Scattering amplitudes from deformed Yangian invariants?

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based on joint work with Niklas Beisert, Marius de Leeuw and Matteo Rosso
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Integrable structures in gauge theories
Durham, April 23rd, 2014

Introduction

- construction of Yangian invariants from the (perimeter) Bethe equations: discussed in the talks by Rouven Frassek and Tomasz Łukowski for the Yangian of different Lie algebras
- choose the (super) Lie algebra suitable for $\mathcal{N} = 4$ super-Yang–Mills theory: $\mathfrak{psu}(2,2|4)$
- on-shell graphs and chains of R-operators encode permutations: nicely visible after promoting $\mathrm{Y}[\mathfrak{psu}(2,2|4)] \to \mathrm{Y}[\mathfrak{u}(2,2|4)] \sim \mathrm{Y}[\mathfrak{gl}(4|4)]$

Questions

- Is there an *algebraic* language for accessing the deformation? [Chicherin Derkachov, Kirschner]
- Is it possible to construct *deformed amplitudes* from deformed building blocks?
- Can the deformation parameter be used as a *regulator* for the one-loop four-point amplitude?

Outline

- *deformed* vs. *undeformed* Yangian invariants in $\mathcal{N}=4$ sYM theory
- R-operator formalism
  R*-operators are close cousins of the operators* $\mathcal{B}$ *from Tomek's talk*
- translate between on-shell graphs, R-chains and permutations
- deformed tree amplitudes from Yangian invariants: MHV and non-MHV
  deformed loop amplitudes from Yangian invariants: four-point and higher
- deformation parameters as a regulator?

Preliminaries

- tree-level S-matrix of $\mathcal{N}=4$ sYM theory is invariant under the infinte-dimensional Yangian algebra $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$ [Drummond Henn, Plefka]
- *Yangian symmetry:* closure of superconformal and dual superconformal symmetry [Drummond, Henn Korchemsky, Sokatchev]
- tree-level *amplitudes* and loop-level *integrands* in $\mathcal{N}=4$ sYM theory are composed from Yangian-invariant building blocks (here represented as on-shell graphs)

$$\mathfrak{J}\,\mathcal{A} = \mathfrak{J}(\mathcal{Y}_1 + \mathcal{Y}_2 + \dots) = 0.$$

[Arkani-Hamed, Bourjaily, Cachazo Caron-Huot, Trnka]

- *deformation:* promote $\mathrm{Y}[\mathfrak{psu}(2,2|4)] \to \mathrm{Y}[\mathfrak{u}(2,2|4)]$,
  introduce central charge c_i
  nonzero c_i shift the helicities of external legs [Ferro, Łukowski, Meneghelli Plefka, Staudacher]

On-shell superspace

- spinor variables $\lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}}$, where $p_\mu = \sigma_{\mu\,\alpha\dot{\alpha}}\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}, \quad \langle ij\rangle = \lambda_i^\alpha\lambda_{j\alpha}, \quad [ij] = \tilde{\lambda}_{i\dot{\alpha}}\tilde{\lambda}_j^{\dot{\alpha}}$
- $\sigma_{\mu\alpha\dot{\alpha}}$ are suitable Pauli matrices $\rightarrow$ different spacetime signatures
- Grassmann variables $\tilde{\eta}^A$ allow to write the $\mathcal{N}=4$ multiplet as a superfield [Nair]

$$\Phi(\lambda,\tilde{\lambda},\tilde{\eta}) := g^+ + \tilde{\eta}^a\psi_a + \frac{1}{2}\tilde{\eta}^a\tilde{\eta}^b\,\phi_{ab} + \frac{1}{3!}\epsilon_{abcd}\,\tilde{\eta}^a\tilde{\eta}^b\tilde{\eta}^c\,\bar{\psi}^d + \frac{1}{4!}\epsilon_{abcd}\,\tilde{\eta}^a\tilde{\eta}^b\tilde{\eta}^c\tilde{\eta}^d\,g^-$$

- n-point Yangian invariants are defined on n copies of on-shell superspace

$$\mathcal{Y}(\{\lambda_i,\tilde{\lambda}_i,\tilde{\eta}_i\}) \in V_1 \otimes \ldots \otimes V_n$$

- number of different Grassmann variables determines the MHV-sector: $k = \frac{\#\tilde{\eta}}{4}$
  k equals the number of negative-helicity gluons in a pure-gluon amplitude
- total momentum P and supermomentum Q are conserved for each Yangian invariant

Yangian symmetry and central extension

- generators of $\mathfrak{psu}(2,2|4)$ (superconformal symmetry) in spinor helicity language, e.g.:

$$\mathfrak{P}^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \quad \text{or} \quad \bar{\mathfrak{Q}}^{\dot{\alpha}}_A = \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\eta}^A}.$$

- generators for level one (and higher levels) satisfy $[\mathfrak{J}^a, \widehat{\mathfrak{J}}^b] = f^{ab}_c \widehat{\mathfrak{J}}^c$.
- consider the central extension to $\mathfrak{u}(2,2|4)$

$$\mathfrak{C} := \lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} - \tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} - \tilde{\eta}^A \frac{\partial}{\partial \tilde{\eta}^A} + 2\,.$$

  eigenvalue c_i vanishes in the undeformed case.
- evaluation representation: $\widehat{\mathfrak{J}}_i \simeq u_i \, \mathfrak{J}_i$
- use the maximally iterated coproduct to construct action of generators on Yangian invariants

$$\Delta^{n-1}(\widehat{\mathfrak{J}}^a) = \sum_{k=1}^{n} \widehat{\mathfrak{J}}^a_k + \tfrac{1}{2} f^a_{bc} \sum_{1 \le i < j \le n} \mathfrak{J}^b_i \, \mathfrak{J}^c_j = \sum_{k=1}^{n} (\tfrac{1}{2} c_i - u_i) \mathfrak{J}^a_k + \tfrac{1}{2} f^a_{bc} \sum_{1 \le i < j \le n} \mathfrak{J}^b_i \, \mathfrak{J}^c_j$$

- Yangian invariance requires

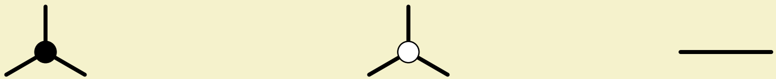
$$\sum_{i=1}^{n} \mathfrak{C}_i \, \mathcal{Y}(\{\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i\}) = 0.$$

Undeformed on-shell graphs

[Arkani-Hamed, Bourjaily, Cachazo Goncharov, Postnikov, Trnka]

- building blocks are glued by integrating over the on-shell superspace of the internal edge

$$\mathcal{A}_{3,\mathsf{MHV}} = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \qquad \mathcal{A}_{3,\overline{\mathsf{MHV}}} = \frac{\delta^4(P)\,\delta^4(\tilde{Q})}{[12][23][31]}, \qquad \int \frac{\mathrm{d}^2\lambda_I\,\mathrm{d}^2\tilde{\lambda}_I}{\mathrm{Vol}[\mathrm{GL}(1)]}\,\mathrm{d}^4\tilde{\eta}_I,$$

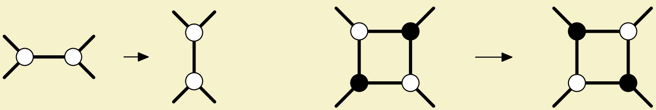


- central charge vanishes for each vertex *individually*:

$$\mathfrak{C}_i \cdot \mathcal{A}_{3,\mathsf{MHV}} = 0, \qquad \mathfrak{C}_i \cdot \mathcal{A}_{3,\overline{\mathsf{MHV}}} = 0 \quad \Rightarrow \quad c_i = 0.$$

- for a general on-shell graph: $n = 3(n_\mathsf{w} + n_\mathsf{b}) - 2n_\mathsf{i}, \quad k = n_\mathsf{w} + 2n_\mathsf{b} - n_\mathsf{i}$.

- representation in terms of on-shell graphs is not unique:
  square move and merger



Deformed on-shell graphs

[Ferro, Łukowski, Meneghelli
Plefka, Staudacher]

- building blocks

$$\mathcal{A}_\bullet = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle^{1+c_1}\langle 23\rangle^{1+c_2}\langle 31\rangle^{1+c_3}}, \qquad \mathcal{A}_\circ = \frac{\delta^4(P)\delta^4(\tilde{Q})}{[12]^{1-c_1}[23]^{1-c_2}[31]^{1-c_3}}.$$

- only the complete vertex is a Yangian invariant, central charge does *not* vanish for each leg individually:

$$\sum_{i=1}^{3} \mathfrak{C}_i \cdot \mathcal{A}_\bullet = 0, \qquad \sum_{i=1}^{3} \mathfrak{C}_i \cdot \mathcal{A}_\circ = 0, \quad \rightarrow \quad c_1 + c_2 + c_3 = 0\,.$$

- Invariance under level-one generators $\hat{\mathfrak{J}}$ requires

$$\begin{aligned} \mathcal{A}_\circ: &\qquad c_1 = u_1 - u_2, \quad c_2 = u_2 - u_3, \quad c_3 = u_3 - u_1 \\ \mathcal{A}_\bullet: &\qquad c_1 = u_1 - u_3, \quad c_2 = u_2 - u_1, \quad c_3 = u_3 - u_2\,. \end{aligned}$$

- Yangian-invariant integration over internal edges A and B requires

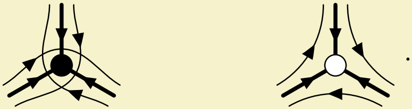
$$c_A = -c_B \quad \text{and} \quad u_A - \tfrac{1}{2}c_A = u_B - \tfrac{1}{2}c_B\,.$$

- for a complete on-shell graph: solve linear system in variables c_i, u_i
- counting argument: # of free variables $\approx n$

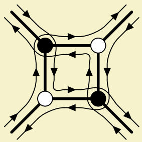
Permutations

[Arkani-Hamed, Bourjaily, Cachazo Goncharov, Postnikov, Trnka][Kanning, Łukowski Staudacher][JB, de Leeuw Rosso]

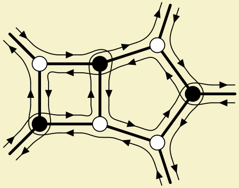
Double-lines indicate the flow of u_i:



Solving the linear system amounts to following the lines and yields a permutation:



$$\rightarrow \{u_1 - c_1, u_2 - c_2, u_3 - c_3, u_4 - c_4\} = \{u_3, u_4, u_1, u_2\}$$



$$\rightarrow \{u_1 - c_1, u_2 - c_2, u_3 - c_3, u_4 - c_4, u_5 - c_5\} = \{u_3, u_4, u_5, u_1, u_2\}\,.$$

Square move and merger are symmetries of the deformed setup: *permutation untouched*

$$c_i = u_i - u_{\sigma(i)}$$

Algebraic approach: R-operator formalism

Two layers of information for deformed amplitudes:
kinematics (BCFW) and Yangian invariance conditions (u_i, c_i).
Is there a representation for Yangian invariants reflecting those properties?

Define the algebra [Chicherin, Derkachov Kirschner]

- canonical supervariables: $\mathbf{x_a} := x_a^{(i)}, \mathbf{p_a} := p_a^{(i)}$, where $i \in \{1, \ldots, M|N\}$ with

$$[x_a^{(i)}, p_b^{(j)}\} = -\delta_{ab}\delta^{ij}.$$

- variables generate algebra $\mathfrak{gl}(M|N)$ with generators $\mathbb{J}^{(ij)} = x^{(i)}p^{(j)}$.

Define two types of operators

- Lax operators & R-operator

$$\mathrm{L}_a(u) = u \cdot \mathbb{1} + \mathbf{x_a} \otimes \mathbf{p_a}, \qquad \mathrm{R}_{ab}(u) = \Gamma(u)\,(\mathbf{p_a} \cdot \mathbf{x_b})^u = \int \frac{\mathrm{d}z}{z^{1+u}} e^{-z(\mathbf{p_a}\cdot\mathbf{x_b})}$$

- R, L satisfy the following relations

$$\mathrm{R}_{12}(u_{12})\mathrm{R}_{23}(u_{13})\mathrm{R}_{12}(u_{23}) = \mathrm{R}_{23}(u_{23})\mathrm{R}_{12}(u_{13})\mathrm{R}_{23}(u_{12})$$
$$\mathrm{R}_{21}(u_{12})\mathrm{L}_1(u_1 + \tfrac{1}{2}c_1)\mathrm{L}_2(u_2 + \tfrac{1}{2}c_2) = \mathrm{L}_1(u_2 + \tfrac{1}{2}c_1)\mathrm{L}_2(u_1 + \tfrac{1}{2}c_2)\mathrm{R}_{21}(u_{12})$$
$$\mathrm{R}_{12}(u_{12})\mathrm{L}_1(u_1 - \tfrac{1}{2}c_1)\mathrm{L}_2(u_2 - \tfrac{1}{2}c_2) = \mathrm{L}_1(u_2 - \tfrac{1}{2}c_1)\mathrm{L}_2(u_1 - \tfrac{1}{2}c_2)\mathrm{R}_{12}(u_{12})$$

Monodromy matrix: $\mathrm{T} = \mathrm{L}_1(v_1) \ldots \mathrm{L}_n(v_n)$
Yangian invariants are defined to be eigenstates of the monodromy matrix

Representation terms of spinor helicity variables

- Identify canonical supervariables with $4|4$-vectors via

$$\mathbf{x} = (\lambda,\, \partial_{\tilde{\lambda}},\, \partial_{\tilde{\eta}}), \qquad \mathbf{p} = (\partial_\lambda,\, -\tilde{\lambda},\, -\tilde{\eta}).$$

- canonical variables act on the infinite-dimensional vector space V_a of functions in spinor-helicity variables $(\lambda_a,\, \tilde{\lambda}_a,\, \tilde{\eta}_a)$
- n-point Yangian invariant is a function in the n-fold tensor product $V = V_1 \otimes \ldots \otimes V_n$.
- Lax matrix is an (8×8)-matrix and maps $\mathrm{L}_a(u) : V_a \to GL(8, V_a)$
- $\mathrm{R}_{ab}(u) : V_a \otimes V_b \to V_a \otimes V_b$

The R-operator acts on a function defined on $V_a \otimes V_b$ as

$$\mathrm{R}_{ab}(u) F(\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i | \lambda_j, \tilde{\lambda}_j, \tilde{\eta}_j) = \int \frac{\mathrm{d}z}{z^{1+u}} F(\lambda_i - z\lambda_j, \tilde{\lambda}_i, \tilde{\eta}_i | \lambda_j, \tilde{\lambda}_j + z\tilde{\lambda}_i, \tilde{\eta}_j + z\tilde{\eta}_i),$$

which is a BCFW-shift. [Britto, Cachazo Feng, Witten]

How to finally build a Yangian invariant ?

A vacuum state is a particular subset of V_a generated by the delta functions

$$\delta_a := \delta^2(\lambda_a) \quad \text{or} \quad \tilde{\delta}_a := \delta^{2|4}(\tilde{\lambda}_a) := \delta^2(\tilde{\lambda}_a)\delta^4(\tilde{\eta}_a).$$

with

$$\mathrm{L}_a(u)\delta_b = (u-1)\,\delta_{ab}\,\delta_b \mathbb{1}, \qquad \mathrm{L}_a(u)\tilde{\delta}_b = u\,\delta_{ab}\,\tilde{\delta}_b \mathbb{1}\,.$$

For an n-point tree-level Yangian invariant of MHV-degree k, start with a tensor product of vaccum states

$$\Omega = \underbrace{\delta \ldots \delta}_{n-k\,\text{times}}\ \underbrace{\tilde{\delta} \ldots \tilde{\delta}}_{k\,\text{times}}$$

and act with $(2n-4)$ R-operators (an R-chain) on it:

$$\mathrm{R}_{a_1 b_1}(v_1)\ldots \mathrm{R}_{a_{2n-4} b_{2n-4}}(v_{2n-4})\Omega\,.$$

From the $2n$ bosonic δ-functions, one will be left with 4 δ-functions after integration $\to$ *momentum conservation*.

Those functions are *not* Yangian invariants in general: checking the eigenstate condition

$$\mathrm{L}_1(w_1)\dots\mathrm{L}_n(w_n)\,\mathrm{R}_{a_1b_1}(v_1)\dots\mathrm{R}_{a_{2n-4}b_{2n-4}}(v_{2n-4})\Omega$$
$$\stackrel{!}{=}\mathrm{R}_{a_1b_1}(v_1)\dots\mathrm{R}_{a_{2n-4}b_{2n-4}}(v_{2n-4})\,\mathrm{L}_1(\sigma'(w_1))\dots\mathrm{L}_n(\sigma'(w_n))\Omega$$

and demanding the *same* eigenvalue for each cyclic permutation of $\mathrm{T}=\mathrm{L}_1(u_1)\dots\mathrm{L}_n(u_n)$ is
$\Rightarrow$ **equivalent to solving the linear system for an on-shell graph.**

Remarks:

- Similar to the situation for on-shell graphs, the R-chain representation is not unique: there are numerous relations between different R-chains.
- for $\mathcal{Y}$ being an Yangian invariant, its *cyclically shifted*, *parity flipped* and *reflected* version will be as well.
- For example: the three-point $\overline{\text{MHV}}$-amplitude can be represented in the following forms

$$\mathrm{R}_{ab}(u)\mathrm{R}_{bc}(v)\delta_a\delta_b\tilde{\delta}_c=\mathrm{R}_{bc}(v-u)\mathrm{R}_{ca}(-u)\tilde{\delta}_a\delta_b\delta_c$$
$$\mathrm{R}_{ab}(u)\mathrm{R}_{bc}(v)\delta_a\delta_b\tilde{\delta}_c=\mathrm{R}_{cb}(-v)\mathrm{R}_{ba}(-u)\tilde{\delta}_a\delta_b\delta_c$$
$$\mathrm{R}_{ab}(u)\mathrm{R}_{bc}(v)\delta_a\delta_b\tilde{\delta}_c=\mathrm{R}_{bc}(v-u)\mathrm{R}_{ac}(u)\delta_a\delta_b\tilde{\delta}_c\ ,$$

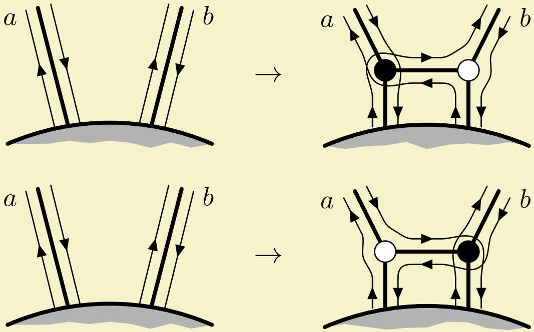
- commuting T through the chain of R-operators needs frequent rewriting
$\rightarrow$ *calculational effort very high*

Translating R-chains into permutations / on-shell graphs

- From the RLL-relations it is clear what R_{ab} does:
  - it is a *swap* altering the permutation - either acting from the left or from the right
  - it applies BCFW-shifts to the kinematic variables.

- vacuum state corresponds to the trivial permutation

- In terms of double-lines



$$\mathrm{R}_{ab} \;\rightarrow\; (ab) \triangleright \begin{pmatrix} \cdot & \cdot & \cdots & \cdot \\ \downarrow & \downarrow & & \downarrow \\ \cdot & \cdot & \cdots & \cdot \end{pmatrix} .$$

$$\mathrm{R}_{ba} \;\rightarrow\; \begin{pmatrix} \cdot & \cdot & \cdots & \cdot \\ \downarrow & \downarrow & & \downarrow \\ \cdot & \cdot & \cdots & \cdot \end{pmatrix} \triangleleft (ba) .$$

Example: five-point invariant

$$\mathrm{R}_{45}\mathrm{R}_{43}\mathrm{R}_{15}\mathrm{R}_{12}\mathrm{R}_{52}\mathrm{R}_{35}\,\delta_1\tilde{\delta}_2\delta_3\delta_4\tilde{\delta}_5\ .$$

$$(35) \triangleright \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}$$

$$(52) \triangleright \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix}$$

$$(12) \triangleright \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix} \triangleleft (15) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$$

$$(45) \triangleright \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix} \triangleleft (34) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}\ .$$

There is a way of decomposing a permutation into a representative series of swaps of minimal length.

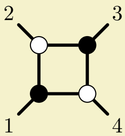
Tree-level: permutation $\leftrightarrow$ class of R-chains

Amplitudes

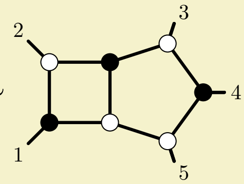
Tree-level

- MHV amplitudes $(k = 2)$ are represented by a *single* on-shell diagram (top-graph):

$\mathcal{A}_{4,\text{MHV}} \sim$



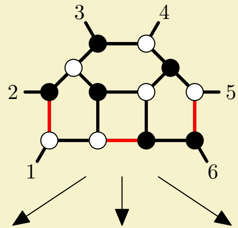
$\mathcal{A}_{5,\text{MHV}} \sim$



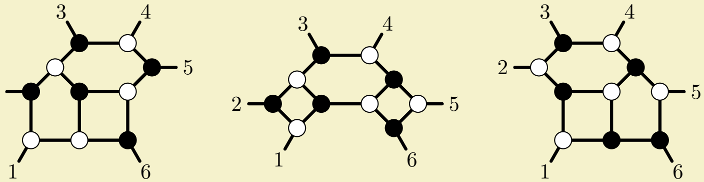
- Permutations are a shift by $k = 2$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix} .$$

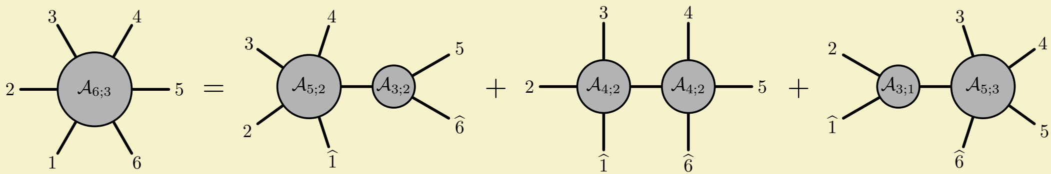
- NMHV amplitudes $(k > 2)$ are composed from channels derived from a top-graph



$\mathcal{A}_{6,\mathsf{NMHV}} \sim$

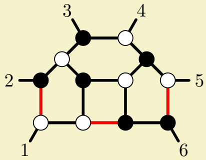


- equivalent to the BCFW-decomposition [Britto, Cachazo Feng, Witten]



What about Yangian invariance of amplitudes with more than one contributing graph?

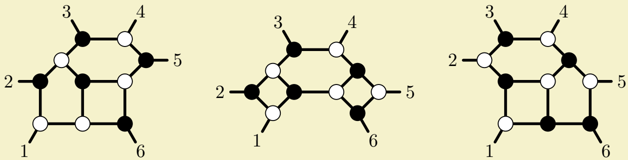
- Impose linear system on top-graph:



$\rightarrow$ 6 free parameters

- decaying into channels requires no central charge flowing along the red lines
  $\rightarrow$ *three additional conditions $\rightarrow$ 3 parameters*

- impose Yangian invariance on each BCFW channel $\rightarrow$ 6 free parameters for each diagram

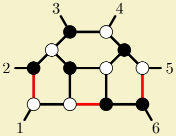


- amplitude: should have the same parameters for each external leg in each channel. *Compatibility* leads to (the same) *three additional conditions*

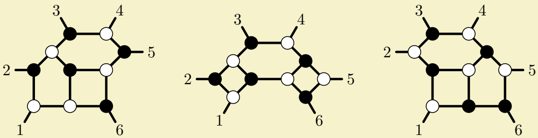
$$u_5 = u_4, \qquad u_2 = u_3. \qquad u_1 = u_6.$$

Reconsider the 6pt NMHV amplitude

$$\mathrm{R}_{34}\mathrm{R}_{45}\mathrm{R}_{23}\mathrm{R}_{34}\color{blue}{\mathrm{R}_{21}}\color{black}{\mathrm{R}_{31}\mathrm{R}_{64}}\color{blue}{\mathrm{R}_{65}\mathrm{R}_{16}}\ \color{black}{\delta_1\delta_2\delta_3\tilde{\delta}_4\tilde{\delta}_5\tilde{\delta}_6}$$



- three channels can be obtained by leaving out the blue R-operators



- permuting the L's through the R's and for each of the cyclic permutations of T and demanding compatibility leads to

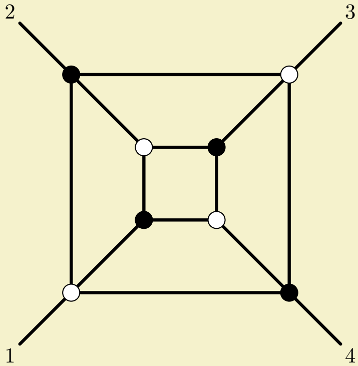
$$u_5 = u_4, \qquad u_2 = u_3. \qquad u_1 = u_6.$$

- complete agreement with the previous calculation

- six-term identity: no deformation possible!

One-loop: undeformed case

- four-point one-loop amplitude contains four additional faces compared to the tree-level amplitude
- four additional R-operators $\to$ four additional integrations
- on-shell conditions do not fix all BCFW-shifts completely



One-loop amplitude

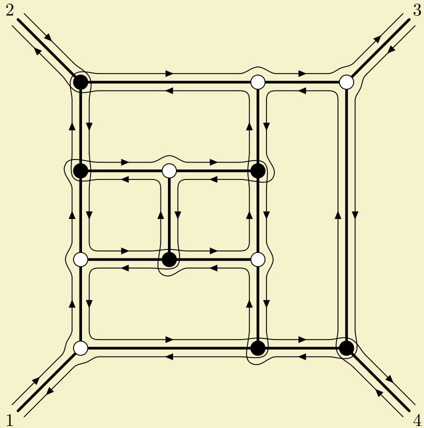
$$\mathcal{A}_4^{\text{1-loop}}(\{a_i\};s,t) = s\,t\,\mathcal{A}_4^{\text{tree}} \cdot I_{\text{box}}(\{a_i\};s,t) \quad \text{where} \quad s = 2k_1 \cdot k_2 \quad \text{and} \quad t = 2k_2 \cdot k_3\,.$$

Box-integral:

$$I_{\mathrm{box}}(\{a_i\};s,t) = \int \frac{\mathrm{d}^4 q}{(q^2)(q+k_1)^2(q+k_1+k_2)^2(q-k_4)^2}.$$

Deformed one-loop amplitudes in the R-operator approach

- integrands for loop amplitudes can be obtained by BCFW recursion
- involves gluing of external legs (forward limit)
- alternatively, represent the final on-shell diagram as a R-chain



One-loop amplitude is produced by

$$R_{43}(u_{43})R_{23}(u_{32})R_{43}(u_{34})R_{21}(u_{21})R_{23}(u_{23})R_{21}(u_{12})R_{41}(u_{41})R_{14}(u_{41})\delta_1\delta_2\tilde{\delta}_3\tilde{\delta}_4$$

which can be shown to yield (along with the tree amplitude)

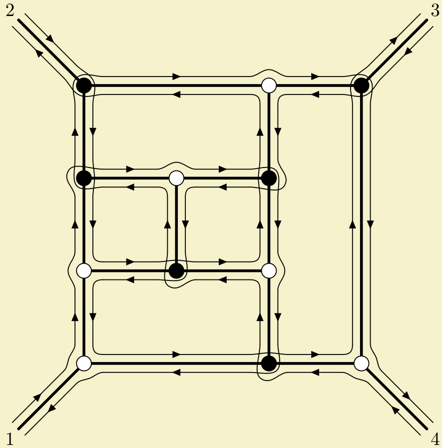
$$I_{\text{box}}(\{a_i\};s,t) = \int \frac{\mathrm{d}^4 q}{(q^2)^{1+a_1}[(q+k_1)^2]^{1+a_2}[(q+k_1+k_2)^2]^{1+a_3}[(q-k_4)^2]^{1+a_4}}.$$

where

$$a_1 = u_4 - u_1, \quad a_2 = u_3 - u_4, \quad a_3 = u_2 - u_3, \quad a_4 = u_1 - u_2\,.$$

Deformed one-loop amplitudes?

- can deformation parameters be employed as regulators? [Ferro, Łukowski, Meneghelli Plefka, Staudacher]
- a deformed amplitude has been calculated from an integrand which is *not* a Yangian invariant
  - undeformed limit: known result
- is there a *"Yangian-invariant regularisation"*?



Deformed box-integral reads:

$$I_{\text{box}}(\{a_i\};s,t)=\int\frac{\mathrm{d}^4q}{(q^2)^{1+a_1}[(q+k_1)^2]^{1+a_2}[(q+k_1+k_2)^2]^{1+a_3}[(q-k_4)^2]^{1+a_4}}.$$

where

$$a_1=u_4-u_1,\quad a_2=u_3-u_4,\quad a_3=u_2-u_3,\quad a_4=u_1-u_2\quad\text{and}\quad \delta:=\sum_i a_i\,.$$

- keep the parameter δ explicit in order to investigate the integral
- convert to Feynman parameters, perform a Mellin-Barnes transformation, carefully choose a contour taking care for pinching to find

$$I_{\mathrm{box}} = \frac{1}{s^{2+\delta}\Gamma(-\delta)} \Bigg\{ \left(\frac{t}{s}\right)^{a_1-1-\delta} \frac{\Gamma(1+\delta-a_1)\Gamma(-\delta+a_1+a_2)\Gamma(-a_1+a_3)\Gamma(-a_2-a_3)}{\Gamma(1+a_2)\Gamma(1+a_3)\Gamma(1+\delta-a_1-a_2-a_3)} + \\ + \left(\frac{t}{s}\right)^{a_3-1-\delta} \frac{\Gamma(1+\delta-a_3)\Gamma(-\delta+a_2+a_3)\Gamma(a_1-a_3)\Gamma(-a_1-a_2)}{\Gamma(1+a_1)\Gamma(1+a_2)\Gamma(1+\delta-a_1-a_2-a_3)} \Bigg\}.$$

- Imposing Yangian invariance $(\delta = 0)$ the function vanishes for generic values of a_1, a_2 and a_3
- in special situations, for example when $a_1 + a_2 = 0$ and $a_2 + a_3 = 0$, the factor $1/\Gamma(-\delta)$ will be cancelled - which allows for the known nonzero result.

A peek on five-point one-loop:

- three diagrams contribute to the integrand:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 1 & 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}.$$

- constraints similar to tree-level NMHV: plugging

$$c_i = u_i - u_{\sigma(i)}$$

simultaneously for all three permutations yields

$$c_i = 0\,.$$

- the relation

$$c_i = u_i - u_{\sigma(i)}$$

is a necessary, but not sufficient condition for defining a loop-level Yangian invariant

- labelling Yangian invariants by permutations only possible for tree-level
for loops: need the on-shell diagram or the R-chain (i.e. non-minimal swap decomposition)

Conclusions

- deformed tree-level amplitudes built from Yangian invariants do not exist beyond the MHV sector
  deformed loop-level amplitudes suffer from similar restrictions
- Can one construct deformed amplitudes from constituent graphs violating Yangian invariance?
- Algebraic formalism leads to the same results as obtained from considering the gluing of Yangian-invariant building blocks into on-shell graphs

Goals & open questions

- Yangian invariant objects are computable: what constraints can be derived from the algebra? Does Yangian symmetry completely fix all tree amplitudes? Is there a formulation of the BCFW recursion relations in an algebraic way?
- Can one understand the holomorphic anomaly and collinear as well as soft limits in the algebraic language?
- 3-pt building blocks are not really *invariant* but violate Yangian invariance for exactly collinear external momenta → *holomorphic anomaly* [**Bargheer, Beisert, Galleas Loebbert, McLoughlin**]
- possibly, the square move and merger are symmetries only up to special kinematical situations

Thanks!