A TALE OF TWO FACTORIZATIONS

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Outline

- Soft-collinear factorization
- The dipole formula
- Regge factorization
- Dipoles at high-energy
- Face to face: one loop
- Face to face: higher orders
- Outlook

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In collaboration with
Vittorio Del Duca
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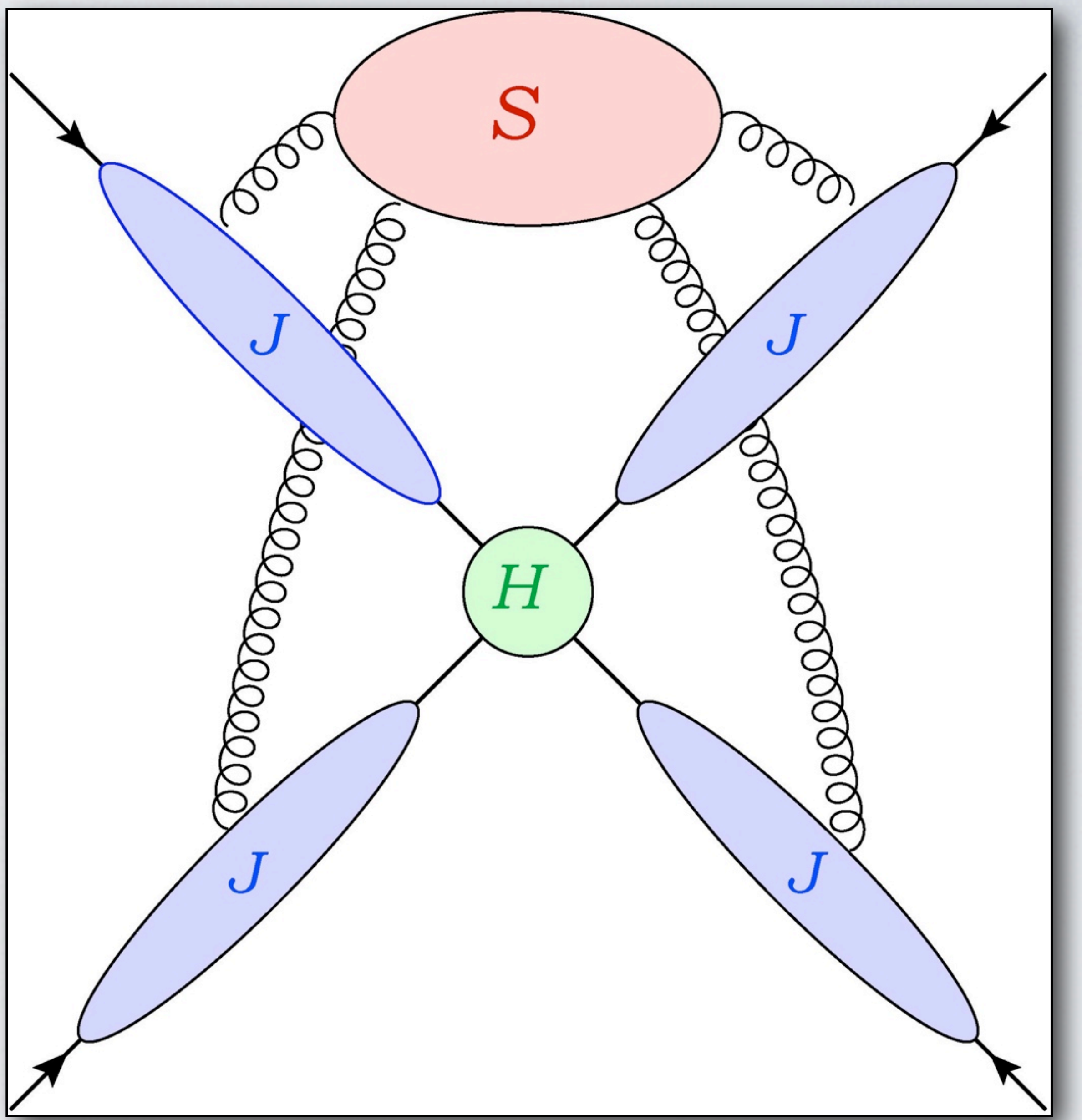
Leonardo Vernazza

SOFT-COLLINEAR FACTORIZATION

Soft-collinear factorization

- Divergences arise in scattering amplitudes from leading regions in loop momentum space.
- Power-counting arguments show that soft gluons decouple from the hard subgraph.
- Ward identities decouple soft gluons from jets and restrict color transfer to the hard part.
- Jet functions J represent color singlet evolution of external hard partons.
- The soft function S is a matrix mixing the available color representations.
- In the planar limit soft exchanges are confined to wedges: S is proportional to the identity.
- Beyond the planar limit S is determined by an anomalous dimension matrix Γ_S.
- The matrix Γ_S correlates color exchange with kinematic dependence.

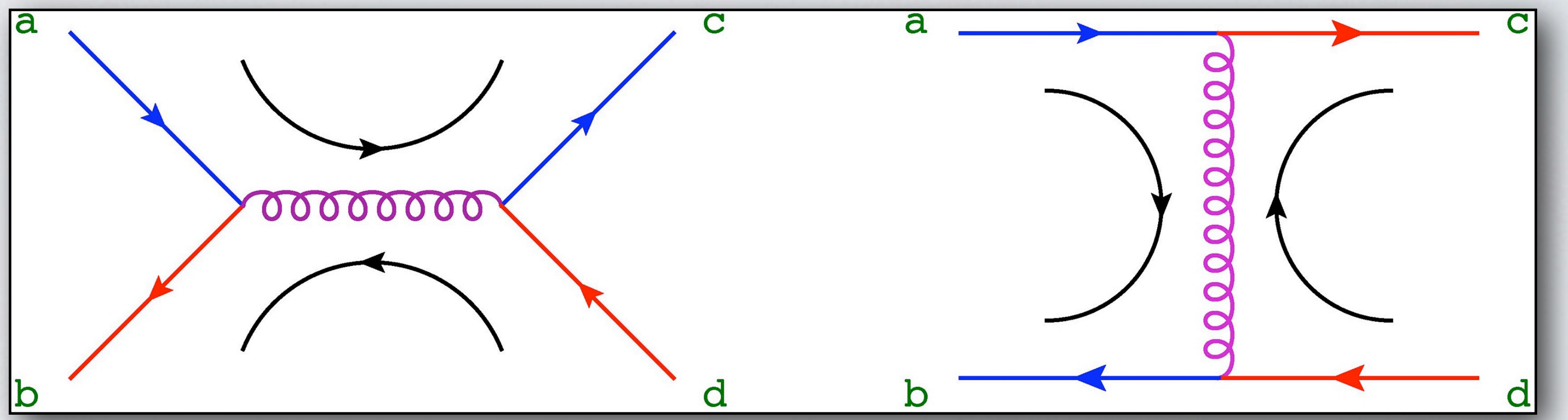


Leading integration regions in loop momentum space for Sudakov factorization

Color flow

In order to understand the matrix structure of the soft function it is sufficient to consider the simple case of quark-antiquark scattering.

At tree level



Tree-level diagrams and color flows for quark-antiquark scattering

For this process only two color structures are possible. A basis in the space of available color tensors is

$$c^{(1)}_{abcd} = \delta_{ab}\delta_{cd}\,, \qquad c^{(2)}_{abcd} = \delta_{ac}\delta_{bd}$$

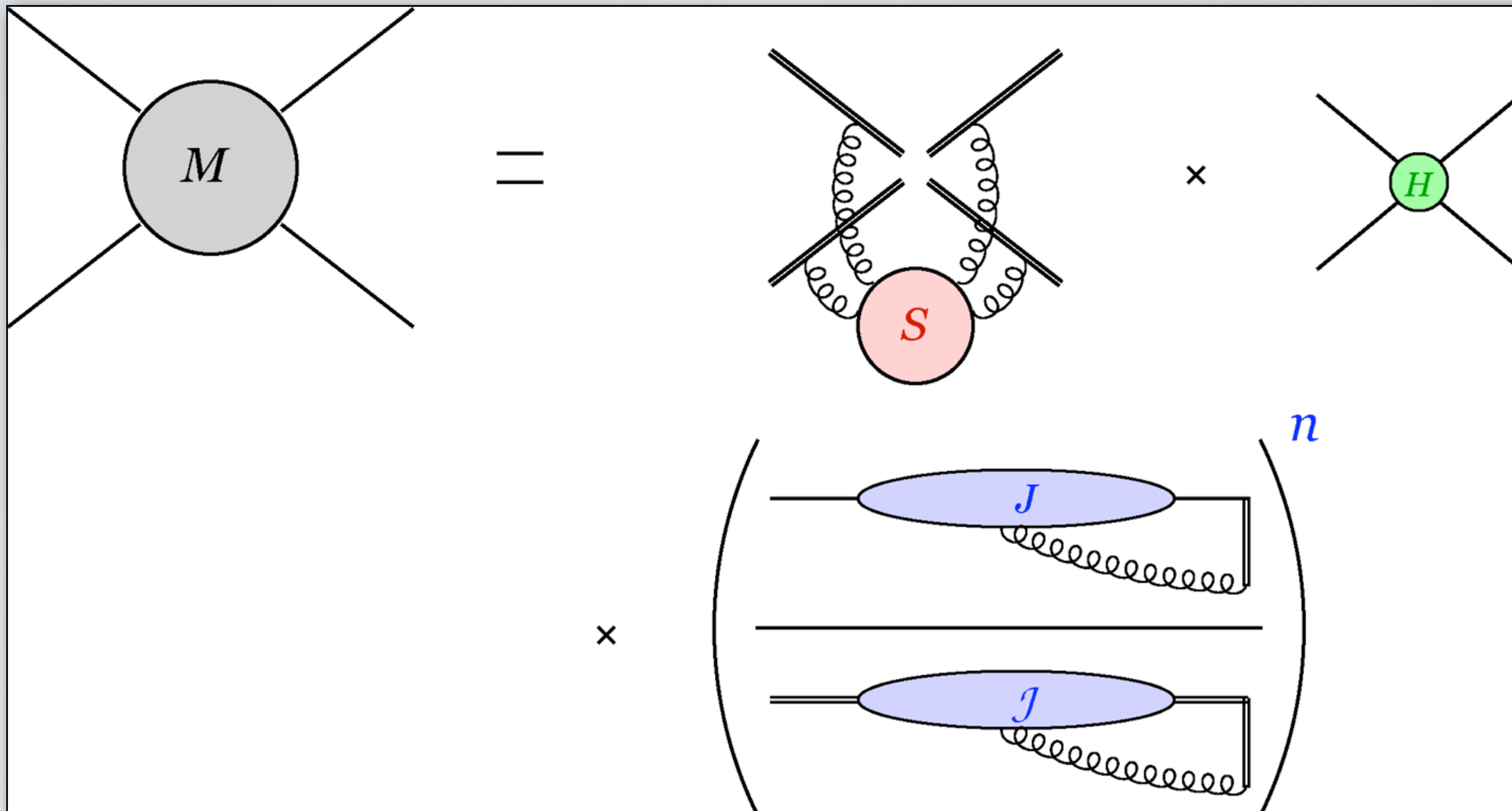
The matrix element is a vector in this space, and the Born cross section is

$$\mathcal{M}_{abcd} = \mathcal{M}_1\, c^{(1)}_{abcd} + \mathcal{M}_2\, c^{(2)}_{abcd} \longrightarrow \sum_{\text{color}} |\mathcal{M}|^2 = \sum_{J,L} \mathcal{M}_J\, \mathcal{M}_L^*\ \text{tr}\left[c^{(J)}_{abcd}\left(c^{(L)}_{abcd}\right)^\dagger\right] \equiv \text{Tr}\Big[HS\Big]_0$$

A virtual soft gluon will reshuffle color and mix the components of this vector

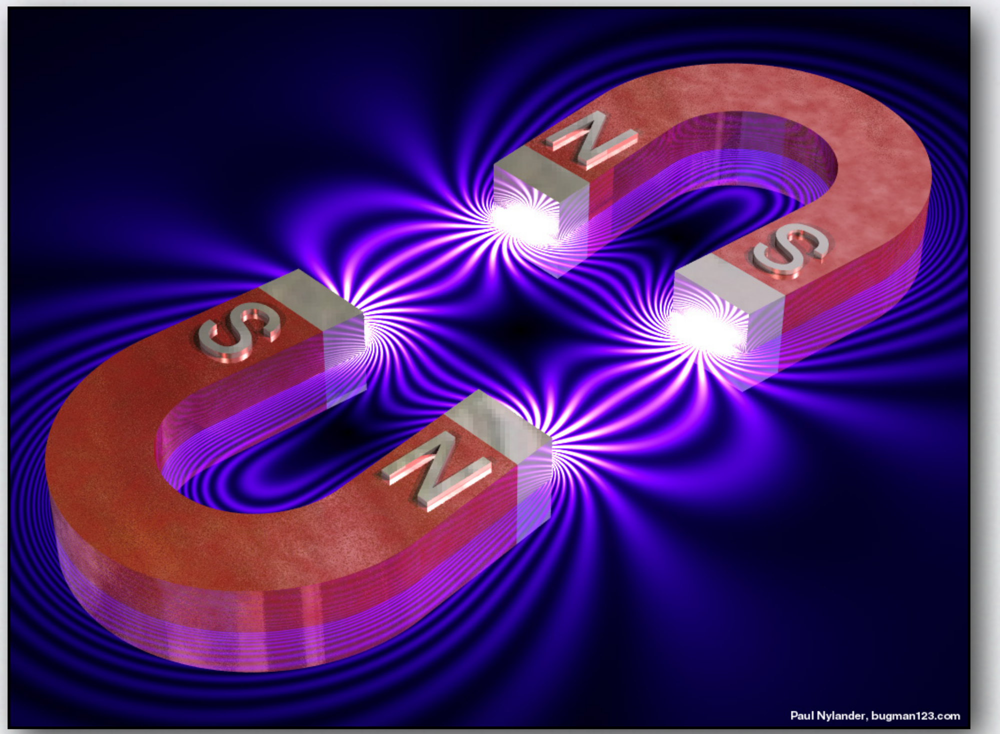
$$\text{QED}: \quad \mathcal{M}_{\text{div}} = S_{\text{div}}\mathcal{M}_{\text{Born}}\,; \qquad \text{QCD}: \quad [\mathcal{M}_{\text{div}}]_J = [S_{\text{div}}]_{JL}\,[\mathcal{M}_{\text{Born}}]_L$$

Soft-collinear factorization: pictorial



A pictorial representation of Sudakov factorization for fixed-angle scattering amplitudes

THE DIPOLE FORMULA



The Dipole Formula

For massless partons, the soft anomalous dimension matrix obeys a set of exact equations that correlate color exchange with kinematics.

The simplest solution to these equations is a sum over color dipoles (Becher, Neubert; Gardi, LM, 09). It gives an ansatz for the all-order singularity structure of all multiparton fixed-angle massless scattering amplitudes: the dipole formula.

- All soft and collinear singularities can be collected in a multiplicative operator Z

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \, \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

- Z contains both soft singularities from S, and collinear ones from the jet functions. It must satisfy its own matrix RG equation

$$\frac{d}{d\ln\mu} Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = - Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) .$$

The matrix Γ has a surprisingly simple dipole structure. It reads

$$\boxed{\Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right)\sum_{j\neq i}\ln\left(\frac{-2\,p_i\cdot p_j}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s(\mu^2)\right) .}$$

Recall that all singularities are generated by integration over the scale of the coupling.

Features of the dipole formula

- All known results for IR divergences of massless gauge theory amplitudes are recovered.
- The absence of multiparton correlations implies remarkable diagrammatic cancellations.
- The color matrix structure is fixed at one loop: path-ordering is not needed.
- All divergences are determined by a handful of anomalous dimensions.
- The cusp anomalous dimension plays a very special role: a universal IR coupling.

Can this be the definitive answer for IR divergences in massless non-abelian gauge theories?

- There are precisely two sources of possible corrections.
  - Quadrupole correlations may enter starting at three loops: they must be tightly constrained functions of conformal cross ratios of parton momenta.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) \,, \qquad \rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_k \, p_j \cdot p_l}$$

  - The cusp anomalous dimension may violate Casimir scaling beyond three loops.

$$\gamma_K^{(i)}(\alpha_s) = C_i \, \widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)$$

  - The functional form of Δ is further constrained by: collinear limits, Bose symmetry, bounds on weights, high-energy constraints. (Becher, Neubert; Dixon, Gardi, LM, 09).
  - A four-loop analysis indicates that Casimir scaling holds (Becher, Neubert, Vernazza).

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$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\rm di}\ldots \left(p_i \ldots\right) + \Delta\left(\rho_{ij}\ldots\right) \qquad \ldots \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_k \, p_j \cdot p_l}$$

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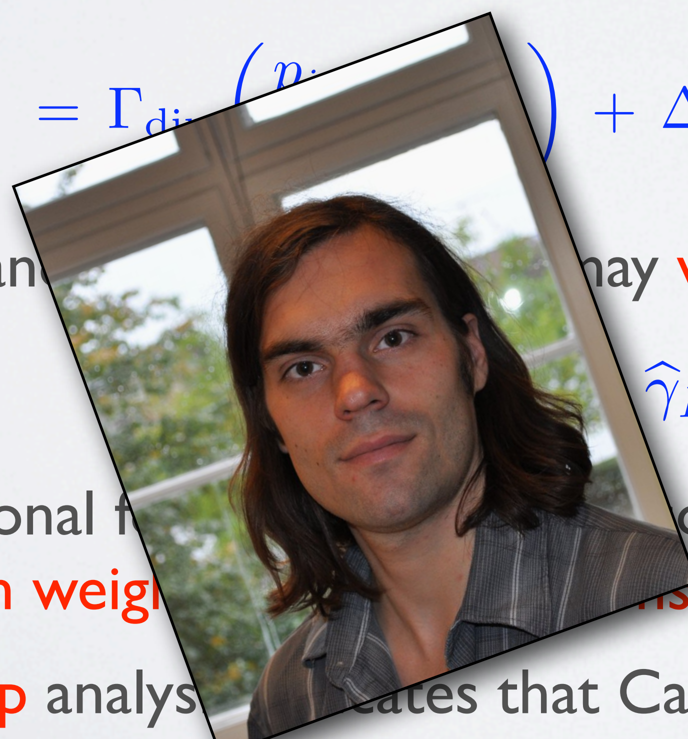

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REGGE FACTORIZATION

Regge Poles

- Studies of the high-energy limit of scattering amplitudes predate the construction of the Standard Model of particle physics.
- A powerful tool in S-matrix theory is the analytic continuation to complex angular momentum. Start with the well known partial wave expansion

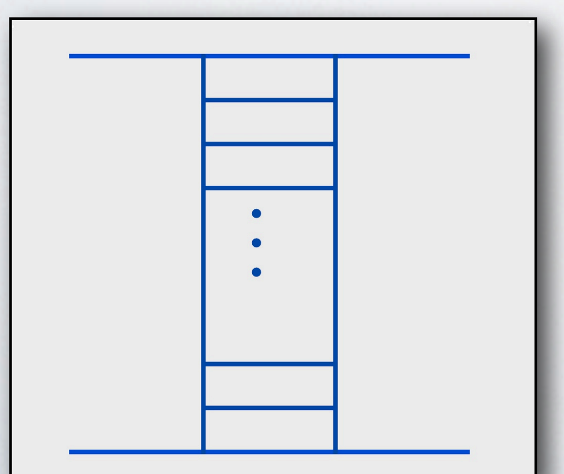
$$A(s,t) = 16\,\pi \sum_{l=0}^{\infty} (2l+1)\; a_l(s)\, P_l(\cos\theta_t)$$

- Moving to the crossed (t-) channel, using dispersion relations and overcoming several technical subtleties one finds a representation for the t-channel partial wave amplitude

$$a_l^{\mathcal{S}}(t) \;=\; \frac{1}{16\pi^2}\int_{\cos\theta_s^0}^{\infty} D^{\mathcal{S}}\,(\cos_{\theta_s}, t)\;\, Q_l(\cos\theta_s)\, d\cos\theta_s$$

- Singularities of $a_l(t)$ in the L plane determine the high-energy behavior of the amplitude: In the case of simple poles one gets

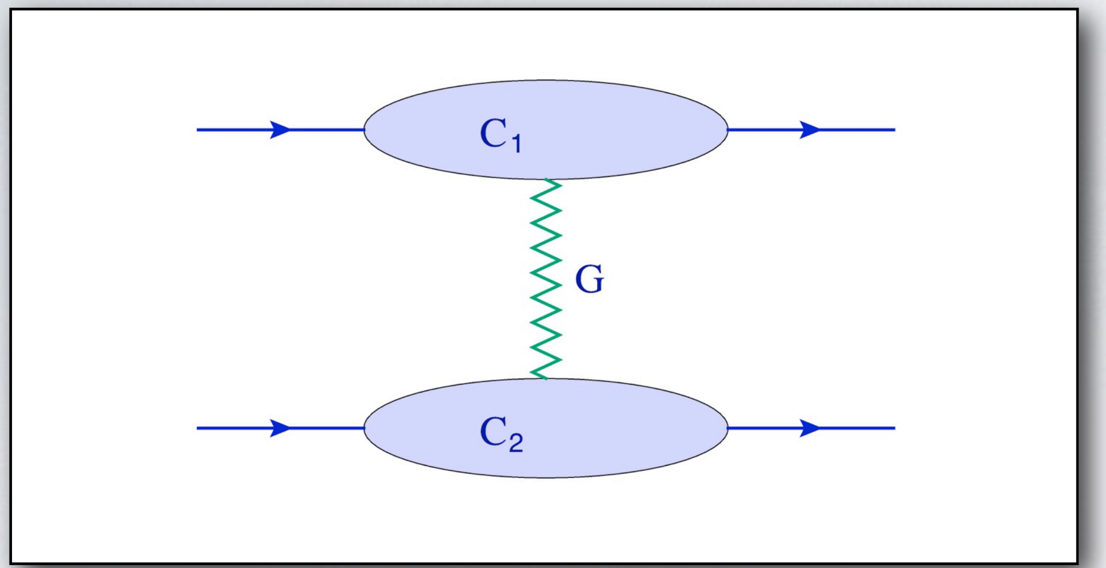
$$a_l^{\mathcal{S}}(t) \sim \frac{1}{l-\alpha(t)} \qquad \longrightarrow \qquad A\,(s,t) \xrightarrow{s\to\infty} f(t)\, s^{\alpha(t)}\,,$$



- The above is derived from the analiticity of the S-matrix, with no reference to a Lagrangian field theory.
- In perturbation theory, the same high-energy behavior is recovered through the summation of ladder diagrams.
- The Regge trajectory $\alpha(t)$ starting at one-loop is given by an IR divergent transverse momentum integral.

Perturbative Reggeization

- In perturbative QCD the high-energy limit is governed by t-channel parton exchange.
- In the t/s → 0 limit gluons in the t-channel `Reggeize' with a computable trajectory.



Quark-quark scattering: the t-channel gluon Reggeizes

- Large logarithms of s/t are generated by a simple replacement of the t-channel propagator,

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

- The Regge trajectory has a perturbative expansion, with IR divergent coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right)$$

- The gluon has been shown to Reggeize at NLL, and the two-loop Regge trajectory is known.
- For example, for gluon-gluon scattering the matrix element obeys Regge factorization

$$\mathcal{M}^{gg\to gg}_{a_1a_2a_3a_4}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1,k_3)\right]\left(\frac{s}{-t}\right)^{\alpha(t)}\left[(T_b)_{a_2a_4}C_{\lambda_2\lambda_4}(k_2,k_4)\right]$$

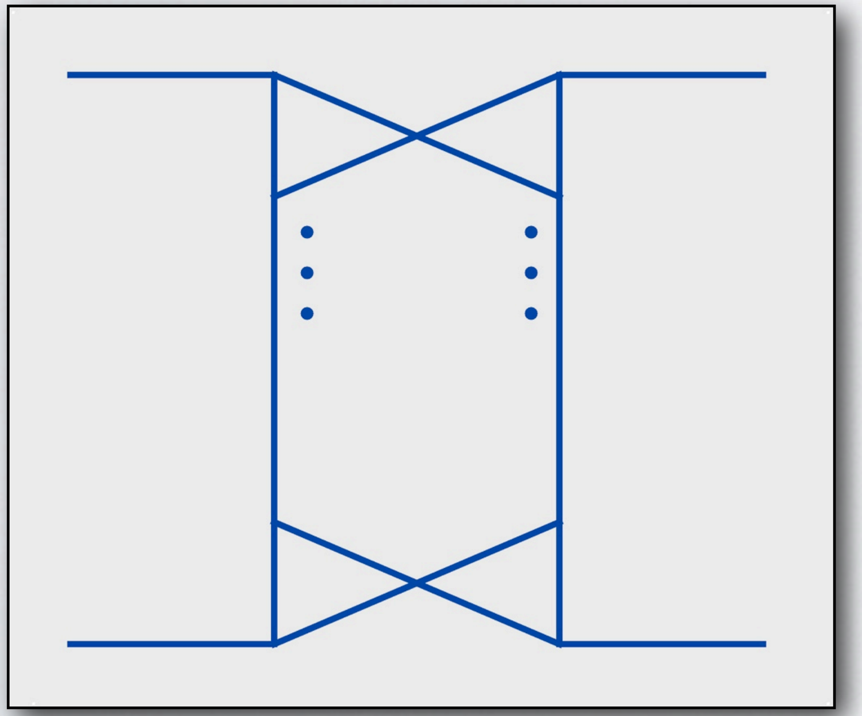
with the perturbative coefficients

$$\alpha^{(1)} = C_A\,\frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A\,\frac{2}{\epsilon} \qquad \alpha^{(2)} = C_A\left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)}\,\frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

Regge Cuts

- One may wonder how the breakdown of simple Regge factorization can be put in the context of the general results of Regge theory.
- Reggeization follows from the assumption that the only singularities in the complex angular momentum plane are isolated poles.
- From the early days of Regge theory it was understood that the the picture would become more intricate in the presence of cuts in the L plane
- Regge cuts can arise when at least two `Reggeons' are exchanged in the t channel (two ladders in perturbation theory)

On general grounds one can show that:



Mandelstam's `double-cross' diagram

- Regge cuts do not arise in the physical region from planar diagrams.
- The first nontrivial contribution from a Regge cut arises from the three-loop non-planar Mandelstam `double-cross' diagram.
- Regge cuts in the physical region arise at leading power in s only if the high energy limit picks up the discontinuity of an energy logarithm.

These properties are in agreement with our findings at three loops and beyond.

HIGH-ENERGY DIPOLES

Del Duca, Duhr, Gardi, LM, White 2011

The dipole formula at high energy

- Introducing `Mandelstam' color operators, and using color and momentum conservation

$$\begin{aligned} \mathbf{T}_s &= \mathbf{T}_1 + \mathbf{T}_2 = -(\mathbf{T}_3 + \mathbf{T}_4) \,, \\ \mathbf{T}_t &= \mathbf{T}_1 + \mathbf{T}_3 = -(\mathbf{T}_2 + \mathbf{T}_4) \,, \\ \mathbf{T}_u &= \mathbf{T}_1 + \mathbf{T}_4 = -(\mathbf{T}_2 + \mathbf{T}_3) \end{aligned}$$

$$s + t + u = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i$$

it is easy to see that the infrared dipole operator Z factorizes in the high-energy limit

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

- The operator Z_1 is s-independent and proportional to the unit matrix in color space.
- Color dependence and s dependence are collected in the factor

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + i\pi\,\mathbf{T}_s^2\right]\right\},$$

where the coupling dependence is (once again!) completely determined by the cusp anomalous dimension and by the β function, through the function (Korchemsky 94-96)

$$K\Big(\alpha_s(\mu^2), \epsilon\Big) \equiv -\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right)$$

- The simple structure of the high-energy operator governs Reggeization and its breaking.

Reggeization of leading logarithms

- At leading logarithmic accuracy, the (imaginary) s-channel contribution can be dropped, and the dipole operator becomes diagonal in a t-channel basis.

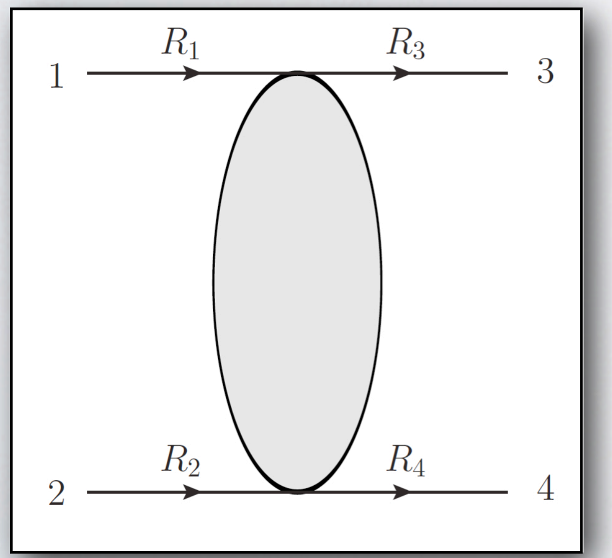
$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big) \ln\left(\frac{s}{-t}\right) \mathbf{T}_t^2\right\} Z_{\mathbf{1}}\, \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

- If, at LO and at leading power in t/s, the scattering is dominated by t-channel exchange, then the hard function is an eigenstate of the color operator $\mathbf{T}_t^2$

$$\mathbf{T}_t^2\, \mathcal{H}^{gg\to gg} \xrightarrow{|t/s|\to 0} C_t\, \mathcal{H}_t^{gg\to gg}$$

- Leading-logarithmic Reggeization for arbitrary t-channel color representations follows

$$\mathcal{M}^{gg\to gg} = \left(\frac{s}{-t}\right)^{C_A\, K\left(\alpha_s(\mu^2),\epsilon\right)} Z_{\mathbf{1}}\, \mathcal{H}_t^{gg\to gg}$$



Scattering for generic color exchange

- The LL Regge trajectory is universal and obeys Casimir scaling.
- Scattering of arbitrary color representations can be analyzed
  Example: let 1 and 2 be antiquarks, 4 a gluon and 3 a sextet; use

$$\overline{\mathbf{3}} \otimes \mathbf{6} = \mathbf{3} \oplus \mathbf{15} \qquad\qquad \overline{\mathbf{3}} \otimes \mathbf{8}_a = \overline{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{15}}$$

LL Reggeization of the **3** and **15** t-channel exchanges follows.

Beyond leading logarithms

- The high-energy infrared operator can be systematically expanded beyond LL, using the Baker-Campbell-Hausdorff formula. At NLL one finds a series of commutators

$$\widetilde{Z}\left(\frac{s}{t},\alpha_s,\epsilon\right)\Big|_{\rm NLL} = \left(\frac{s}{-t}\right)^{K(\alpha_s,\epsilon)\,\mathbf{T}_t^2}\Bigg\{1+{\rm i}\,\pi K\left(\alpha_s,\epsilon\right)\left[\mathbf{T}_s^2-\frac{K\left(\alpha_s,\epsilon\right)}{2!}\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right] + \frac{K^2\left(\alpha_s,\epsilon\right)}{3!}\ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]+\ldots\right]\Bigg\}$$

- The real part of the amplitude Reggeizes also at NLL for arbitrary t-channel exchanges.
- At NNLL Reggeization generically breaks down also for the real part of the amplitude.
  - At two loops, terms that are non-logarithmic and non-diagonal in a t-channel basis arise

$$\mathcal{E}_0\left(\alpha_s,\epsilon\right) \equiv -\frac{1}{2}\pi^2\,K^2\left(\alpha_s,\epsilon\right)\,\left(\mathbf{T}_s^2\right)^2$$

  - At three loops, the first Reggeization-breaking logarithms of s/t arise, generated by

$$\mathcal{E}_1\left(\frac{s}{t},\alpha_s,\epsilon\right) \equiv -\frac{\pi^2}{3}\,K^3(\alpha_s,\epsilon)\,\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]$$

- **NOTE**
  - In the planar limit ($N_C \to \infty$) all commutators vanish and Reggeization holds also beyond NLL (as perhaps expected from string theory).
  - Possible quadrupole corrections to the dipole formula cannot come to the rescue.

FACE-OFF: ONE LOOP

Master formulas: infrared

- We consider quark and gluon four-point amplitudes in QCD. Soft-collinear factorization leads to a \`master formula' for these amplitudes valid to leading power in t/s.

$$\mathcal{M}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s,\epsilon\right) = \mathcal{Z}_{1,\mathbf{R}}\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right)\exp\left[-\mathrm{i}\,\frac{\pi}{2}\,K\left(\alpha_s,\epsilon\right)\,\mathcal{C}_{\rm tot}\right]$$
$$\times\exp\left\{K\left(\alpha_s,\epsilon\right)\left[\log\left(\frac{s}{-t}\right)\mathbf{T}_t^2+\mathrm{i}\pi\mathbf{T}_s^2\right]\right\}\mathcal{H}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s,\epsilon\right)+\mathcal{O}\left(\frac{t}{s}\right)$$

- We have made explicit an important \`Coulomb' phase, where $\mathcal{C}_{\rm tot} = \sum_{i=1}^{4}\mathcal{C}_{[i]}$

- The color-singlet $Z_{1,R}$ factor is real and collects factors associated with the four \`jets'.

$$\mathcal{Z}_{1,\mathbf{R}}\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right) = \exp\left\{\frac{1}{2}\left[K\left(\alpha_s,\epsilon\right)\log\left(\frac{-t}{\mu^2}\right)+D\left(\alpha_s,\epsilon\right)\right]\mathcal{C}_{\rm tot}+\sum_{i=1}^{4}B_i\left(\alpha_s,\epsilon\right)\right\}$$

- Each function in the exponent has an expansion known to three loops. For example

$$K(\alpha_s,\epsilon) = \frac{\alpha_s}{\pi}\,\frac{\widehat{\gamma}_K^{(1)}}{4\epsilon}+\left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\widehat{\gamma}_K^{(2)}}{8\epsilon}-\frac{b_0\,\widehat{\gamma}_K^{(1)}}{32\epsilon^2}\right)+\left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\widehat{\gamma}_K^{(3)}}{12\epsilon}-\frac{b_0\,\widehat{\gamma}_K^{(2)}+b_1\,\widehat{\gamma}_K^{(1)}}{48\epsilon^2}+\frac{b_0^2\,\widehat{\gamma}_K^{(1)}}{192\epsilon^3}\right)+\mathcal{O}(\alpha_s^4)\,,$$

Master formulas: high-energy

- Regge factorization, under the assumption of `only poles' in the L plane, and including crossing information, also leads to a `master formula' for color-octet t-channel exchange.

$$\mathcal{M}_{ab}^{[8]}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s,\epsilon\right) = 2\pi\alpha_s\, H_{ab}^{(0),[8]}\Bigg[C_a\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right) A_+\left(\frac{s}{t},\alpha_s,\epsilon\right) C_b\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right)$$

$$+\,\kappa\, C_a\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right) A_-\left(\frac{s}{t},\alpha_s,\epsilon\right) C_b\left(\frac{t}{\mu^2},\alpha_s,\epsilon\right) + \mathcal{R}_{ab}^{[8]}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s,\epsilon\right) + \mathcal{O}\left(\frac{t}{s}\right)\Bigg],$$

- Here the signature factor $\kappa = \dfrac{4 - N_c^2}{N_c^2}$ for quarks, while $\kappa = 1$ for gluons.

- The Regge trajectory appears in the (anti)symmetrized factors

$$A_\pm\left(\frac{s}{t},\alpha_s,\epsilon\right) = \left(\frac{-s}{-t}\right)^{\alpha(t)} \pm \left(\frac{s}{-t}\right)^{\alpha(t)}$$

- Regge factorization is proved only at LL and at NLL for the real part of the amplitude, but it is valid for finite terms as well.

- We have introduced a color-octet non-factorizing remainder function $\mathcal{R}_{ab}^{[8]}$.

- The remainder function starts at NNLL for the real part, but could in principle have NLL imaginary parts. They will turn out to vanish.

Finite order expansions

- To proceed, we expand all factors in powers of the coupling and of the high-energy logarithm.

$$\widetilde{\mathcal{Z}}\left(\frac{s}{t}, \alpha_s, \epsilon\right) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \left(\frac{\alpha_s}{\pi}\right)^n \log^i\left(\frac{s}{-t}\right) \widetilde{Z}^{(n),i}(\epsilon) \,,$$

$$\mathcal{R}_{ab}^{[8]}\left(\frac{s}{\mu^2}, \frac{t}{\mu^2}, \alpha_s, \epsilon\right) = \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \left(\frac{\alpha_s}{\pi}\right)^n \log^k\left(\frac{s}{-t}\right) R_{ab}^{(n),i,[8]}\left(\frac{t}{\mu^2}, \epsilon\right) \,,$$

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- At one loop, for example, soft-collinear factorization yields

$$M^{(1),0} = \left[Z_{1,\mathbf{R}}^{(1)} + i\pi K_1 \left(\mathbf{T}_s^2 - \frac{1}{2}\mathcal{C}_{\rm tot}\right)\right] H^{(0)} + H^{(1),0},$$

$$M^{(1),1} = K_1 \mathbf{T}_t^2 H^{(0)} + H^{(1),1} \, ,$$

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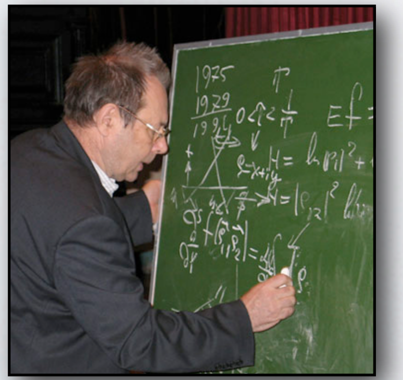
- For the octet component of the same matrix elements, Regge factorization yields



$$\begin{aligned} M_{ab}^{(1),0,[8]} &= \left[C_a^{(1)} + C_b^{(1)} - i\frac{\pi}{2}(1+\kappa)\alpha^{(1)}\right] H_{ab}^{(0),[8]}\,, \\ M_{ab}^{(1),1,[8]} &= \alpha^{(1)} H_{ab}^{(0),[8]}\,, \end{aligned}$$

One-loop results

- A comparison of the two factorizations, even in this simple setting, yields interesting results.
- Obviously, the LL hard imaginary part vanishes $$\mathrm{Im}\left[H^{(1),1,[8]}\right] = 0\,.$$
- The Regge trajectory has the expected expression

$$\alpha^{(1)} = \frac{K_1 \mathbf{T}_t^2 H^{(0)}}{H^{(0),[8]}} + \mathrm{Re}\left[H^{(1),1,[8]}\right] = C_A K_1 + \mathcal{O}(\epsilon)\,.$$

- The NLL hard imaginary part shows interesting structure

$$\mathrm{Im}\left[H^{(1),0,[8]}\right] = -\frac{\pi}{2}(1+\kappa)\,\mathrm{Re}\left[H^{(1),1,[8]}\right] - \frac{\pi}{2}K_1\left(\left[\mathcal{C}_{\rm tot} + 2\mathbf{T}_s^2 + (1+\kappa)\mathbf{T}_t^2\right]H^{(0)}\right)^{[8]}$$

- Finiteness of H requires a \`curious identity' which is indeed satisfied for both quarks and gluons $$\left[\mathcal{C}_{\rm tot} + 2\mathbf{T}_s^2 + (1+\kappa)\mathbf{T}_t^2\right]_{[8],[8]} = 0$$
- One-loop impact factors can be computed $$C_a^{(1)} = \frac{1}{2}Z^{(1)}_{1,\mathbf{R},a} + \frac{1}{2}\widehat{H}^{(1),0,[8]}_{aa}\,,$$
- NLL hard real parts are constrained

$$\mathrm{Re}\left(\widehat{H}^{(1),0,[8]}_{qg}\right) = \frac{1}{2}\left[\mathrm{Re}\left(\widehat{H}^{(1),0,[8]}_{gg}\right) + \mathrm{Re}\left(\widehat{H}^{(1),0,[8]}_{qq}\right)\right]$$

FACE-OFF: HIGHER ORDERS

Two-loop results

- At two loops, for leading logarithms, we find the expected pattern of exponentiation

$$\mathrm{Im}\left[H^{(2),2,[8]}\right] = 0\,, \qquad \mathrm{Re}\left[\widehat{H}^{(2),2,[8]}\right] = \frac{1}{2}\,\mathrm{Re}\left[\widehat{H}^{(1),1,[8]}\right]^2 = \mathcal{O}\left(\epsilon^2\right)\,.$$

- At NLL, the hard imaginary part appears to have a more elaborate structure

$$\begin{aligned}\mathrm{Im}\left[H^{(2),1,[8]}\right] &= -\frac{\pi}{2}K_1^2\left[\left(\{\mathbf{T}_t^2,\mathbf{T}_s^2\} - \mathcal{C}_{\rm tot}\mathbf{T}_t^2\right)H^{(0)}\right]^{[8]} - \pi K_1\left[\left(\mathbf{T}_s^2 - \frac{\mathcal{C}_{\rm tot}}{2}\right)H^{(1),1}\right]^{[8]} \\ &\quad - \frac{\pi}{2}(1+\kappa)\left(\alpha^{(1)}\right)^2 H^{(0),[8]}\end{aligned}$$

- We can however use a simple color identity, octet dominance at tree level, and the one-loop `curious identity', to get

$$\left[\{\mathbf{T}_t^2,\mathbf{T}_s^2\}H^{(0)}\right]^{[8]} = 2\,C_A\left[\mathbf{T}_s^2\right]_{[8],[8]}H^{(0),[8]} \quad\rightarrow\quad \mathrm{Im}\left[H^{(2),1,[8]}\right] = \mathcal{O}\left(\epsilon^2\right)$$

- The Regge trajectory can be extracted from the NLL real part, and has the predicted form

$$\alpha^{(2)} = C_A K_2 + \mathrm{Re}\left[\hat{H}_{ab}^{(2),1,[8]}\right] + \mathcal{O}(\epsilon)\,.$$

- The NNLL hard imaginary part is related to the NLL hard real part

$$\mathrm{Im}\left[H^{(2),0,[8]}\right] = -\frac{\pi}{2}(1+\kappa)\,\mathrm{Re}\left[H^{(2),1,[8]}\right]$$

Two loops: universality breaking

- At two loops and at NNLL the impact factors display universality breaking

$$
\begin{aligned}
C_a^{(2)} \quad = \quad & \frac{1}{2} Z^{(2)}_{1,\mathbf{R},aa} - \frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},aa}\right)^2 + \frac{1}{4} Z^{(1)}_{1,\mathbf{R},aa}\,\mathrm{Re}\left[\hat{H}^{(1),0,[8]}_{aa}\right] - \frac{1}{2} R^{(2),0,[8]}_{aa} \\
& - \frac{\pi^2}{4} K_1^2 \left\{ \left[\left(\mathbf{T}^2_{s,aa}\right)^2\right]_{[8],[8]} - \mathcal{C}_{\mathrm{tot},aa}\left[\mathbf{T}^2_{s,aa}\right]_{[8],[8]} + \frac{1}{4}\mathcal{C}^2_{\mathrm{tot},aa} - \frac{(1+\kappa)N_c^2}{2} \right\} + \mathcal{O}\left(\epsilon^0\right)
\end{aligned}
$$

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$$C_a^{(2)} = \boxed{\frac{1}{2} Z^{(2)}_{1,\mathbf{R},aa} - \frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},aa}\right)^2 + \frac{1}{4} Z^{(1)}_{1,\mathbf{R},aa}\,\mathrm{Re}\left[\hat{H}^{(1),0,[8]}_{aa}\right]} - \frac{1}{2} R^{(2),0,[8]}_{aa}$$
$$- \frac{\pi^2}{4} K_1^2 \left\{ \left[\left(\mathbf{T}^2_{s,aa}\right)^2\right]_{[8],[8]} - \mathcal{C}_{\mathrm{tot},aa}\left[\mathbf{T}^2_{s,aa}\right]_{[8],[8]} + \frac{1}{4}\mathcal{C}^2_{\mathrm{tot},aa} - \frac{(1+\kappa)N_c^2}{2} \right\} + \mathcal{O}\left(\epsilon^0\right)$$

Universal, real, color singlet, from jets

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$$C_a^{(2)} = \boxed{\frac{1}{2} Z^{(2)}_{1,\mathbf{R},aa} - \frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},aa}\right)^2 + \frac{1}{4} Z^{(1)}_{1,\mathbf{R},aa}\,\mathrm{Re}\left[\hat{H}^{(1),0,[8]}_{aa}\right]} - \frac{1}{2} R^{(2),0,[8]}_{aa}$$
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Define as proper impact factor

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Non universal, from phases, color-mixed

Two loops: universality breaking

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$$
\begin{aligned}
C_a^{(2)} &= \boxed{\frac{1}{2} Z^{(2)}_{1,\mathbf{R},aa} - \frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},aa}\right)^2 + \frac{1}{4} Z^{(1)}_{1,\mathbf{R},aa}\, \mathrm{Re}\left[\hat{H}^{(1),0,[8]}_{aa}\right]} - \frac{1}{2} R^{(2),0,[8]}_{aa} \\
&\quad \boxed{- \frac{\pi^2}{4} K_1^2 \left\{ \left[\left(\mathbf{T}^2_{s,aa}\right)^2\right]_{[8],[8]} - \mathcal{C}_{\mathrm{tot},aa} \left[\mathbf{T}^2_{s,aa}\right]_{[8],[8]} + \frac{1}{4} \mathcal{C}^2_{\mathrm{tot},aa} - \frac{(1+\kappa) N_c^2}{2} \right\}} + \mathcal{O}\left(\epsilon^0\right)
\end{aligned}
$$

Universal, real, color singlet, from jets

Define as proper impact factor

Non universal, from phases, color-mixed

Assign to non-factorizing remainder

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Assign to non-factorizing remainder

- Quark and gluon impact factors derived from qq and gg amplitudes must properly form the qg amplitude. Assuming factorization this fails (Del Duca and Glover 2001). Indeed defining

$$\begin{aligned} \Delta_{(2),0,[8]} &= M^{(2),0,[8]}_{qg} - \left[C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{4}(1+\kappa)\left(\alpha^{(1)}\right)^2\right] H^{(0),[8]}_{qg} \\ &= \widetilde{R}^{(2),0,[8]}_{qg} - \frac{1}{2}\left(\widetilde{R}^{(2),0,[8]}_{qq} + \widetilde{R}^{(2),0,[8]}_{gg}\right) \end{aligned}$$

Two loops: universality breaking

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$$C_a^{(2)} = \boxed{\frac{1}{2} Z^{(2)}_{1,\mathbf{R},aa} - \frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},aa}\right)^2 + \frac{1}{4} Z^{(1)}_{1,\mathbf{R},aa}\,\mathrm{Re}\left[\hat{H}^{(1),0,[8]}_{aa}\right]} - \frac{1}{2} R^{(2),0,[8]}_{aa}$$
$$\boxed{-\frac{\pi^2}{4} K_1^2 \left\{ \left[\left(\mathbf{T}^2_{s,aa}\right)^2\right]_{[8],[8]} - \mathcal{C}_{\mathrm{tot},aa}\left[\mathbf{T}^2_{s,aa}\right]_{[8],[8]} + \frac{1}{4}\mathcal{C}^2_{\mathrm{tot},aa} - \frac{(1+\kappa)N_c^2}{2}\right\}} + \mathcal{O}\left(\epsilon^0\right)$$

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$$\Delta_{(2),0,[8]} = M^{(2),0,[8]}_{qg} - \left[C_q^{(2)} + C_g^{(2)} + C_q^{(1)} C_g^{(1)} - \frac{\pi^2}{4}(1+\kappa)\left(\alpha^{(1)}\right)^2\right] H^{(0),[8]}_{qg}$$
$$= \widetilde{R}^{(2),0,[8]}_{qg} - \frac{1}{2}\left(\widetilde{R}^{(2),0,[8]}_{qq} + \widetilde{R}^{(2),0,[8]}_{gg}\right)$$

we find that

$$\Delta_{(2),0,[8]} = \frac{\pi^2 K_1^2}{2}\left[\frac{3}{2}\left(\frac{N_c^2+1}{N_c^2}\right)\right] = \frac{\pi^2}{\epsilon^2}\frac{3}{16}\left(\frac{N_c^2+1}{N_c^2}\right) \qquad \checkmark$$

Three loops: factorization breaking

- At three loops, the NLL hard imaginary part requires a new color identity, but ends up being constrained as at lower orders ... generalization is to be expected.

$$\left[\left(\left(\mathbf{T}_t^2\right)^2\mathbf{T}_s^2+\mathbf{T}_t^2\mathbf{T}_s^2\mathbf{T}_t^2+\mathbf{T}_s^2\left(\mathbf{T}_t^2\right)^2\right)H^{(0)}\right]^{[8]}=3N_c^2\left(\mathbf{T}_s^2\right)_{[8],[8]}H^{(0),[8]}\quad\rightarrow\quad\mathrm{Im}\left[\widehat{H}^{(3),2,[8]}\right]=\mathcal{O}\left(\epsilon^3\right)$$

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- The real contribution at NNLL should give the Regge trajectory at three loops. As expected, fitting the coefficient of the single log we find a non-universal result. Using

$$\left[\left(\mathbf{T}_t^2 \left(\mathbf{T}_s^2\right)^2 + \mathbf{T}_s^2 \mathbf{T}_t^2 \mathbf{T}_s^2 + \left(\mathbf{T}_s\right)^2 \mathbf{T}_t^2\right) H^{(0)}\right]^{[8]} = \sum_n \left(2N_c + \mathcal{C}_{[n]}\right) \left|\left(\mathbf{T}_s^2\right)_{[8],[n]}\right|^2 H^{(0),[8]}$$

we find

$$\begin{aligned}
\alpha_{\text{fit}}^{(3)} \quad = \quad & C_A K_3 + \frac{\pi^2 K_1^3}{2}\Bigg[\mathcal{C}_{\text{tot},ij} N_c \left(\mathbf{T}_{s,ij}^2\right)_{[8],[8]} - \frac{\mathcal{C}_{\text{tot},ij}^2 N_c}{4} + \frac{1+\kappa}{2} N_c^3 \\
& - \frac{1}{3}\sum_n \left(2N_c + \mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_{s,ij}^2\right)_{[8],n}\right|^2\Bigg] - R^{(3),1,[8]} + \mathcal{O}\left(\epsilon^{-2}\right)
\end{aligned}$$

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$$\left[\left(\left(\mathbf{T}_t^2\right)^2 \mathbf{T}_s^2 + \mathbf{T}_t^2\mathbf{T}_s^2\mathbf{T}_t^2 + \mathbf{T}_s^2\left(\mathbf{T}_t^2\right)^2\right) H^{(0)}\right]^{[8]} = 3N_c^2 \left(\mathbf{T}_s^2\right)_{[8],[8]} H^{(0),[8]} \quad \rightarrow \quad \mathrm{Im}\left[\widehat{H}^{(3),2,[8]}\right] = \mathcal{O}\left(\epsilon^3\right)$$

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we find

$$\alpha_{\rm fit}^{(3)} = \boxed{C_A K_3} + \boxed{\frac{\pi^2 K_1^3}{2}\left[\mathcal{C}_{\mathrm{tot},ij} N_c \left(\mathbf{T}_{s,ij}^2\right)_{[8],[8]} - \frac{\mathcal{C}_{\mathrm{tot},ij}^2 N_c}{4} + \frac{1+\kappa}{2} N_c^3 - \frac{1}{3}\sum_n \left(2N_c + \mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_{s,ij}^2\right)_{[8],n}\right|^2\right]} - R^{(3),1,[8]} + \mathcal{O}\left(\epsilon^{-2}\right)$$

universal

non universal

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$$\left[\left(\left(\mathbf{T}_t^2\right)^2\mathbf{T}_s^2+\mathbf{T}_t^2\mathbf{T}_s^2\mathbf{T}_t^2+\mathbf{T}_s^2\left(\mathbf{T}_t^2\right)^2\right)H^{(0)}\right]^{[8]}=3N_c^2\left(\mathbf{T}_s^2\right)_{[8],[8]}H^{(0),[8]}\quad\rightarrow\quad\mathrm{Im}\left[\widehat{H}^{(3),2,[8]}\right]=\mathcal{O}\left(\epsilon^3\right)$$

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$$\left[\left(\mathbf{T}_t^2\left(\mathbf{T}_s^2\right)^2+\mathbf{T}_s^2\mathbf{T}_t^2\mathbf{T}_s^2+\left(\mathbf{T}_s\right)^2\mathbf{T}_t^2\right)H^{(0)}\right]^{[8]}=\sum_n\left(2N_c+\mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_s^2\right)_{[8],[n]}\right|^2H^{(0),[8]}$$

we find

$$\alpha_{\mathrm{fit}}^{(3)}=\boxed{C_AK_3}+\frac{\pi^2K_1^3}{2}\left[\mathcal{C}_{\mathrm{tot},ij}N_c\left(\mathbf{T}_{s,ij}^2\right)_{[8],[8]}-\frac{\mathcal{C}_{\mathrm{tot},ij}^2N_c}{4}+\frac{1+\kappa}{2}N_c^3\right.$$
$$\left.-\frac{1}{3}\sum_n\left(2N_c+\mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_{s,ij}^2\right)_{[8],n}\right|^2\right]-R^{(3),1,[8]}+\mathcal{O}\left(\epsilon^{-2}\right)$$

universal

non universal

- We define then a single-logarithmic three-loop non-factorizing remainder as

$$\widetilde{R}_{ij}^{(3),1,[8]}=\pi^2K_1^3\left[\mathcal{C}_{\mathrm{tot},ij}N_c\left(\mathbf{T}_{s,ij}^2\right)_{[8],[8]}-\frac{\mathcal{C}_{\mathrm{tot},ij}^2N_c}{4}\right.$$
$$\left.+\frac{1+\kappa}{2}N_c^3-\frac{1}{3}\sum_n\left(2N_c+\mathcal{C}_{[n]}\right)\left|\left(\mathbf{T}_{s,ij}^2\right)_{[8],n}\right|^2\right]+\mathcal{O}\left(\epsilon^{-2}\right)$$

Predictions at three loops and beyond

- Explicitly, the non-factorizing remainders at three loops for qq, gg and qg amplitudes are

$$
\begin{aligned}
\widetilde{R}_{qq}^{(3),1,[8]} &= \frac{2}{3}\pi^2 K_1^3 \, \frac{2N_c^2-5}{N_c} = \frac{\pi^2}{\epsilon^3}\,\frac{2N_c^2-5}{12N_c}\,, \\
\widetilde{R}_{gg}^{(3),1,[8]} &= -\frac{16}{3}\pi^2 K_1^3 \, N_c = -\frac{\pi^2}{\epsilon^3}\,\frac{2}{3}\,N_c \\
\widetilde{R}_{qg}^{(3),1,[8]} &= -\frac{1}{3}\pi^2 K_1^3 N_c = -\frac{\pi^2}{\epsilon^3}\,\frac{N_c}{24}\,.
\end{aligned}
$$

Note that all remainders are subleading in N_c as they must.

- Furthermore, one can prove a sequence of all-order identities for the hard parts

$$
\begin{aligned}
\mathrm{Im}(\hat{H}^{(n),n,[8]}) &= 0 \\
\mathrm{Re}(\hat{H}^{(n),n,[8]}) &= \frac{1}{n!}\left(\hat{H}^{(1),1,[8]}\right)^n = O(\epsilon^n) \\
\mathrm{Im}(\hat{H}^{(n),n-1,[8]}) &= -\pi\frac{1+\kappa}{2}\left(n\hat{H}^{(n),n,[8]}\right) = O(\epsilon^n) \\
\mathrm{Re}(\hat{H}^{(n),n-1,[8]}) &= \mathrm{Re}(\hat{H}^{(2),1})\hat{H}^{(n-2),n-2} + (2-n)\mathrm{Re}(\hat{H}^{(1),0,[8]})\hat{H}^{(n-1),n-1} \\
&= \mathcal{O}(\epsilon^{n-2})
\end{aligned}
$$

proving a `strong vanishing' of the hard part up to NLL.

OUTLOOK

Summary

- The dipole formula may encode all infrared singularities for any massless gauge theory, a natural generalization of the planar limit.
- The study of possible corrections to the dipole formula is under way.
- The high-energy limit of the dipole formula provides insights into Reggeization and beyond, at least for divergent contributions to the amplitude.
- Leading logarithmic Reggeization is proved for generic color representations.
- Regge factorization generically breaks down at NNLL, with computable corrections which may be related to Regge cuts in the angular momentum plane.
- We have studied in detail the combined consequences of Regge and soft-collinear factorizations on four-point multi-loop QCD amplitudes
- Infrared information provides a natural way to identify non-universal contributions beyond Regge factorization: these could be the leading terms of a Regge-cut resummation.
- We recover from first principles the non-factorizing non-logarithmic two-loop remainder of Del Duca and Glover.
- We explicitly predict the leading non-factorizing high-energy logarithms at three loops for qq, gg and qg amplitudes in QCD.
- All-order identities strongly constrain the hard, finite contributions to the amplitude at LL and NLL, and weaker constraints at NNLL and beyond.
- Similar results can be derived for multi-parton amplitudes in multi-Regge kinematics.

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