Conformal Field Theories in more than Two Dimensions

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Durham December 17th 2013

Inspired by





supported by

- Pre Modern, where did it start
- Contemporary Modern
 - I. When does conformal symmetry arise?
 - 2. Conformal Kinematics, Null Cone
 - 3. Quantum Fields
 - 4. Three point functions, Operator Product Expansion
 - 5. Four point functions
 - 6. Bootstrap
 - 7. Superconformal
 - 8. Minkowski Space Methods

• Post Modern, where might it go

THE PRINCIPLE OF RELATIVITY IN ELECTRODYNAMICS AND AN EXTENSION THEREOF

By E. CUNNINGHAM.

[Received May 1st, 1909.]*

THE TRANSFORMATION OF THE ELECTRODYNAMICAL EQUATIONS

By H. BATEMAN.

[Received March 7th, 1909.—Read March 11th, 1909.—Received, in revised form, July 22nd, 1909.]

ANNALS OF MATHEMATICS Vol. 37, No. 2, April, 1936

WAVE EQUATIONS IN CONFORMAL SPACE

By P. A. M. DIRAC

(Received May 18, 1935)

What is Conformal Symmetry, Conformal transformations preserve angles



Two dimensions are different

$$\begin{array}{ll} x \rightarrow x' & dx'^2 \propto dx^2 \\ x'^{\mu} = x^{\mu} + v^{\mu} & \partial_{\mu}v_{\nu} + \partial_{\nu}v_{\mu} = 2\sigma \eta_{\mu\nu} \\ x^{\mu} = a^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu} + \lambda \, x^{\mu} + b^{\mu}x^2 - 2b \cdot x \, x^{\mu} \\ {}_{\text{translation Lorentz scale special conformal}} \\ v^{\mu} & \text{ is a conformal Killing vector } \frac{1}{2}(d+1)(d+2) & \text{ parameters} \end{array}$$

Conformal field theories are obtained by RG flow to non trivial IR limit where the beta functions vanish

For gauge theories this is expected for a restricted range of flavours defining the conformal window

QCD
$$SU(2)$$
 $3 \lesssim N_f \le 10$
 $SU(3)$ $7-8 \lesssim N_f \le 16$

The lower bound is a strong coupling problem depending on lattice calculations

For large $N_f, N_c, N_f = \frac{11}{2}N_c - \epsilon$ there is the weakly coupled Banks Zaks fixed point In $\mathcal{N} = 1$ SQCD by Seiberg duality conformal window is $\frac{3}{2}N_c < N_f < 3N_c$

weakly coupled magnetic theory weakly coupled electric theory

 $\mathcal{N}=4$ conformal for any g_{YM}

Conserved current for any conformal Killing vector $J_v^{\mu} = T^{\mu\nu}v_{\nu} \text{ if } \partial_{\mu}T^{\mu\nu} = 0, \quad \eta_{\mu\nu}T^{\mu\nu} = 0$

In general in a QFT expect

lf

$$\eta_{\mu\nu}T^{\mu\nu} = \beta^{I}\mathcal{O}_{I} + \partial_{\mu}J^{\mu}$$
$$\beta^{I} = 0$$

there is a conserved current for scale transformations

$$S^{\mu} = T^{\mu\nu} x_{\nu} - J^{\mu}, \quad \partial_{\mu} S^{\mu} = 0$$

but conformal invariance is broken

 $\begin{array}{ll} \mbox{Ways out:} & J^{\mu} = \partial_{\nu} L^{\mu\nu} & \mbox{can redefine} & T^{\mu\nu} \\ \mbox{to make it traceless} \\ \mbox{or if} & \partial_{\mu} J^{\mu} = r^{I} \mathcal{O}_{I} & \mbox{then get a CFT if} & B^{I} = \beta^{I} + r^{I} = 0 \end{array}$

This reflects potential arbitrariness in the definition of beta functions beyond a choice of scheme

Many authors have discussed whether there are scale but not conformal invariant theories

It is very likely that for unitary theories scale invariance does imply conformal symmetry

For non unitary theories there are counterexamples

In two dimensions it is a theorem

Polchinksi Riva Cardy Dorigoni Rychkov Fortin Grinstein Stergiou Nakayama lackiw Pi **El-Showk** Dymarsky Komargodski Schwimmer Theisen

Conformal transformations are nonlinear

$$(x-y)^2 \to \Omega(x)\Omega(y) (x-y)^2$$

Need 4 points to construct an invariant It is often simpler to use homogeneous coords

$$\begin{aligned} X^{\mu}, X^{d}, X^{d+1} & X \sim \lambda X \\ &- \frac{1}{2}X \cdot X = \eta_{\mu\nu}X^{\mu}X^{\nu} + (X^{d})^{2} - (X^{d+1})^{2} = 0 \end{aligned}^{\text{cone (Dirac)}} \\ x^{\mu} &= \frac{X^{\mu}}{X^{+}}, \quad X^{+} = X^{d} + X^{d+1}, \quad (x - y)^{2} = \frac{X \cdot Y}{X^{+}Y^{+}} \end{aligned}$$

Conformal group defined by linear SO(d,2)transformations preserving the null cone SO(d+1,1) AdS_{d+1} may be defined by $X \cdot X = -1$ and so has the conformal group as its isometry group

The boundary of AdS_{d+1} is the projective null cone



This is just the tip of the AdS/CFT correspondence

The inversion operation plays a crucial role in CFTs

like time reversal

$$x^{\mu} \to x^{\mu}/x^2, \quad X^{d+1} \to -X^{d+1}$$

Algebra and Fields Conformal Generators $M_{AB} = -M_{BA}$ contain P_{μ} $M_{\mu\nu}$ D K_{μ} translation Lorentz scale special conformal $[D, P_{\mu}] = P_{\mu} \quad [D, K_{\mu}] = -K_{\mu}$ Radial Quantisation, treat Das the Hamiltonian evolving in $\tau = \log |x|$ Fields are labelled by D eigenvalue Δ and spin $\phi(x) = \Phi(X), \quad \Phi(\lambda X) = \lambda^{-\Delta} \Phi(X)$ adjoint $\bar{\phi}(x) = (x^2)^{-\Delta} \phi(x/x^2)$

states $|\phi\rangle = \phi(0)|0\rangle$ $\langle \phi| = \langle 0|\bar{\phi}(0)$ correspond to fields $P_{\mu}^{\ \dagger} = K_{\mu}$ Require the fields generate unitary positive energy representations for a unitary CFT

D has positive eigenvalues, zero on the vacuum

Conformal primary $K_{\mu}|\phi\rangle = 0$

Descendants generated by action $\prod (P_{\mu})^{n_{\mu}} |\phi\rangle$ of momentum operators

For unitarity there are constraints on Δ and the spin

For symmetric tensor fields of rank $\ell \qquad \phi_{\mu_1...\mu_\ell}$ $\Delta \geq \frac{1}{2}(d-2) \quad \ell = 0, \quad \Delta \geq \ell + d - 2 \quad \ell = 1, 2, ...$

Mack Dobrev

If equality the representation truncates $\ell = 0$ free field $\ell > 0$ conserved current

For fields with spin

 $\phi_{\mu_{1}...\mu_{\ell}}(x)$ $\Phi_{A_{1}...A_{\ell}}(X), \quad X^{2} = 0$ $X^{A_{r}} \Phi_{A_{1}...A_{\ell}}(X) = 0, \quad r = 1...\ell$ $\Phi_{A_{1}...A_{\ell}}(X) \sim \Phi_{A_{1}...A_{\ell}}(X) + X_{A_{r}} \Psi_{A_{1}...\hat{A}_{r}...A_{\ell}}(X), \quad r = 1...\ell$

$$\bar{\phi}^{\mu_1...\mu_\ell}(x) = (x^2)^{-\Delta} \prod_{r=1...\ell} I^{\mu_r\nu_r}(x) \phi_{\nu_1...\nu_\ell}(x/x^2)$$

$$I^{\mu\nu}(x) = \eta^{\mu\nu} - 2 \, \frac{x^{\mu} x^{\nu}}{x^2} \qquad \qquad \ln$$

nversion tensor

Two point functions define the normalisation of the fields

$$\begin{aligned} \langle \phi(x) \, \phi(y) \rangle &= \frac{1}{(x-y)^{2\Delta}} \quad \Rightarrow \quad \langle \phi | \phi \rangle = 1 \\ \langle \Phi(X) \, \Phi(Y) \rangle &= \frac{1}{(X \cdot Y)^{\Delta}} \end{aligned}$$

Generalisations to spin are straightforward and involve the inversion tensor

Three point functions for primary operators in CFTs are determined up to a finite number of coefficients

Three points can be mapped to any three points by a conformal transformation

Scalars

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = \frac{c_{123}}{x_{12}^{2\delta_3} x_{23}^{2\delta_1} x_{31}^{2\delta_2}} \\ x_{12} = x_1 - x_2 \qquad \delta_3 = \frac{1}{2} (\Delta_1 + \Delta_2 - \Delta_3) \\ \langle \phi_1 | \phi_2(x) | \phi_3 \rangle |_{|x|=1} = c_{123}$$

Generalisations to spin are a linear algebraic problem

Number of independent terms for spins ℓ_1, ℓ_2, ℓ_3 equal to the number of on shell amplitudes in d+1 dim Minkowski space

Conserved currents correspond to amplitudes for massless particles

For parity conserving amplitudes number is $\min \ell_i + 1$

Zhiboedov 2012

Hence for the em tensor three point function

 $\langle T_{\mu\nu}(x_1) T_{\sigma\rho}(x_2) T_{\alpha\beta}(x_3) \rangle$

there are three independent terms

These correspond to the number of free CFTs in four dimensions, scalars, fermions, vectors

Three point functions for higher spin currents correspond to those of free theories in even dimensions

The spectrum of operators, scale dimensions and spins, and the three point functions determine a CFT through the operator product expansion (OPE)

The product of any two conformal primary fields is given by an expansion in terms of conformal primaries and their descendants

The expansion is convergent and is determined by reproducing the three point functions

$$\phi(x)\phi(0) = \sum_{\mathcal{O}} c_{\phi\phi\mathcal{O}} \frac{1}{(x^2)^{\frac{1}{2}(2\Delta_{\phi} - \Delta + \ell)}} C_{\Delta,\ell}(x,\partial)^{\mu_1\dots\mu_{\ell}} \mathcal{O}_{\mu_1\dots\mu_{\ell}}(0)$$
$$C_{\Delta,\ell}(x,0)^{\mu_1\dots\mu_{\ell}} = x^{\mu_1}\dots x^{\mu_{\ell}}$$

 $\phi \times \phi = \sum_{\mathcal{O}} \mathcal{O}_{\Delta,\ell}$

for two scalars $\,{\cal O}\,\,$ symmetric traceless tensor scale dimension $\,\Delta\,$

The OPE applied to the four point functions gives non trivial constraints on the spectrum of operators

$$\langle \phi(x_1) \, \phi(x_2) \, \phi(x_3) \, \phi(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_{\phi}}} \, F(u, v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \qquad \text{two conforma}_{\text{invariants}}$$

Crossing symmetry

$$F(u,v) = F(u/v, 1/v) \qquad 1 \leftrightarrow 2$$
$$= \left(\frac{u}{v}\right)^{\Delta_{\phi}} F(v,u) \qquad 2 \leftrightarrow 4$$

$$\begin{split} & \mathsf{The \ OPE \ gives} \\ & \langle \phi(x_1) \ \phi(x_2) \ \phi(x_3) \ \phi(x_4) \rangle \\ & = \sum_{\mathcal{O}} \frac{c_{\phi\phi\mathcal{O}}^2}{(x_{12}^2 x_{34}^2)^{\frac{1}{2}(2\Delta_{\phi} - \Delta + \ell)}} \\ & \times C_{\Delta,\ell}(x_{12}, \partial_2)^{\mu_1 \dots \mu_\ell} C_{\Delta,\ell}(x_{34}, \partial_4)^{\nu_1 \dots \nu_\ell} \langle \mathcal{O}_{\mu_1 \dots \mu_\ell}(x_2) \ \mathcal{O}_{\nu_1 \dots \nu_\ell}(x_4) \rangle \\ & \mathsf{or} \quad F(u, v) = 1 + \sum_{\mathcal{O}} c_{\phi\phi\mathcal{O}}^2 \ G_{\Delta,\ell}(u, v) \qquad \underset{u \to 0 \ v \to 1}{\mathsf{converges as}} \\ & \underset{u \to 0 \ v \to 1}{\mathsf{converges as}} \end{split}$$

This is analogous to the partial wave expansion for S-matrix amplitudes, conformal blocks replaced by Legendre polynomials or their generalisations

The large order behaviour of $c_{\phi\phi\mathcal{O}}$ is constrained by reproducing the behaviour as $v \to 0$ $u \to 1$ Pappadopulo,Rychkov Espin, Rattazzi, 2012

In two dimensions for minimal models this can be restricted to a finite sum if the conformal blocks are extended to Virasoro conformal blocks, related to the infinite dimensional Virasoro algebra

New variables $u = z\overline{z}$ $v = (1 - z)(1 - \overline{z})$ $F(u, v) = \mathcal{F}(z, \overline{z}) = \mathcal{F}(\overline{z}, z)$

Restrict coords to a plane

$$\langle \phi | \phi(x_3) \phi(x_2) | \phi \rangle = \frac{1}{(z \, \overline{z})^{\Delta_{\phi}}} \, \mathcal{F}(z, \overline{z})$$
$$x_3 = (1, 1) \qquad x_2 = (z, \overline{z}) \quad x_2^2 = z\overline{z}$$



Conformal blocks are non polynomial and rather non trivial functions

In principle they are solutions of second order PDEs

$$\frac{1}{2}M_{AB}M^{BA}G_{\Delta,\ell}(u,v) = (\Delta(\Delta-d) + \ell(\ell+d-2))G_{\Delta,\ell}(u,v)$$
$$G_{\Delta,\ell}(u,v) = u^{\frac{1}{2}(\Delta-\ell)}(1-v)^{\ell}(1+O(u,1-v))$$

Conformal blocks were discussed by Ferrara, Gatto, Grillo, Parisi in the 1970's who obtained results in particular limits. More general expressions were obtained quite recently but we still lack compact formulae for arbitrary d

In four dimensions there is a nice solution in terms of single variable hypergeometric functions

$$G_{\Delta,\ell}(u,v) = \frac{z\bar{z}}{z-\bar{z}} \left(g_{\frac{1}{2}(\Delta+\ell)}(z) g_{\frac{1}{2}(\Delta-\ell)-1}(\bar{z}) - g_{\frac{1}{2}(\Delta+\ell)}(\bar{z}) g_{\frac{1}{2}(\Delta-\ell)-1}(z) \right)$$
$$u = z\bar{z} \qquad v = (1-z)(1-\bar{z})$$

Baron von Munchausen 1720-1797 fantasist and hero of the romantic age

Bootstrap!



Münchhaufen

the Baron escapes a swamp by pulling on his hair in later myth his bootstraps

as in The Surprising Adventures of Baron von Munchhausen, published in English in 1785

Bootstrap equation

Combining crossing with the conformal block expansion

proposed by Polyakov in the context of CFTs in 1971 revived by Rychkov, Rattazzi, Tonni,Vichi in 2008 mini Milner prize to Rychkov in 2013



There is a region in the neighbourhood of the crossing symmetric point

$$u = v = \frac{1}{4}$$
 $z = \bar{z} = \frac{1}{2}$

where this expansion converges.

Can truncate the expansion by considering a finite Taylor expansion around $z = \overline{z} = \frac{1}{2}$ in powers of $z + \overline{z} - 1$, $(z - \overline{z})^2$

 $F_{\Delta,\ell}^{\Delta_{\phi}}(z,\bar{z}) \to F_{\Delta,\ell;n,m}^{\Delta_{\phi}}$ $\delta_{n0}\delta_{m0} = \sum_{\Delta,\ell} a_{\Delta,\ell} F_{\Delta,\ell;n,m}^{\Delta_{\phi}} \quad a_{\Delta,\ell} \ge 0$

Truncate the spectrum of operators,

These equations do not have solutions unless there are restrictions or bounds on Δ, ℓ

This is a problem in linear programming







The x axis is Δ_{σ} the y axis is the Δ lowest in the OPE

| Operator | Spin l | \mathbb{Z}_2 | Δ |
|---------------------------|----------|----------------|---------------|
| σ | 0 | _ | 0.5182(3) |
| σ' | 0 | _ | $\gtrsim 4.5$ |
| ε | 0 | + | 1.413(1) |
| ε' | 0 | + | 3.84(4) |
| ε'' | 0 | + | 4.67(11) |
| $T_{\mu\nu}$ | 2 | + | 3 |
| $C_{\mu\nu\kappa\lambda}$ | 4 | + | 5.0208(12) |

Superconformal Symmetry

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Extension of conformal group to include supersymmetry

In addition to the usual fermionic charges Q, \bar{Q} there are additional charges S, S

$$\begin{array}{ll} \{Q,\bar{Q}\}=P & \{S,\bar{S}\}=K & \{Q,S\}=M+D+R\\ & \text{Lorentz} & \text{Scale} & \text{R-charge} \end{array}\\ \textbf{R-symmetry is an essential part of the superconformal group}\\ \mathcal{N}=1 & U(1)_R & \mathcal{N}=2 & U(2)_R & \mathcal{N}=4 & SU(4)_R\\ \textbf{For } |\phi\rangle \text{ a superconformal primary } K|\phi\rangle=S|\phi\rangle=\bar{S}|\phi\rangle=0\\ \textbf{Supermultiplet generated by} & \prod_{n,j,k}P^n Q^j \bar{Q}^k |\phi\rangle \end{array}$$

Shortening conditions

One or more of the Q, \bar{Q} acting on $|\phi\rangle$ may give zero This gives rise to short or semi-short $\frac{1}{n}$ -BPS multiplets n=2,4,8,16

Such multiplets are protected $\ \Delta$ is determined in terms of $\ \ell$ and R-symmetry representation

For short multiplets $\ell = 0$ and the R-symmetry representations are restricted

For 4 point functions the conformal partial wave expansion extends to one in terms of superconformal blocks

$$\begin{split} \mathcal{N} &= 4 \qquad \mathcal{O}_{20}^{I} \text{ half BPS 20 dim short supermultiplet} \begin{array}{l} \overset{\text{Rastellit}}{\overset{\text{van Rees}}{\overset{\text{van Rees}}{\overset{\text{constrained}}{\overset{\overset{\text{constrained}}{\overset{\text{constrained}}{\overset{\text{constrained}}{\overset{\text{constrained}}{\overset{\overset{\text{constrained}}{\overset{\text{constrained}}{\overset{\text{constrained}}{\overset{\overset{\text{constrained}}}{\overset{\overset{\text{constrained}}{\overset{\overset{\text{constrained}}{\overset{\overset{\text{constrained}}{\overset{\overset{\text{constrained}}}$$

protected short multiplet contribution

$$a = \frac{1}{4} \dim G$$
 G gauge group

Beem

Expand F(u, v) over the sum of protected and unprotected superconformal blocks with positive coefficients

Get constraints on potential Δ_{ℓ}



FIG. 1. Exclusion plots in the space of leading twist gaps Δ_0 , Δ_2 , and Δ_4 . Central charges a = 3/4, a = 15/4, and $a = \infty$ are shown, corresponding to $\mathcal{N} = 4$ SYM with gauge group SU(2), SU(4), and $SU(\infty)$, respectively. The area outside of a cube-shaped region is excluded.

Conjecture: the corner of the cube corresponds to a self dual point under $Sl(2,\mathbb{Z})$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} = e^{\frac{1}{2}\pi i}, e^{\frac{1}{3}\pi i}$$

Such self dual points are difficult to access by other methods

Minkowski approach

$$\phi(x)\phi(0) = \sum_{\mathcal{O}} c_{\phi\phi\mathcal{O}} \frac{1}{(x^2)^{\frac{1}{2}(2\Delta_{\phi} - \Delta + \ell)}} C_{\Delta,\ell}(x,\partial)^{\mu_1...\mu_{\ell}} \mathcal{O}_{\mu_1...\mu_{\ell}}(0)$$

$$C_{\Delta,\ell}(x,0)^{\mu_1...\mu_{\ell}} = x^{\mu_1} \dots x^{\mu_{\ell}}$$

In the light cone limit $x^2 \to 0$ the OPE is essentially an expansion in which operators of equal twist τ contribute equally

$$\tau_\ell = \Delta - \ell$$

This limit is relevant for deep inelastic scattering where the twist controls the approach to scaling in the deep inelastic limit

Positivity requires that the twist is a convex function

Nachtmann 1973

$$\frac{\tau_{\ell_3} - \tau_{\ell_1}}{\ell_3 - \ell_1} \le \frac{\tau_{\ell_2} - \tau_{\ell_1}}{\ell_2 - \ell_1} \quad \ell_1 < \ell_2 < \ell_3$$

Wednesday, 18 December 13

Such results have been refined using a bootstrap type approach relating s and t channel expansions in the light cone limit

$$\begin{array}{ll} \langle \mathcal{O}_{1}(x_{1}) \, \mathcal{O}_{2}(x_{2}) \, \bar{\mathcal{O}}_{1}(x_{3}) \, \bar{\mathcal{O}}_{2}(x_{4}) \rangle & & & & & \\ \text{Trispatrick} \\ \text{Trisp$$

 au_{\min} is the minimum twist of operators appearing in the t channel expansion au^*

C is also calculable



- Future possibilities in CFTs
 - Understand the origin of kinks in bootstrap
 bounds in terms of decoupling of particular
 states. Maybe there is a more analytic approach
 - Show there are no non trivial CFTs for d>6.
 This is true for SCFTs for algebraic, representation theory reasons (Nahm).
 - 3. Are there CFTs with large anomalous dimensions or which are not a deformation of a free Lagrangian theory? This might be relevant to extending ideas of naturalness.
- 4. Construct conformal blocks with external spins. No succinct formulae exist as yet.
 Hopefully there will be interesting physically relevant results before mathematicians take over.

Two dimensional CFTs are not always a good guide to higher dimensions. We have no idea as to any classification in d=3,4

Perhaps we will have a new yellow book but not yet



My thanks to the organisers for this meeting and many others I would like to commemorate Francis Dolan with nearly all my work on CFTs was done IRD HIGH ENERGY LABORATORY

INFORMAL THEORETICAL PHYSICS MEETING, 4th - 6th January,

PROGRAMME

| Monday, 4th January, 1971 | | SPEAKER | CHAIRMAN | | |
|------------------------------|--|---------------|--------------------|--|--|
| | | | | | |
| 14.00 | The Nucleus as a Platform for Elementary Particle Experiments | K. Gottfried | G. H. Stafford | | |
| 15.30 | TEA | | | | |
| 16.00 | Dips, bumps and cuts | G. Ringland | R. J. N. Phillips | | |
| 17.40 | Coaches depart for Oxford and Didcot | | | | |
| Tuesday, 5th January, 1971 | | | | | |
| 9.45 | Recent results in High Energy Scattering | J. Allaby | G. Manning | | |
| 11.15 | COFFEE | | | | |
| 11.45 | Asymptotic Collisions | R. Eden | G. Manning | | |
| 13.15 | LUNCH | | | | |
| 14.15 | The World at Extremely High Energies | T.T.Wu | J. C. Polkinghorne | | |
| 15.45 | TEA | | | | |
| 16.15 | How is the Pomeron? | R. Roberts | J. C. Polkinghorne | | |
| 18.00 | Coaches depart for Oxford and Didcot | | | | |
| Wednesday, 6th January, 1971 | | | | | |
| 9.45 | Duality | M. Kugler | R. H. Dalitz | | |
| 11.15 | COFFEE | | | | |
| 11.45 | Bumps and dips in Meson Spectroscopy | D. Sutherland | R. H. Dalítz | | |
| 13.15 | LUNCH | | | | |
| 14.15 | Deep inelastic electron scattering | H. Osborn | J. G. Taylor | | |
| 15.45 | TEA | | | | |
| 16.15 | Analyticity and the $\pi\pi$ interaction | A. Martin | J. G. Taylor | | |
| 17.45 | Meeting ends | | | | |
| | | | | | |

ALL LECTURES WILL BE HELD IN THE LECTURE HALL, BUILDING R-22