s)$ (also at 5 loops) /this talk/

- construction of a self-consistent prediction for $\alpha_s(M_Z)$ from $\alpha_s(M_\tau)$ **and** the decoupling equation for α_s (known to 4 loops /K.Ch., J.Kühn and Ch. Sturm; Y. Schröder and M. Steinhauser (2006)/)

- lattice (description of running vertexes and propagators for intermediate momentum transfer)

$$\mathcal{L}_R^{QCD} = -\frac{1}{4} Z_3 \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{1}{2} g\, Z_1^{3g} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) \left(A_\mu \times A_\nu\right)^a - \frac{1}{4} g^2\, Z_1^{4g} \left(A_\mu \times A_\nu\right.$$
$$+ \quad Z_3^c\, \partial_\nu \bar{c} \left(\partial_\nu c\right) + g\, Z_1^{ccg}\, \partial^\mu \bar{c} \left(A \times c\right) + Z_2\, \bar{\psi} \mathrm{i} \partial\!\!\!/ \psi - Z_{\bar{\psi}\psi} m_f\, \bar{\psi}\psi + g Z_1^{\psi\psi g}\, \bar{\psi} A\!\!\!/ \psi$$

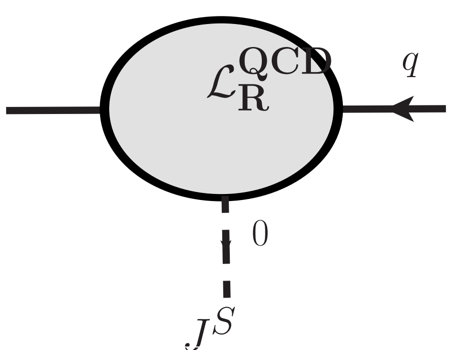
Minimal sets of Z-factors to compute β and γ_m: Z_3, Z_3^c, Z_1^{ccg} and $Z_3, Z_2, Z_{\bar{\psi}\psi}$

Most important property of Z-factors (in minimal schemes based on CDR): they depend *only* on $\epsilon = 2 - D/2$ (J. Collins, 1975). This leads to tremendous simplifications in calculations $\rightarrow$ multiloop completely analytical calculations are really possible.

Let us concentrate on $Z_{\bar{\psi}\psi}$ and consider consider vertex function

$$\Gamma(a_s, q^2) = Z_{\bar{\psi}\psi} + Z_{\bar{\psi}\psi}\, \delta\Gamma(a_s, q^2)$$

of the scalar quark current



$= \Gamma(a_s, q^2)$

Suppose we want to compute L-loop contribution to $Z_{\bar{\psi}\psi}$. There are (at least) 4 ways to do it:

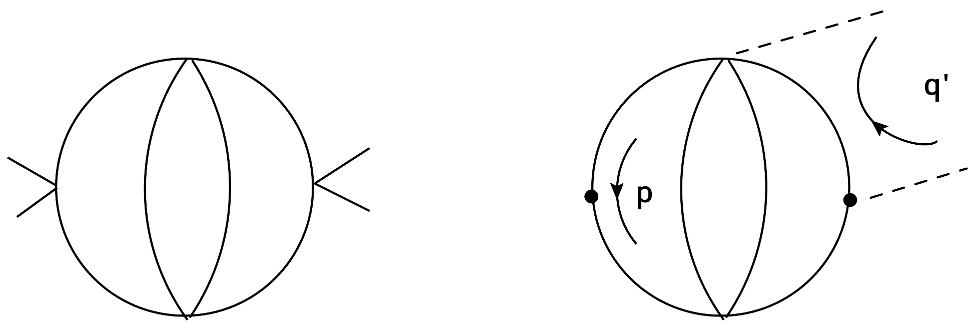
1. set 1 of 2 ext. momenta to zero $\rightarrow$ (the poles of) L-loop p-integrals (massless propagators) tobe computed. That is how first 2-loop RG calculations in QCD were done /D. R.T Jones, 1974/.

2. set **all ext.** momenta to zero and introduce a universal mass to **all** propagators (including gluon!) $\rightarrow$ (the poles of) L-loop m-integrals (massive tadpoles) to be computed. That is how the first 4-loop calculation of the QCD β-function was done /van Ritbergen, T., Vermaseren, J. and Larin, S. (1997)/

3. set **all ext.** momenta to zero and introduce a mass into only **one** (*but properly chosen to avoid IR singularities*) propagator $\rightarrow$ (L-1)-loop p-integrals (including their finite part) to be computed /A. Vladimirov (1978)/ That is how the first 3-loop calculation of the QCD β-function was done /Tarasov, O., Vladimirov, A. and Zharkov, A. (1980)/. Problems: difficult to automatize; not applicable to all diagrams.

4. the same as 3. but IR singularities are removed recursively with so-called R^*-operation /K.Ch. V. Smirnov (1984)/

An example of a diagram which can not be comuted with the 3-rd method



Here two well-separated IR divergencies in loop-integration makes problems. One, of course, could regulate it with a small "auxiliary" mass: $\frac{1}{p^4} \to \frac{1}{(p^2+m^2)^2}$ but that will complicates integration, leading to a 2-scale integral.

The idea how to overcome the problem (in fact, it came from the Bogolyubov's distributional approach to QFT) is very simple: to subtract the unwanted IR divergency with the help of an IR counterterm but now local in *position space*:

$$\boldsymbol{\frac{1}{p^4} \;\to\; \frac{1}{(p^4)} \;-\; \frac{c}{\epsilon}\delta^D(p)}$$

with the constant $\boldsymbol{c}$ choosen such that there would be no IR poles coming from the integration region of small momentum p.

After such a replacement no IR poles survive and integrations are made easily.

At 5-loop level only the 4-th way is currently feasible

with the use of the following tools:

- global solution of the combinatorics of R^* operation (rather involved and problem specific)

- the Baikov's way of doing reduction with the help of $1/D$ expansion of the corresponding coefficient functions in front of masters (analytically known from **two!** independent calculations /K.Ch, P.Baikov (2010), R. Lee, V. Smirnov (2012)/

- **ParFORM and T-FORM:**

M. Tentyukov et al. "ParFORM: Parallel Version of the Symbolic Manipulation Program", PoS ACAT2010 (2010) 072

M. Tentyukov, H. M. Staudenmaier, and J. A. M. Vermaseren. "ParFORM: Recent development". *Nucl. Instrum. Meth.*, A559:224–228, 2006.

M. Tentyukov and J. A. M. Vermaseren. "The multithreaded version of FORM", hep-ph/0702279"

in order to effectively implement the $1/D$ expansion

Calculation of the ghost field anomalous dimension $\gamma_3^c = \sum_{i=0}^{\infty} (\gamma_3^c)_i \left(\frac{\alpha_s}{4\pi}\right)^{i+1}$ at 5 loops has been just finished:

$$
\begin{aligned}
(\gamma_3^c)_4 = & \frac{193301287}{2048} + \frac{19562145}{128}\zeta_3 + \frac{2060829}{128}\zeta_3^2 - \frac{1101573}{16}\zeta_4 - \frac{66632427}{128}\zeta_5 + \frac{36327825}{256}\zeta_6 + \frac{140900823}{512} \\
+ \quad & n_f \left[-\frac{633704171}{27648} - \frac{5166473}{144}\zeta_3 - \frac{233519}{64}\zeta_3^2 + \frac{764949}{32}\zeta_4 + \frac{32902291}{384}\zeta_5 - \frac{4123825}{128}\zeta_6 - \frac{14425075}{384}\zeta_7 \right] \\
+ \quad & n_f^2 \left[\frac{1326547}{3456} + \frac{1739167}{864}\zeta_3 + \frac{2659}{6}\zeta_3^2 - \frac{13485}{8}\zeta_4 - \frac{8074}{9}\zeta_5 + \frac{16775}{12}\zeta_6 \right] \\
+ \quad & n_f^3 \left[\frac{342895}{7776} + \frac{1211}{18}\zeta_3 + \frac{5}{2}\zeta_4 - \frac{284}{3}\zeta_5 \right] + n_f^4 \left[-\frac{65}{108} - \frac{20}{27}\zeta_3 + \frac{4}{3}\zeta_4 \right]
\end{aligned}
$$

Numerically ($a_s \equiv \frac{\alpha_s}{\pi}$):

$$
\gamma_3^c(n_f = 3) = \frac{3}{8}\left(a_s + 2.4375\, a_s^2 + 4.8867\, a_s^3 + 19.980\, a_s^4 + 122.246\, a_s^5\right)
$$

For generic n_f:

$$
\begin{aligned}
\gamma_3^c = & \frac{3}{8}\Big\{ a_s + a_s^2 (3.063 - 0.208 n_f) + a_s^3 \left(10.556 - 1.768 n_f - 0.0405 n_f^2\right) \\
+ \quad & a_s^4 \left(49.325 - 10.957 n_f + 0.36562 n_f^2 + 0.0087 n_f^3\right) \\
& + a_s^5 \left(283.632 - 70.979 n_f + 5.498 n_f^2 + 0.0769 n_f^3 - 0.000128038 n_f^4\right) \Big\}
\end{aligned}
$$

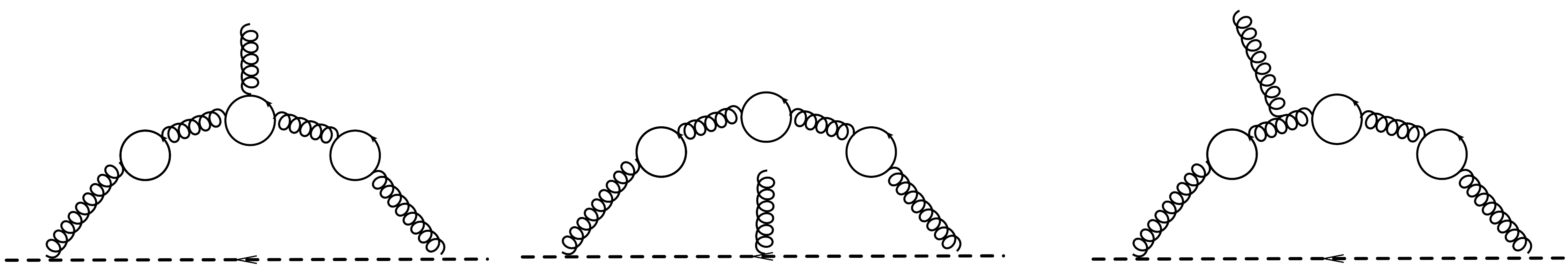
Calculation of the anomalous dimension of gluon-ghost-ghost vertex

$$\gamma_1^{ccg} = \sum_{i=0}^{\infty} (\gamma_1^{ccg})_i \left(\frac{\alpha_s}{4\pi}\right)^{i+1}$$

is under way; the first result is ready (in "next^2-to-renormalon" approximation, Feynman gauge):

$$(\gamma_1^{ccg})_4 = n_f^3 \left[\frac{2989}{864} + \frac{5}{3}\zeta_3 - 6\zeta_4\right] + n_f^2 \left[-\frac{572723}{2304} - \frac{8105}{16}\zeta_3 + \frac{3789}{32}\zeta_4 + \frac{2109}{8}\zeta_5\right] + \mathcal{O}(n_f^1$$

Note that the leading renormalon contribution $\approx n_f^4\, a_s^5$ vanishes *(in any gauge!)* due to the Taylor theorem which states, in paricularly, that $\gamma_1^{ccg} \equiv 0$ in the Landau gauge



Quark Mass Anomalous Dimension $\gamma_m = -\sum_{i\geq 0} \gamma_i \, a_s^i$: history

3-loops: /O, Tarasov (82, with IRR reduced to 2-loop p-integrals);

3-loops: /S. Larin/ (92; direct evaluation of 3-loop p-integrals with MINCER)

4-loops: /K. Chetyrkin/ (97; with R^*-operation all FI's were reduced to 3-loop p-integrals; the latter were performed with MINCER)

4-loops: /J.A.M. Vermaseren, S.A. Larin, T. van Ritbergen/ (97; direct evaluation of the **completely massive 4-loop tadpoles** /via a kind of Laporta machine (?)/)

$$\gamma_0 = 1 \quad \gamma_1 = \frac{1}{16}\left\{ \frac{202}{3} + n_f \left[-\frac{20}{9} \right] \right\}, \gamma_2 = \frac{1}{64}\left\{ 1249 + n_f \left[-\frac{2216}{27} - \frac{160}{3}\,\zeta(3) \right] + n_f^2 \left[-\frac{140}{81} \right] \right\}$$

$$\begin{aligned}
\gamma_3 = \frac{1}{256}\Big\{ & \frac{4603055}{162} + \frac{135680}{27}\,\zeta(3) - 8800\,\zeta(5) \\
+ \quad & n_f \left[-\frac{91723}{27} - \frac{34192}{9}\,\zeta(3) + 880\,\zeta(4) + \frac{18400}{9}\,\zeta(5) \right] \\
+ \quad & n_f^2 \left[\frac{5242}{243} + \frac{800}{9}\,\zeta(3) - \frac{160}{3}\,\zeta(4) \right] + n_f^3 \left[-\frac{332}{243} + \frac{64}{27}\,\zeta(3) \right] \Big\}.
\end{aligned}$$

5 loop term in $\gamma_m = -\sum_{-i\geq 0} \gamma_i \, a_s^i$

New result (preliminary)

$$
\begin{aligned}
\gamma_4 = & \frac{-1}{4^5}\Bigg\{ -\frac{99512327}{162} - \frac{46402466}{243}\zeta_3 - 96800\,\zeta_3^2 + \frac{698126}{9}\zeta_4 \\
& + \frac{231757160}{243}\zeta_5 - 242000\,\zeta_6 - 412720\,\zeta_7 \\
+ \quad & n_f \left[\frac{150736283}{1458} + \frac{12538016}{81}\zeta_3 + \frac{75680}{9}\zeta_3^2 - \frac{2038742}{27}\zeta_4 \right. \\
& \left. - \frac{49876180}{243}\zeta_5 + \frac{638000}{9}\zeta_6 + \frac{1820000}{27}\zeta_7 \right] \\
+ \quad & n_f^2 \left[-\frac{1320742}{729} - \frac{2010824}{243}\zeta_3 - \frac{46400}{27}\zeta_3^2 + \frac{166300}{27}\zeta_4 + \frac{264040}{81}\zeta_5 - \frac{92000}{27}\zeta_6 \right] \\
+ \quad & \boxed{ n_f^3 \left[-\tfrac{91865}{1458} - \tfrac{12848}{81}\zeta_3 - \tfrac{448}{9}\zeta_4 + \tfrac{5120}{27}\zeta_5 \right] + n_f^4 \left[\tfrac{260}{243} + \tfrac{320}{243}\zeta_3 - \tfrac{64}{27}\zeta_4 \right] } \Bigg\}
\end{aligned}
$$

Boxed terms are in full agreement with predication made on the base of the $1/n_f$ method /M. Ciuchini, S.E. Derkachov, J.A. Gracey, A.N. Manashov, (2000)/

Numerical result:

$$\gamma_4^{exact} = 559.71 - 143.6\, n_f + 7.4824\, n_f^2 + 0.1083\, n_f^3 - 0.00008535\, n_f^4$$

should be compared with a prediciton

$$\gamma_4^{APAP} = 530 - 143\, n_f + 6.67\, n_f^2 + 0.037\, n_f^3 - \boxed{0.00008535\, n_f^4}$$

which is 15 years old result (obtained with the "Asymptotic Pade Approximants" /APAP/ method) by J. Ellis, I. Jack, D.R.T. Jones, M. Karliner, M. A. Samuel, Phys. Rev. D57 (1998) 2665

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Unfortunately, this strikingly good agreement does not survive for fixed values of n_f:

n_f	3	4	5	6
$\gamma_4^{\mathrm{exact}}$	198.899	111.579	41.807	-9.777
γ_4^{APAP}	162	67	-13.7	-80.0

due to severe cancellations between dfferent powers of n_f

The mass evolution is described by equation $\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}$ where

$$\begin{aligned} c(x) &= \exp\left\{\int \frac{dx'}{x'}\frac{\gamma_m(x')}{\beta(x')}\right\} = (x)^{\bar{\gamma}_0}\left\{1 + (\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)x\right. \\ &+ \frac{1}{2}\left[(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0\right] x^2 \\ &+ \left[\frac{1}{6}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^3 + \frac{1}{2}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)(\bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0)\right. \\ &\left.\left. + \frac{1}{3}\left(\bar{\gamma}_3 - \bar{\beta}_1{}^3\bar{\gamma}_0 + 2\bar{\beta}_1\bar{\beta}_2\bar{\gamma}_0 - \bar{\beta}_3\bar{\gamma}_0 + \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_2\right)\right] x^3 + \mathcal{O}(x^4)\right\} \end{aligned}$$

$\bar{\gamma}_i = \gamma_i/\beta_0$, $\bar{\beta}_i = \beta_i/\beta_0$

Important concept: RGI mass

$$m^{RGI} \equiv m(\mu_0)/c(a_s(\mu_0))$$

is μ and *scheme* independent; in *any* (mass-independent) scheme

$$\lim_{\mu\to\infty} a_s(\mu)^{-\bar{\gamma}_0}\, m(\mu) = m^{RGI}$$

The function $c(x)$ is used, e.g, by the **ALPHA** lattice collaboration to find the ($\overline{\mathrm{MS}}$) mass of the strange quark at a lower scale from the RGI mass determined from lattice simulations

Example (setting $a_s(\mu = 2\,\mathrm{GeV}) = \frac{\alpha_s(\mu)}{\pi} = .1$; h counts loops)

$$m_s(2\,\mathrm{GeV}) = \hat{m}_s \cdot (a_s(2\,\mathrm{GeV}))^{\frac{4}{9}} \cdot$$

$$\left(1 + 0.0895\,h^2 + 0.0137\,h^3 + 0.00195\,h^4 + (0.00157 - .000011\,\overline{\beta}_4)\,h^5\right)$$

$$\beta(n_f = 3) = -\left(\beta_0 = \frac{4}{9}\right) \cdot \left\{a_s + 1.777\,a_s^2 + 4.4711\,a_s^3 + 20.990\,a_s^4 + \overline{\beta}_4\,a_s^5\right\}$$

It is natural to estimate $\overline{\beta}_4$ as sitting in the interval 50 – 100 Note that for any reasonable value of $\overline{\beta}_4$ (positive and ≤ 200) the (apparent) convergency of the above series is quite good even at rather small energy scale

Higgs Decay into $\bar{b}b$ quarks

$$\Gamma(H \to \bar{f}f) = \frac{G_F\, M_H}{4\sqrt{2}\pi}\, m_f^2(\mu)\, R^S(s = M_H^2, \mu)$$

R^S is the spectral density of the scalar correlator and is known to α_s^4 /P. Baikov, J. Kühn, K.Ch. (2006)/

$$\begin{aligned} R^S(s = M_H^2, \mu = M_H) &= 1 + 5.667\, a_s + 29.147\, a_s^2 + 41.758\, a_s^3 - 825.7\, a_s^4 \\ &= 1 + 0.2041 + 0.0379 + 0.0020 - 0.00140 \end{aligned}$$

where we set $a_s = \alpha_s/\pi = 0.0360$ (for the Higgs mass value $M_H = 125$ GeV and $\alpha_s(M_Z) = 0.118$)

$m_b(\mu = M_H)$ is to be obtained with RG running from $m_b(\mu = 10\,\mathrm{GeV})$ and, thus, depends on β and γ_m:

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.4 \cdot 10^{-4} (\bar{b}_4 = 0)| - 4.3 \cdot 10^{-4} (\bar{b}_4 = 100)| - 7.3 \cdot 10^{-4} (\bar{b}_4 = 200)$$

If we set $\mu = M_H$, then the combined effect of $\mathcal{O}(\alpha_s^4)$ terms as coming from the 5-loop running and 4-loop contribition to R^S on

$$\Gamma(H \to \bar{b}b) = \frac{G_F\, M_H}{4\sqrt{2}\pi}\, m_f^2(M_H)\, R^S(s = M_H^2, M_H)$$

is around -2‰. This should be contrasted to the parametric uncertainties

as coming from* $\alpha_s(M_Z)$ (± 6 ‰) and** $m_b^2(\mu = 10\,\mathrm{GeV})$ (± 9 ‰)

(we neglect higher order QCD corrections)

Conclusion: our α_s^4 terms are of no phenomenological relevancy at present. BUT, the situation could be different if the project of TLEP*** is implemented. For instance, the uncertainity in $\alpha_s(M_Z)$ will be reduced to ± 2‰ ...

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* A. Pich, "Review of α_s detreminations", arXiv:1303.2262
** K. Ch., J. H. Kühn, A. Maier, P. Maierhöfer, P. Marquard, M. Steinhauser, C. Sturm, "Charm and Bottom Quark Masses: an Update", arXiv:0907.2110
*** M. Bice et al., "First Look at the Physics Case of TLEP", arXiv:1308.6176

Concluding Notes I:

- R^* + Baikov Algorithm to reduce 4-loop p-integrals + Form (J. Vermaseren, M. Tentyukov + ...) + known 4-loop masters (P. Baikov, K.Ch.) $\Longrightarrow$ the 5-loop RG functions are *in principle* doable in *any* model.

- But: global representation of neccessary IR subtractions (that is on the level of Green functions) strongly depends on the problem and is not always easy.

- The 5-loop quark anomalous dimension γ_m QCD is finished. The phenomenological implications are not not very dramatic.

- The 5-loop QCD β-function is significantly more complicated; first results are expected in a year or so.

Concluding Notes II:

- Truly remarkable fact: N=4 SYM theory seems to be simpler than QCD: "Konishi" (anomalous dimension of a specific operator in N=4 SYM) in 5-loop has been recenltly computed with a via IRR + p-intergrals + Laporta machine + a lot of ingenuity; the result confirms the prediction from non-perturbative methods ("Five-loop Konishi in N=4 SYM", B. Eden, P. Heslop, G. Korchemsky, V. Smirnov, E. Sokatchev, arXiv:1202.5733)

- There are some theoretical problems requiring analytical evaluation of 6-loop anomalous dimensions: e.g. "Konishi" in 6-loop is already available from non-perturbative methods:

  Six and seven loop Konishi from Luscher corrections. Z. Bajnok, R. Janik e-Print: arXiv:1209.0791

  Here the main problem is the very reduction to masters (the way to compute the resulting masters is known /K.Ch. and Baikov, 2010). BUT: shear # of contributing diagrams in "normal" gauge theories would presumably be prohibitively large for, say, QCD 6-loop β-function.