Number Theory of Radiative Corrections

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Topics

(1) The ubiquitous multiple polylogarithm (MPL)
(2) An MPL in quantum field theory (QFT): unsolved puzzle (Schwinger, 1949)
(3) Three-loop 6-point MPLs in a superfluous model (Dixon et al, 10 Aug 2013)
(4) Beyond MPL: a small toolkit of modular forms
(5) BMPL: an elliptic dilogarithm (Bloch and Vanhove, 23 Sept 2013)
(6) B^{2}MPL: modular forms in massless QFT (Brown and Schnetz)
(7) B^{3}MPL: modular forms in massive QFT (with Francis Brown)

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1 The ubiquitous MPL

An MPL is defined, recursively, by a **word** and a complex variable at the upper limit of an integration that prepends a **letter** to a shorter word:

$$G(\mathbf{aw}, z) = \int_0^z \frac{\mathrm{d}t}{t - a} G(\mathbf{w}, t) \; ; \quad G(\{\mathbf{0}\}_n, t) = \frac{\log^n(t)}{n!}.$$

This definition subsumes harmonic polylogarithms (HPLs), nested sums of the form

$$L_{s_1,\ldots,s_k}(z_1, \ldots, z_k) = \sum_{n_1>n_2>\ldots>n_k>0} \prod_{j=1}^{k} \frac{z_j^{n_j}}{n_j^{s_j}}$$

and hence subsumes the MZV datamine, where $z_j^2 = 1$ was studied with Johannes Blümlein and Jos Vermaseren (BBV), and also my favourite MPLs, with $z_j^6 = 1$, in `arXiv:hep-th/9803091`.

Good maths: a ring with a shuffle algebra, solved by Lyndon words, and a co-product leading to a Hopf algebra. For MZVs: a double-shuffle algebra.

Utility: several talks at this conference exploit this mathematical structure, to good effect, in the service of successful standard-model phenomenology.

2 The first MPL in QFT: an unsolved puzzle

Following the Durham LMS conference on polylogs, in June 2013, Jianqiang Zhao, Freeman Dyson (né 15 December 1923), Johannes Blümlein and I sought the oldest MPL in QFT. Our best candidate comes from
Julian Schwinger, Phys. Rev. 79 (1949) 790–817
with Eq. (2.97) on page 812 giving

$$f(\theta) = \frac{1}{\sin(\theta/2)} \int_{\cos(\theta/2)}^{1} \frac{F(x) - F(-x)}{\sqrt{x^2 - \cos^2(\theta/2)}} \mathrm{d}x \ ; \quad F(x) = \frac{\log(1+x) - \log(2)}{1 - x}$$

with $f(\pi) = \frac{1}{2}\zeta(2)$ evaluated by Julian. PSLQ gives the unproven evaluation

$$f(\pi/2) \stackrel{?}{=} \frac{1}{2}\zeta(2) + \frac{1}{2}\log^2(2).$$

Moreover, I conjecture a general evaluation in terms of an MPL:

$$f(\theta) \stackrel{?}{=} \frac{\zeta(2) + G(\mathbf{01}, \cos^2(\theta/2))}{1 - \cos(\theta)}.$$

Lumley Castle ghost-free MPL challenge: A liquid prize for a **proof** of this formula, while Nigel Glover's malt whisky stocks persist.

3 Three-loop 6-point MPLs in a superfluous model

Imagine the (infrared pathological) on–shell limit of a Yang–Mills theory with $N = 4$ superfluous symmetries (SUSYs) combined with Gerard 'tHooft's trick of letting the coupling tend to zero and the number of colours tend to infinity, in such a manner that only the very sparse set of planar diagrams survives.Then we need no ultra-violet renormalization, no dimensional transmutation and probably obtain no mass gap and hence no physics. But the maths is beautiful. Computing no Feynman diagrams, Lance Dixon, James Drummond, Matt von Hippel and Jeffrey Pennington announced in `http://arxiv.org/abs/1308.2276`, last month, an evaluation of an infra-red regularized 3-loop 6-point amplitude, with conformal cross–ratios $\{u, v, w\}$, in terms of MPLs with words of up to 6 letters in the 11-letter alphabet

$$\{0, 1, u, v, w, 1 - u, 1 - v, 1 - w, y_u, y_v, y_w\}$$

$$y_u = \frac{u - z_+}{u - z_-}\,; \quad y_v = \frac{v - z_+}{v - z_-}\,; \quad y_w = \frac{w - z_+}{w - z_-}\,;$$

$$2z_\pm = -1 + u + v + w \pm \sqrt{(-1 + u + v + w)^2 - 4uvw}\,.$$

They identified a basis of 69 irreducible MPLs with less than 6 letters. There are 105 irreducible MPLs with precisely 6 letters. They were able to provide a complete reduction to MPLs of a 3-loop 6-point remainder function, in this planar $N = 4$ super–Yang–Mills model, without needing to identify a complete 6-letter basis.

4 Beyond MPL: a small toolkit of modular forms

For $|q| < 1$, let

$$\eta(q) \equiv q^{1/24} \prod_{n>0} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}$$

then for $\Im z > 0$,

$$\eta(\exp(2\pi \mathrm{i} z)) = (\mathrm{i}/z)^{1/2} \eta(\exp(-2\pi \mathrm{i}/z)).$$

If $f(z) = (\sqrt{-N}/z)^w f(-N/z)$, we say that f is a modular form of weight w and level N. Example with weight 12 and level 1:

$$\eta(q)^{24} = \sum_{n>0} A(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \ldots$$

Its Fourier coefficients are multiplicative: $A(mn) = A(m)A(n)$ for $\gcd(m,n) = 1$, and are determined by $A(p)$ at the primes p:

$$L(s) \equiv \sum_{n>0} \frac{A(n)}{n^s} = \prod_p \frac{1}{1 - A(p)p^{-s} + p^{11-2s}}.$$

Moreover, we can analytically continue to values inside the critical strip:

$$\Lambda(s) \equiv \frac{\Gamma(s)}{(2\pi)^s} L(s) = \sum_{n>0} A(n) \int_1^\infty \mathrm{d}x \left(x^{s-1} + x^{11-s}\right) \exp(-2\pi x) = \Lambda(12 - s).$$

4.1 Multiplicative modular forms from eta-products

For brevity, let $\eta_n \equiv \eta(q^n)$. Here are some multiplicative modular forms identified in quantum field theory

form	weight	level	QFT
$\eta_1^3\eta_7^3$	3	7	BS
$\eta_1^2\eta_2\eta_4\eta_8^2$	3	8	BS
$\eta_2^3\eta_6^3$	3	12	BS + BFT + BBBG + BV
$\eta_1^4\eta_5^4$	4	5	BS
$\eta_1^2\eta_2^2\eta_3^2\eta_6^2$	4	6	BS + BB
$\eta_1^4\eta_2^2\eta_4^4$	5	4	BS
$\eta_1^6\eta_3^6$	6	3	BS
η_2^{12}	6	4	BS
$\eta_1^8\eta_2^8$	8	2	BS
η_1^{24}	12	1	BK

by Bailey, Borwein, Broadhurst, Glasser (BBBG), Bloch, Vanhove (BV), Broadhurst, Brown (BB), Broadhurst, Fleischer, Tarasov (BFT), Broadhurst, Kreimer (BK), and Brown, Schnetz (BS).
Comment: QFT seems blind to Birch and Swinnerton–Dyer: nothing at weight 2.

5 BMPL: an elliptic dilogarithm

Consider the two-loop massive sunrise diagram in $D = 2$ spacetime dimensions:

$$I(p^2, m_1, m_2, m_3) \equiv \frac{1}{\pi^2} \left(\prod_{k=1}^{3} \int \frac{\mathrm{d}^2 q_k}{q_k^2 - m_k^2 + \mathrm{i}\epsilon} \right) \delta^{(2)}(p - q_1 - q_2 - q_3).$$

Following BBBG, we obtain an efficient result from the imaginary part on the cut:

$$I(w^2, m_1, m_2, m_3) = 8\pi \int_{m_1+m_2+m_3}^{\infty} \frac{A(x) x \mathrm{d}x}{x^2 - w^2}$$

with an elliptic integral

$$A(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(w) - F(-w)}\right)}$$

that is the reciprocal of an **arithmetic–geometric mean** with

$F(w) = (w+m_1+m_2+m_3)(w+m_1-m_2-m_3)(w-m_1+m_2-m_3)(w-m_1-m_2+m_3)$.

From the complementary elliptic integral

$$B(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(-w)}\right)}$$

we obtain the **elliptic nome**

$$q(w) \equiv \exp(-\pi B(w)/A(w)).$$

5.1 Differential equation in the equal–mass case

Now set $m_1 = m_2 = m_3 = 1$. Then $F(w) = (w+3)(w-1)^3$ and the differential equation, found with Jochem Fleischer (sadly deceased in April 2013) and Oleg Tarasov (BFT) in 1993, is

$$-\left(\frac{q(w)}{q'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right)^2\left(\frac{I(w^2,1,1,1)}{24\sqrt{3}A(w)}\right) = \frac{w^2(w^2-1)(w^2-9)A(w)^3}{9\sqrt{3}}.$$

Regarding w and $A(w)$ as functions of q, we have a parametric solution

$$\frac{w}{3} = \left(\frac{\eta_3}{\eta_1}\right)^4\left(\frac{\eta_2}{\eta_6}\right)^2, \quad 4\sqrt{3}A = \frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2}.$$

Moreover, the two algebraic relations between $\{\eta_1, \eta_2, \eta_3, \eta_6\}$ give

$$\frac{w^2-1}{8} = \left(\frac{\eta_2}{\eta_1}\right)^9\left(\frac{\eta_3}{\eta_6}\right)^3, \quad \frac{w^2-9}{72} = \left(\frac{\eta_6}{\eta_1}\right)^5\frac{\eta_2}{\eta_3}.$$

Hence the BFT differential equation reduces to

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2\left(\frac{I}{24\sqrt{3}A}\right) = \frac{w}{3}f_{3,12} = \left(\frac{\eta_3^3}{\eta_1}\right)^3 + \left(\frac{\eta_6^3}{\eta_2}\right)^3$$

where, remarkably, $f_{3,12} \equiv (\eta_2\eta_6)^3$ is a weight-3 level-12 modular form found in massless ϕ^4 theory by Brown and Schnetz.

5.2 Bloch–Vanhove elliptic dilogarithm

Define a character with $\chi(n) = \pm 1$ for $n = \pm 1 \bmod 6$ and $\chi(n) = 0$ otherwise. Then

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{I}{24\sqrt{3}A}\right) = \sum_{n>0} \frac{n^2(q^n - q^{5n})}{1-q^{6n}} = \sum_{n>0}\sum_{k>0} n^2\chi(k)q^{nk}.$$

Integrate twice and use the known imaginary part on the cut, to obtain

$$\frac{I(w^2,1,1,1)}{4A(w)} = L(-1) - L\left(\mathrm{e}^{-\pi B(w)/A(w)}\right), \quad L(q) = \pi\log(-q) + \sum_{k>0}\frac{6\sqrt{3}\chi(k)q^k}{k^2(1-q^k)},$$

where the Clausen value $L(-1) = -5\,\mathrm{Cl}_2(\pi/3)$ makes $I(1,1,1,1)$ finite.

For the deeper meaning, please consult `http://arxiv.org/abs/1309.5865` by Spencer Bloch and Pierre Vanhove. Here, I have sought merely to find a short route from the differential equation of BFT to the final BV formula. By doing so, I omit their intuition and perspective, which are more important than the result.

6 $\mathbf{B^2MPL}$: modular forms in massless QFT

In 1995, Dirk Kreimer and I evaluted all periods for ϕ^4 primitive divergences up to 6 loops. At 7 loops we lacked three evaluations. Of these, I have evaluated two:

$$\begin{aligned} P_{7,8} &= \frac{22383}{20}\zeta(11) - \frac{4572}{5}[\zeta(3)\zeta(5,3) - \zeta(3,5,3)] - 700\zeta(3)^2\zeta(5) \\ &\quad + 1792\zeta(3)\left[\frac{27}{80}\zeta(5,3) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right], \\ P_{7,9} &= \frac{92943}{160}\zeta(11) - \frac{3381}{20}[\zeta(3)\zeta(5,3) - \zeta(3,5,3)] - \frac{1155}{4}\zeta(3)^2\zeta(5) \\ &\quad + 896\zeta(3)\left[\frac{27}{80}\zeta(5,3) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right]. \end{aligned}$$

The period $P_{7,11}$ in the Schnetz census has not been reduced to MZVs. Francis Brown suggests that it might eventually be reduced to polylogs of weight 11 at sixth roots of unity. Erik Panzer is working on this very demanding problem.

In April 2013, Francis Brown and Oliver Schnetz (BS) announced a study that classifies obstructions to polylogarithmic evaluations of ϕ^4 counterterms at 8, 9 and 10 loops. In 16 cases they were able to exhibit a modular form, inferred from study of the Symanzik polynomial, modulo a selection of primes. Here I select $f_{3,12} \equiv (\eta_2\eta_6)^3$ and $f_{4,6} \equiv (\eta_1\eta_2\eta_3\eta_6)^2$. The first figured in the Bloch–Vanhove (BV) sunrise diagram. Now I use the second. **Comment:** massless and massive diagrams communicate.

7 B^{3}MPL: modular forms in massive QFT

This week, Ettore Remiddi sent encouraging news about Stefano Laporta's epic efforts to achieve thousands of digits of accuracy for the very large number of master integrals that contribute to the 4-loop magnetic moment of the electron.

Here I examine the number theory of the most demanding case: with 5 fermions in the intermediate state. As in the Bloch–Vanhove case, I avoid UV problems by working in $D = 2$ dimensions, where the number theory is expected to be the same as for $D = 4$. Contracting the photons lines and replacing the fermion lines by massive scalar propagators we arrive at the on-shell sunrise integral $S_{6,4}$, where

$$S_{N,L} \equiv 2^L \int_0^\infty I_0(y)^{N-L-1} K_0(y)^{L+1} y \mathrm{d}y.$$

Here N is the total number of Bessel functions and L is the number of loops. For convergence, we require that $L < N \leq 2L+2$. With $N = 2L+2$ we require that $L > 1$. BBBG proved that:

$$S_{1,0} = S_{2,1} = 1, \quad S_{3,1} = \frac{2\pi}{3\sqrt{3}}, \quad S_{3,2} = \frac{4\,\mathrm{Cl}_2(\pi/3)}{\sqrt{3}}, \quad S_{4,2} = \frac{\pi^2}{4}, \quad S_{4,3} = 7\zeta(3),$$

$$S_{5,2} = \frac{\pi^2}{8}\left(\sqrt{15}-\sqrt{3}\right)\left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\sqrt{15}\pi n^2}\right)^4 = \frac{\sqrt{3}}{120\pi}\,\Gamma(1/15)\Gamma(2/15)\Gamma(4/15)\Gamma(8/15)$$

where the final product of Gamma values results from the Chowla–Selberg theorem.

Remark: York Schröder needed a counterterm in 3-dimensional lattice field theory for which we used Chowla–Selberg to obtain the product $\Gamma^2(1/24)\Gamma^2(11/24)$.

BBBG also conjectured (and checked to 1000 digits) that

$$S_{5,3} = \frac{4\pi}{\sqrt{15}} S_{5,2}, \quad S_{6,4} = \frac{4\pi^2}{3} S_{6,2}, \quad S_{8,5} = \frac{18\pi^2}{7} S_{8,3}.$$

7.1 Sunrise at 3 loops from a modular form of weight 3

Let $L_{3,15}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{3,15} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$$

with weight 3 and level 15. Then I conjecture (and have checked to 1000 digits) that

$$S_{5,2} = 3L_{3,15}(2), \quad S_{5,3} = \frac{8\pi^2}{15} L_{3,15}(1),$$

where $S_{5,3}$ is the 5-Bessel moment giving the on–shell 3–loop sunrise diagram.

7.2 Sunrise at 4 loops from a modular form of weight 4

Let $L_{4,6}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. Then I conjecture (and have checked to 1000 digits) that

$$S_{6,2} = 6L_{4,6}(2), \quad S_{6,3} = 12L_{4,6}(3), \quad S_{6,4} = 8\pi^2 L_{4,6}(2),$$

where $S_{6,4}$ is the 6-Bessel moment giving the on–shell 4–loop sunrise diagram.

7.3 Almost sunrise at 6 loops from a modular form of weight 6

Let $L_{6,6}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. Then I conjecture (and have checked to 1000 digits) that

$$S_{8,3} = 8L_{6,6}(3), \quad S_{8,4} = 36L_{4,6}(4), \quad S_{8,5} = 216L_{4,6}(5),$$

but no result for $S_{8,6}$, the 8-Bessel moment for the on–shell 6–loop sunrise diagram.

Epilogue

Thus far, with rough and all–unable pen,
Our bending author hath pursued the story,
In little room confining mighty men,
Mangling by starts the full course of their glory.